

## Coincidence Points and Common Fixed Points for Expansive Type Mappings in $b$ -Metric Spaces

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**ABSTRACT.** The main purpose of this paper is to obtain sufficient conditions for existence of points of coincidence and common fixed points for a pair of self mappings satisfying some expansive type conditions in  $b$ -metric spaces. Finally, we investigate that the equivalence of one of these results in the context of cone  $b$ -metric spaces cannot be obtained by the techniques using scalarization function. Our results extend and generalize several well known comparable results in the existing literature.

**Keywords:**  $b$ -Metric space, Scalarization function, Point of coincidence, Common fixed point.

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### 1. INTRODUCTION

Fixed point theory plays an important role in applications of many branches of mathematics. There has been a number of generalizations of the usual notion of a metric space. One such generalization is a  $b$ -metric space introduced and studied by Bakhtin [5] and Czerwik [8]. After that a series of articles have been dedicated to the improvement of fixed point theory. In [15], Huang and Zhang introduced the concept of cone metric spaces as a generalization of metric spaces and proved some important fixed point theorems in such spaces. In most of those articles, the authors used normality property of cones in their results.

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Recently, Hussain and Shah[16] introduced the concept of cone  $b$ -metric spaces and studied some topological properties. The aim of this work is to establish sufficient conditions for existence of points of coincidence and common fixed points for a pair of self mappings satisfying some expansive type conditions in the setting of  $b$ -metric spaces. In fact, Theorem 3.1 and its corollaries, as well as Theorem 4.11 are respectively variations of the results of [23] in  $b$ -metric spaces and cone  $b$ -metric spaces. Moreover, we investigate that the equivalence of one of these results in the context of cone  $b$ -metric spaces can be obtained by the techniques using scalarization function and the other cannot be obtained by the same techniques.

## 2. PRELIMINARIES

In this section we need to recall some basic notations, definitions, and necessary results from existing literature.

**Definition 2.1.** [8] Let  $X$  be a nonempty set and  $s \geq 1$  be a given real number. A function  $d : X \times X \rightarrow \mathbb{R}^+$  is said to be a  $b$ -metric on  $X$  if the following conditions hold:

- (i)  $d(x, y) = 0$  if and only if  $x = y$ ;
- (ii)  $d(x, y) = d(y, x)$  for all  $x, y \in X$ ;
- (iii)  $d(x, y) \leq s(d(x, z) + d(z, y))$  for all  $x, y, z \in X$ .

The pair  $(X, d)$  is called a  $b$ -metric space.

Observe that if  $s = 1$ , then the ordinary triangle inequality in a metric space is satisfied, however it does not hold true when  $s > 1$ . Thus the class of  $b$ -metric spaces is effectively larger than that of the ordinary metric spaces. That is, every metric space is a  $b$ -metric space, but the converse need not be true. The following example illustrates the above remarks.

**EXAMPLE 2.2.** Let  $X = \{-1, 0, 1\}$ . Define  $d : X \times X \rightarrow \mathbb{R}^+$  by  $d(x, y) = d(y, x)$  for all  $x, y \in X$ ,  $d(x, x) = 0$ ,  $x \in X$  and  $d(-1, 0) = 3$ ,  $d(-1, 1) = d(0, 1) = 1$ . Then  $(X, d)$  is a  $b$ -metric space, but not a metric space since the triangle inequality is not satisfied. Indeed, we have that

$$d(-1, 1) + d(1, 0) = 1 + 1 = 2 < 3 = d(-1, 0).$$

It is easy to verify that  $s = \frac{3}{2}$ .

**EXAMPLE 2.3.** [27] Let  $(X, d)$  be a metric space and  $\rho(x, y) = (d(x, y))^p$ , where  $p > 1$  is a real number. Then  $\rho$  is a  $b$ -metric with  $s = 2^{p-1}$ .

**Definition 2.4.** [7] Let  $(X, d)$  be a  $b$ -metric space,  $x \in X$  and  $(x_n)$  be a sequence in  $X$ . Then

- (i)  $(x_n)$  converges to  $x$  if and only if  $\lim_{n \rightarrow \infty} d(x_n, x) = 0$ . We denote this by  $\lim_{n \rightarrow \infty} x_n = x$  or  $x_n \rightarrow x (n \rightarrow \infty)$ .

