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Coincidence Points and Common Fixed Points for Expansive Type Mappings in b-Metric Spaces

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Abstract. The main purpose of this paper is to obtain sufficient conditions for existence of points of coincidence and common fixed points for a pair of self mappings satisfying some expansive type conditions in b-metric spaces. Finally, we investigate that the equivalence of one of these results in the context of cone b-metric spaces cannot be obtained by the techniques using scalarization function. Our results extend and generalize several well known comparable results in the existing literature.

Keywords: b-Metric space, Scalarization function, Point of coincidence, Common fixed point.

2000 Mathematics subject classification: 54H25, 47H10.

1. Introduction

Fixed point theory plays an important role in applications of many branches of mathematics. There has been a number of generalizations of the usual notion of a metric space. One such generalization is a b-metric space introduced and studied by Bakhtin [5] and Czerwik [8]. After that a series of articles have been dedicated to the improvement of fixed point theory. In [15], Huang and Zhang introduced the concept of cone metric spaces as a generalization of metric spaces and proved some important fixed point theorems in such spaces. In most of those articles, the authors used normality property of cones in their results.

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Recently, Hussain and Shah[16] introduced the concept of cone b-metric spaces and studied some topological properties. The aim of this work is to establish sufficient conditions for existence of points of coincidence and common fixed points for a pair of self mappings satisfying some expansive type conditions in the setting of b-metric spaces. In fact, Theorem 3.1 and its corollaries, as well as Theorem 4.11 are respectively variations of the results of [23] in b-metric spaces and cone b-metric spaces. Moreover, we investigate that the equivalence of one of these results in the context of cone b-metric spaces can be obtained by the techniques using scalarization function and the other cannot be obtained by the same techniques.

2. Preliminaries

In this section we need to recall some basic notations, definitions, and necessary results from existing literature.

Definition 2.1. [8] Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}^+$ is said to be a b-metric on X if the following conditions hold:

(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, y) \leq s\,(d(x, z) + d(z, y))$ for all $x, y, z \in X$.

The pair (X, d) is called a b-metric space.

Observe that if $s = 1$, then the ordinary triangle inequality in a metric space is satisfied, however it does not hold true when $s > 1$. Thus the class of b-metric spaces is effectively larger than that of the ordinary metric spaces. That is, every metric space is a b-metric space, but the converse need not be true. The following example illustrates the above remarks.

Example 2.2. Let $X = \{-1, 0, 1\}$. Define $d : X \times X \to \mathbb{R}^+$ by $d(x, y) = d(y, x)$ for all $x, y \in X$, $d(x, x) = 0$, $x \in X$ and $d(-1, 0) = 3,\ d(-1, 1) = d(0, 1) = 1$. Then (X, d) is a b-metric space, but not a metric space since the triangle inequality is not satisfied. Indeed, we have that

$$d(-1, 1) + d(1, 0) = 1 + 1 = 2 < 3 = d(-1, 0).$$

It is easy to verify that $s = \frac{3}{2}$.

Example 2.3. [27] Let (X, d) be a metric space and $\rho(x, y) = (d(x, y))^p$, where $p > 1$ is a real number. Then ρ is a b-metric with $s = 2^{p-1}$.

Definition 2.4. [7] Let (X, d) be a b-metric space, $x \in X$ and (x_n) be a sequence in X. Then

(i) (x_n) converges to x if and only if $\lim\limits_{n\to\infty} d