- (ii)  $(x_n)$  is Cauchy if and only if  $\lim_{n,m \rightarrow \infty} d(x_n, x_m) = 0$ .
- (iii)  $(X, d)$  is complete if and only if every Cauchy sequence in  $X$  is convergent.

**Remark 2.5.** [7] In a  $b$ -metric space  $(X, d)$ , the following assertions hold:

- (i) A convergent sequence has a unique limit.
- (ii) Each convergent sequence is Cauchy.
- (iii) In general, a  $b$ -metric is not continuous.

The following example shows that a  $b$ -metric need not be continuous.

**EXAMPLE 2.6.** [18] Let  $X = \mathbb{N} \cup \{\infty\}$  and let  $d : X \times X \rightarrow \mathbb{R}$  be defined by

$$d(m, n) = \begin{cases} 0, & \text{if } m = n, \\ \left| \frac{1}{m} - \frac{1}{n} \right|, & \text{if one of } m, n \text{ is even and the other is even or } \infty, \\ 5, & \text{if one of } m, n \text{ is odd and the other is odd (and } m \neq n) \text{ or } \infty, \\ 2, & \text{otherwise.} \end{cases}$$

Then considering all possible cases, it can be checked that for all  $m, n, p \in X$ , we have

$$d(m, p) \leq \frac{5}{2}(d(m, n) + d(n, p)).$$

Then,  $(X, d)$  is a  $b$ -metric space (with  $s = \frac{5}{2}$ ). Let  $x_n = 2n$  for each  $n \in \mathbb{N}$ . Then

$$d(2n, \infty) = \frac{1}{2n} \rightarrow 0 \text{ as } n \rightarrow \infty,$$

that is,  $x_n \rightarrow \infty$ , but  $d(x_n, 1) = 2 \not\rightarrow 5 = d(\infty, 1)$  as  $n \rightarrow \infty$ .

**Theorem 2.7.** [4] Let  $(X, d)$  be a  $b$ -metric space and suppose that  $(x_n)$  and  $(y_n)$  converge to  $x, y \in X$ , respectively. Then, we have

$$\frac{1}{s^2}d(x, y) \leq \liminf_{n \rightarrow \infty} d(x_n, y_n) \leq \limsup_{n \rightarrow \infty} d(x_n, y_n) \leq s^2d(x, y).$$

In particular, if  $x = y$ , then  $\lim_{n \rightarrow \infty} d(x_n, y_n) = 0$ .

Moreover, for each  $z \in X$ , we have

$$\frac{1}{s}d(x, z) \leq \liminf_{n \rightarrow \infty} d(x_n, z) \leq \limsup_{n \rightarrow \infty} d(x_n, z) \leq sd(x, z).$$

**Definition 2.8.** Let  $(X, d)$  be a  $b$ -metric space with the coefficient  $s \geq 1$  and let  $T : X \rightarrow X$  be a given mapping. We say that  $T$  is continuous at  $x_0 \in X$  if for every sequence  $(x_n)$  in  $X$ , we have  $x_n \rightarrow x_0$  as  $n \rightarrow \infty$  then  $Tx_n \rightarrow Tx_0$  as  $n \rightarrow \infty$ . If  $T$  is continuous at each point  $x_0 \in X$ , then we say that  $T$  is continuous on  $X$ .

**Definition 2.9.** Let  $(X, d)$  be a  $b$ -metric space with the coefficient  $s \geq 1$ . A mapping  $T : X \rightarrow X$  is called expansive if there exists a real constant  $k > s$  such that

$$d(Tx, Ty) \geq k d(x, y)$$

for all  $x, y \in X$ .

**Definition 2.10.** [1] Let  $T$  and  $S$  be self mappings of a set  $X$ . If  $y = Tx = Sx$  for some  $x$  in  $X$ , then  $x$  is called a coincidence point of  $T$  and  $S$  and  $y$  is called a point of coincidence of  $T$  and  $S$ .

**Definition 2.11.** [22] The mappings  $T, S : X \rightarrow X$  are weakly compatible, if for every  $x \in X$ , the following holds:

$$T(Sx) = S(Tx) \text{ whenever } Sx = Tx.$$

**Proposition 2.12.** [1] Let  $S$  and  $T$  be weakly compatible selfmaps of a nonempty set  $X$ . If  $S$  and  $T$  have a unique point of coincidence  $y = Sx = Tx$ , then  $y$  is the unique common fixed point of  $S$  and  $T$ .

### 3. MAIN RESULTS

In this section, we prove some point of coincidence and common fixed point results in  $b$ -metric spaces.

**Theorem 3.1.** Let  $(X, d)$  be a  $b$ -metric space with the coefficient  $s \geq 1$ . Suppose the mappings  $f, g : X \rightarrow X$  satisfy the condition

$$d(fx, fy) \geq \alpha d(gx, gy) + \beta d(fx, gx) + \gamma d(fy, gy) \quad (3.1)$$

for all  $x, y \in X$ , where  $\alpha, \beta, \gamma$  are nonnegative real numbers with  $\alpha + \beta + \gamma > s$ . Assume the following hypotheses:

(i)  $\beta < 1$  and  $\alpha \neq 0$ , (ii)  $g(X) \subseteq f(X)$ , (iii)  $f(X)$  or  $g(X)$  is complete.

Then  $f$  and  $g$  have a point of coincidence in  $X$ . Moreover, if  $\alpha > 1$ , then the point of coincidence is unique. If  $f$  and  $g$  are weakly compatible and  $\alpha > 1$ , then  $f$  and  $g$  have a unique common fixed point in  $X$ .

*Proof.* Let  $x_0 \in X$  and choose  $x_1 \in X$  such that  $gx_0 = fx_1$ . This is possible since  $g(X) \subseteq f(X)$ . Continuing this process, we can construct a sequence  $(x_n)$  in  $X$  such that  $fx_n = gx_{n-1}$ , for all  $n \geq 1$ .

By (3.1), we have

$$\begin{aligned} d(gx_{n-1}, gx_n) &= d(fx_n, fx_{n+1}) \\ &\geq \alpha d(gx_n, gx_{n+1}) + \beta d(fx_n, gx_n) + \gamma d(fx_{n+1}, gx_{n+1}) \\ &= \alpha d(gx_n, gx_{n+1}) + \beta d(gx_{n-1}, gx_n) + \gamma d(gx_n, gx_{n+1}) \end{aligned}$$

which gives that

$$d(gx_n, gx_{n+1}) \leq \lambda d(gx_{n-1}, gx_n)$$

where  $\lambda = \frac{1-\beta}{\alpha+\gamma}$ . It is easy to see that  $\lambda \in (0, \frac{1}{s})$ .

By induction, we get that

$$d(gx_n, gx_{n+1}) \leq \lambda^n d(gx_0, gx_1) \quad (3.2)$$

for all  $n \geq 0$ .

For  $m, n \in \mathbb{N}$  with  $m > n$ , we have by repeated use of (3.2)

$$\begin{aligned} d(gx_n, gx_m) &\leq s[d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_m)] \\ &\leq sd(gx_n, gx_{n+1}) + s^2d(gx_{n+1}, gx_{n+2}) + \cdots \\ &\quad + s^{m-n-1}[d(gx_{m-2}, gx_{m-1}) + d(gx_{m-1}, gx_m)] \\ &\leq [s\lambda^n + s^2\lambda^{n+1} + \cdots + s^{m-n-1}\lambda^{m-2} + s^{m-n-1}\lambda^{m-1}]d(gx_0, gx_1) \\ &\leq [s\lambda^n + s^2\lambda^{n+1} + \cdots + s^{m-n-1}\lambda^{m-2} + s^{m-n}\lambda^{m-1}]d(gx_0, gx_1) \\ &= s\lambda^n [1 + s\lambda + (s\lambda)^2 + \cdots + (s\lambda)^{m-n-2} + (s\lambda)^{m-n-1}]d(gx_0, gx_1) \\ &\leq \frac{s\lambda^n}{1-s\lambda} d(gx_0, gx_1). \end{aligned}$$

So  $(gx_n)$  is a Cauchy sequence in  $g(X)$ . Suppose that  $g(X)$  is a complete subspace of  $X$ . Then there exists  $y \in g(X) \subseteq f(X)$  such that  $gx_n \rightarrow y$  and also  $fx_n \rightarrow y$ . In case,  $f(X)$  is complete, this holds also with  $y \in f(X)$ . Let  $u \in X$  be such that  $fu = y$ .

By (3.1), we have

$$\begin{aligned} d(gx_{n-1}, fu) &= d(fx_n, fu) \\ &\geq \alpha d(gx_n, gu) + \beta d(fx_n, gx_n) + \gamma d(fu, gu) \\ &\geq \alpha d(gx_n, gu). \end{aligned}$$

If  $\alpha \neq 0$ , then

$$d(gx_n, gu) \leq \frac{1}{\alpha} d(gx_{n-1}, fu).$$

Therefore,

$$\begin{aligned} d(y, gu) &\leq s[d(y, gx_n) + d(gx_n, gu)] \\ &\leq s[d(y, gx_n) + \frac{1}{\alpha} d(gx_{n-1}, fu)] \\ &= s[d(y, gx_n) + \frac{1}{\alpha} d(fx_n, fu)]. \end{aligned}$$

Taking limit as  $n \rightarrow \infty$ , we have  $d(y, gu) = 0$ , i.e.,  $gu = y$  and hence  $fu = gu = y$ . Therefore,  $y$  is a point of coincidence of  $f$  and  $g$ .

Now we suppose that  $\alpha > 1$ . Let  $v$  be another point of coincidence of  $f$  and  $g$ . So  $fx = gx = v$  for some  $x \in X$ . Then

$$d(y, v) = d(fu, fx) \geq \alpha d(gu, gx) + \beta d(fu, gu) + \gamma d(fx, gx) = \alpha d(y, v),$$

which implies that

$$d(y, v) \leq \frac{1}{\alpha} d(y, v).$$

Since  $\alpha > 1$ , we have  $d(v, y) = 0$  i.e.,  $v = y$ . Therefore,  $f$  and  $g$  have a unique point of coincidence in  $X$ .

If  $f$  and  $g$  are weakly compatible, then by Proposition 2.12,  $f$  and  $g$  have a unique common fixed point in  $X$ .  $\square$

**Corollary 3.2.** *Let  $(X, d)$  be a  $b$ -metric space with the coefficient  $s \geq 1$ . Suppose the mappings  $f, g : X \rightarrow X$  satisfy the condition*

$$d(fx, fy) \geq \alpha d(gx, gy)$$

*for all  $x, y \in X$ , where  $\alpha > s$  is a constant. If  $g(X) \subseteq f(X)$  and  $f(X)$  or  $g(X)$  is complete, then  $f$  and  $g$  have a unique point of coincidence in  $X$ . Moreover, if  $f$  and  $g$  are weakly compatible, then  $f$  and  $g$  have a unique common fixed point in  $X$ .*

*Proof.* It follows by taking  $\beta = \gamma = 0$  in Theorem 3.1.  $\square$

The following Corollary is the  $b$ -metric version of Banach's contraction principle.

**Corollary 3.3.** *Let  $(X, d)$  be a complete  $b$ -metric space with the coefficient  $s \geq 1$ . Suppose the mapping  $g : X \rightarrow X$  satisfies the contractive condition*

$$d(gx, gy) \leq \lambda d(x, y)$$

*for all  $x, y \in X$ , where  $\lambda \in (0, \frac{1}{s})$  is a constant. Then  $g$  has a unique fixed point in  $X$ . Furthermore, the iterative sequence  $(g^n x)$  converges to the fixed point.*

*Proof.* It follows by taking  $\beta = \gamma = 0$  and  $f = I$ , the identity mapping on  $X$ , in Theorem 3.1.  $\square$

**Corollary 3.4.** *Let  $(X, d)$  be a complete  $b$ -metric space with the coefficient  $s \geq 1$ . Suppose the mapping  $f : X \rightarrow X$  is onto and satisfies*

$$d(fx, fy) \geq \alpha d(x, y)$$

*for all  $x, y \in X$ , where  $\alpha > s$  is a constant. Then  $f$  has a unique fixed point in  $X$ .*

*Proof.* Taking  $g = I$  and  $\beta = \gamma = 0$  in Theorem 3.1, we obtain the desired result.  $\square$

**Corollary 3.5.** *Let  $(X, d)$  be a complete  $b$ -metric space with the coefficient  $s \geq 1$ . Suppose the mapping  $f : X \rightarrow X$  is onto and satisfies the condition*

$$d(fx, fy) \geq \alpha d(x, y) + \beta d(fx, x) + \gamma d(fy, y)$$

*for all  $x, y \in X$ , where  $\alpha, \beta, \gamma$  are nonnegative real numbers with  $\alpha \neq 0$ ,  $\beta < 1$ ,  $\alpha + \beta + \gamma > s$ . Then  $f$  has a fixed point in  $X$ . Moreover, if  $\alpha > 1$ , then the fixed point of  $f$  is unique.*

*Proof.* It follows by taking  $g = I$  in Theorem 3.1. □

**Theorem 3.6.** *Let  $(X, d)$  be a complete  $b$ -metric space with the coefficient  $s \geq 1$ . Suppose the mappings  $S, T : X \rightarrow X$  satisfy the following conditions:*

$$d(T(Sx), Sx) + \frac{k}{s} d(T(Sx), x) \geq \alpha d(Sx, x) \quad (3.3)$$

*and*

$$d(S(Tx), Tx) + \frac{k}{s} d(S(Tx), x) \geq \beta d(Tx, x) \quad (3.4)$$

*for all  $x \in X$ , where  $\alpha, \beta, k$  are nonnegative real numbers with  $\alpha > s + (1 + s)k$  and  $\beta > s + (1 + s)k$ . If  $S$  and  $T$  are continuous and surjective, then  $S$  and  $T$  have a common fixed point in  $X$ .*

*Proof.* Let  $x_0 \in X$  be arbitrary and choose  $x_1 \in X$  such that  $x_0 = Tx_1$ . This is possible since  $T$  is surjective. Since  $S$  is also surjective, there exists  $x_2 \in X$  such that  $x_1 = Sx_2$ . Continuing this process, we can construct a sequence  $(x_n)$  in  $X$  such that  $x_{2n} = Tx_{2n+1}$  and  $x_{2n-1} = Sx_{2n}$  for all  $n \in \mathbb{N}$ .

Using (3.3), we have for  $n \in \mathbb{N} \cup \{0\}$

$$d(T(Sx_{2n+2}), Sx_{2n+2}) + \frac{k}{s} d(T(Sx_{2n+2}), x_{2n+2}) \geq \alpha d(Sx_{2n+2}, x_{2n+2})$$

which implies that

$$d(x_{2n}, x_{2n+1}) + \frac{k}{s} d(x_{2n}, x_{2n+2}) \geq \alpha d(x_{2n+1}, x_{2n+2}).$$

Hence, we have

$$\alpha d(x_{2n+1}, x_{2n+2}) \leq d(x_{2n}, x_{2n+1}) + kd(x_{2n}, x_{2n+1}) + kd(x_{2n+1}, x_{2n+2}).$$

Therefore,

$$d(x_{2n+1}, x_{2n+2}) \leq \frac{1+k}{\alpha-k} d(x_{2n}, x_{2n+1}). \quad (3.5)$$

Using (3.4) and by an argument similar to that used above, we obtain that

$$d(x_{2n}, x_{2n+1}) \leq \frac{1+k}{\beta-k} d(x_{2n-1}, x_{2n}). \quad (3.6)$$

Let  $\lambda = \max \left\{ \frac{1+k}{\alpha-k}, \frac{1+k}{\beta-k} \right\}$ . It is easy to see that  $\lambda \in (0, \frac{1}{s})$ .  
Combining (3.5) and (3.6), we get

$$d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_n) \quad (3.7)$$

for all  $n \geq 1$ . By repeated application of (3.7), we obtain

$$d(x_n, x_{n+1}) \leq \lambda^n d(x_0, x_1).$$

By an argument similar to that used in Theorem 3.1, it follows that  $(x_n)$  is a Cauchy sequence in  $X$ . Since  $X$  is complete, there exists  $u \in X$  such that  $x_n \rightarrow u$  as  $n \rightarrow \infty$ . Now,  $x_{2n+1} \rightarrow u$  and  $x_{2n} \rightarrow u$  as  $n \rightarrow \infty$ . The continuity of  $S$  and  $T$  imply that  $Tx_{2n+1} \rightarrow Tu$  and  $Sx_{2n} \rightarrow Su$  as  $n \rightarrow \infty$  i.e.,  $x_{2n} \rightarrow Tu$  and  $x_{2n-1} \rightarrow Su$  as  $n \rightarrow \infty$ . The uniqueness of limit yields that  $u = Su = Tu$ . Hence,  $u$  is a common fixed point of  $S$  and  $T$ .  $\square$

**Corollary 3.7.** *Let  $(X, d)$  be a complete  $b$ -metric space with the coefficient  $s \geq 1$ . Let  $T : X \rightarrow X$  be a continuous surjective mapping such that*

$$d(T^2x, Tx) + \frac{k}{s}d(T^2x, x) \geq \alpha d(Tx, x)$$

*for all  $x \in X$ , where  $\alpha, k$  are nonnegative real numbers with  $\alpha > s + (1+s)k$ . Then  $T$  has a fixed point in  $X$ .*

*Proof.* It follows from Theorem 3.6 by taking  $S = T$  and  $\beta = \alpha$ .  $\square$

**Corollary 3.8.** *Let  $(X, d)$  be a complete  $b$ -metric space with the coefficient  $s \geq 1$ . Let  $T : X \rightarrow X$  be a continuous surjective mapping such that*

$$d(T^2x, Tx) \geq \alpha d(Tx, x)$$

*for all  $x \in X$ , where  $\alpha > s$  is a constant. Then  $T$  has a fixed point in  $X$ .*

*Proof.* It follows from Theorem 3.6 by taking  $S = T$  and  $\beta = \alpha$ ,  $k = 0$ .  $\square$

We conclude with some examples.

**EXAMPLE 3.9.** Let  $X = [0, 1]$  and  $p > 1$  be a constant. We define  $d : X \times X \rightarrow \mathbb{R}^+$  as

$$d(x, y) = |x - y|^p \text{ for all } x, y \in X.$$

Then  $(X, d)$  is a  $b$ -metric space with the coefficient  $s = 2^{p-1}$ . Let us define  $f, g : X \rightarrow X$  as  $fx = \frac{x}{3}$  and  $gx = \frac{x}{9} - \frac{x^2}{27}$  for all  $x \in X$ . Then, for every  $x, y \in X$  one has  $d(fx, fy) \geq 3^p d(gx, gy)$  i.e., the condition (3.1) holds for  $\alpha = 3^p, \beta = \gamma = 0$ . Thus, we have all the conditions of Theorem 3.1 and  $0 \in X$  is the unique common fixed point of  $f$  and  $g$ .



EXAMPLE 3.10. Let  $X = [0, \infty)$ . We define  $d : X \times X \rightarrow \mathbb{R}^+$  as

$$d(x, y) = |x - y|^2 \text{ for all } x, y \in X.$$

Then  $(X, d)$  is a complete  $b$ -metric space with the coefficient  $s = 2$ . Let us define  $S, T : X \rightarrow X$  as  $Sx = 3x$  and  $Tx = 4x$  for all  $x \in X$ . Then, the conditions (3.3) and (3.4) hold for  $\alpha = \beta = 3 + 3k > s + (1 + s)k$ , where  $k$  is a nonnegative real number. We see that all the conditions of Theorem 3.6 are satisfied and  $0 \in X$  is a common fixed point of  $S$  and  $T$ .

#### 4. SCALARIZATION FUNCTIONS AND FIXED POINTS

Let  $E$  be a real Banach space and  $\theta$  denote the zero element in  $E$ . A cone  $P$  is a subset of  $E$  such that

- (i)  $P$  is closed, nonempty and  $P \neq \{\theta\}$ ;
- (ii)  $a, b \in \mathbb{R}, a, b \geq 0, x, y \in P \Rightarrow ax + by \in P$ ;
- (iii)  $P \cap (-P) = \{\theta\}$ .

For any cone  $P \subseteq E$ , we can define a partial ordering  $\preceq$  on  $E$  with respect to  $P$  by  $x \preceq y$  (equivalently,  $y \succeq x$ ) if and only if  $y - x \in P$ . We shall write  $x \prec y$  (equivalently,  $y \succ x$ ) if  $x \preceq y$  and  $x \neq y$ , while  $x \ll y$  will stand for  $y - x \in \text{int}(P)$ , where  $\text{int}(P)$  denotes the interior of  $P$ . Throughout this section, we suppose that  $E$  is a real Banach space,  $P$  is a cone in  $E$  with  $\text{int}(P) \neq \emptyset$  and  $\preceq$  is a partial ordering on  $E$  with respect to  $P$ .

**Definition 4.1.** [15] Let  $X$  be a nonempty set. Suppose the mapping  $d : X \times X \rightarrow E$  satisfies

- (i)  $\theta \preceq d(x, y)$  for all  $x, y \in X$  and  $d(x, y) = \theta$  if and only if  $x = y$ ;
- (ii)  $d(x, y) = d(y, x)$  for all  $x, y \in X$ ;
- (iii)  $d(x, y) \preceq d(x, z) + d(z, y)$  for all  $x, y, z \in X$ .

Then  $d$  is called a cone metric on  $X$ , and  $(X, d)$  is called a cone metric space.

**Definition 4.2.** [16] Let  $X$  be a nonempty set and  $E$  a real Banach space with cone  $P$ . A vector valued function  $p : X \times X \rightarrow E$  is said to be a cone  $b$ -metric function on  $X$  with the constant  $s \geq 1$  if the following conditions are satisfied:

- (i)  $\theta \preceq p(x, y)$  for all  $x, y \in X$  and  $p(x, y) = \theta$  if and only if  $x = y$ ;
- (ii)  $p(x, y) = p(y, x)$  for all  $x, y \in X$ ;
- (iii)  $p(x, y) \preceq s(p(x, z) + p(z, y))$  for all  $x, y, z \in X$ .

The pair  $(X, p)$  is called a cone  $b$ -metric space.

**Definition 4.3.** [11, 12, 13] The nonlinear scalarization function  $\xi_e : E \rightarrow \mathbb{R}$ , where  $e \in \text{int}(P)$  is defined as follows:

$$\xi_e(y) = \inf\{r \in \mathbb{R} : y \in re - P\} \text{ for all } y \in E.$$

**Lemma 4.4.** [11, 12, 13] For each  $r \in \mathbb{R}$  and  $y \in E$ , the following statements are satisfied:

- (i)  $\xi_e(y) \leq r \iff y \in re - P$ ,
- (ii)  $\xi_e(y) > r \iff y \notin re - P$ ,
- (iii)  $\xi_e(y) \geq r \iff y \notin re - \text{int}(P)$ ,
- (iv)  $\xi_e(y) < r \iff y \in re - \text{int}(P)$ ,
- (v)  $\xi_e(\cdot)$  is positively homogeneous and continuous on  $E$ ,
- (vi) if  $y_1 \in y_2 + P$  (i.e.  $y_2 \preceq y_1$ ), then  $\xi_e(y_2) \leq \xi_e(y_1)$ ,
- (vii)  $\xi_e(y_1 + y_2) \leq \xi_e(y_1) + \xi_e(y_2)$  for all  $y_1, y_2 \in E$ .

**Remark 4.5.** [13]

- (a) Clearly  $\xi_e(\theta) = 0$ .
- (b) It is worth mentioning that the reverse statement of (vi) in Lemma 4.4 does not hold in general.

**Theorem 4.6.** [13] Let  $(X, p)$  be a cone  $b$ -metric space. Then,  $d_p : X \times X \rightarrow [0, \infty)$  defined by  $d_p = \xi_e \circ p$  is a  $b$ -metric.

**Definition 4.7.** [16] Let  $(X, p)$  be a cone  $b$ -metric space,  $x \in X$  and  $(x_n)$  be a sequence in  $X$ . Then

- (i)  $(x_n)$  converges to  $x$  whenever, for every  $c \in E$  with  $\theta \ll c$ , there is a natural number  $n_0$  such that for all  $n > n_0$ ,  $p(x_n, x) \ll c$ . We denote this by  $\lim_{n \rightarrow \infty} x_n = x$  or  $x_n \rightarrow x$  ( $n \rightarrow \infty$ );
- (ii)  $(x_n)$  is a Cauchy sequence whenever, for every  $c \in E$  with  $\theta \ll c$ , there is a natural number  $n_0$  such that for all  $n, m > n_0$ ,  $p(x_n, x_m) \ll c$ ;
- (iii)  $(X, p)$  is a complete cone  $b$ -metric space if every Cauchy sequence is convergent.

**Definition 4.8.** Let  $(X, p)$  be a cone  $b$ -metric space and let  $T : X \rightarrow X$  be a given mapping. We say that  $T$  is continuous at  $x_0 \in X$  if for every sequence  $(x_n)$  in  $X$ , we have  $x_n \rightarrow x_0$  as  $n \rightarrow \infty$  then  $Tx_n \rightarrow Tx_0$  as  $n \rightarrow \infty$ . If  $T$  is continuous at each point  $x_0 \in X$ , then we say that  $T$  is continuous on  $X$ .

**Theorem 4.9.** [13] Let  $(X, p)$  be a cone  $b$ -metric space,  $x \in X$  and  $(x_n)$  be a sequence in  $X$ . Set  $d_p = \xi_e \circ p$ . Then the following statements hold:

- (i)  $(x_n)$  converges to  $x$  in cone  $b$ -metric space  $(X, p)$  if and only if  $d_p(x_n, x) \rightarrow 0$  as  $n \rightarrow \infty$ ,
- (ii)  $(x_n)$  is a Cauchy sequence in cone  $b$ -metric space  $(X, p)$  if and only if  $(x_n)$  is a Cauchy sequence in  $(X, d_p)$ ,

- (iii)  $(X, p)$  is a complete cone  $b$ -metric space if and only if  $(X, d_p)$  is a complete  $b$ -metric space.

**Theorem 4.10.** Let  $(X, p)$  be a complete cone  $b$ -metric space with the coefficient  $s \geq 1$ . Suppose the mappings  $S, T : X \rightarrow X$  satisfy the following conditions:

$$p(T(Sx), Sx) + \frac{k}{s}p(T(Sx), x) \succeq \alpha p(Sx, x) \quad (4.1)$$

and

$$p(S(Tx), Tx) + \frac{k}{s}p(S(Tx), x) \succeq \beta p(Tx, x) \quad (4.2)$$

for all  $x \in X$ , where  $\alpha, \beta, k$  are nonnegative real numbers with  $\alpha > s + (1+s)k$  and  $\beta > s + (1+s)k$ . If  $S$  and  $T$  are continuous and surjective, then  $S$  and  $T$  have a common fixed point in  $X$ .

*Proof.* Taking  $d_p = \xi_e \circ p$ , it follows that  $d_p$  is a  $b$ -metric on  $X$ . Using Theorem 4.9, we conclude that  $(X, d_p)$  is a complete  $b$ -metric space and  $S, T$  are continuous on  $(X, d_p)$ . By applying Lemma 4.4, we obtain from (4.1) and (4.2) that

$$d_p(T(Sx), Sx) + \frac{k}{s}d_p(T(Sx), x) \geq \alpha d_p(Sx, x)$$

and

$$d_p(S(Tx), Tx) + \frac{k}{s}d_p(S(Tx), x) \geq \beta d_p(Tx, x)$$

for all  $x \in X$ , where  $\alpha, \beta, k$  are nonnegative real numbers with  $\alpha > s + (1+s)k$  and  $\beta > s + (1+s)k$ .

Now, Theorem 3.6 applies to obtain the desired result.  $\square$

Following a similar argument as in Theorem 3.1, we can derive the following theorem.

**Theorem 4.11.** Let  $(X, p)$  be a cone  $b$ -metric space with the coefficient  $s \geq 1$ . Suppose the mappings  $f, g : X \rightarrow X$  satisfy

$$p(fx, fy) \succeq \alpha p(gx, gy) + \beta p(fx, gx) + \gamma p(fy, gy)$$

for all  $x, y \in X$ , where  $\alpha, \beta, \gamma$  are nonnegative real numbers with  $\alpha + \beta + \gamma > s$ . Assume the following hypotheses:

- (i)  $\beta < 1$  and  $\alpha \neq 0$ , (ii)  $g(X) \subseteq f(X)$ , (iii)  $f(X)$  or  $g(X)$  is complete.

Then  $f$  and  $g$  have a point of coincidence in  $X$ . Moreover, if  $\alpha > 1$ , then the point of coincidence is unique. If  $f$  and  $g$  are weakly compatible and  $\alpha > 1$ , then  $f$  and  $g$  have a unique common fixed point in  $X$ .

**Remark 4.12.** We observe that the last theorem cannot be derived by the techniques using scalarization function. In fact, Lemma 4.4 does not imply that

$$\xi_e(p(fx, fy)) \geq \alpha \xi_e(p(gx, gy)) + \beta \xi_e(p(fx, gx)) + \gamma \xi_e(p(fy, gy))$$

for all  $x, y \in X$ , where  $\alpha, \beta, \gamma$  are nonnegative real numbers with  $\alpha + \beta + \gamma > s$ .  
or, equivalently,

$$d_p(fx, fy) \geq \alpha d_p(gx, gy) + \beta d_p(fx, gx) + \gamma d_p(fy, gy).$$

for all  $x, y \in X$ , where  $\alpha, \beta, \gamma$  are nonnegative real numbers with  $\alpha + \beta + \gamma > s$ .

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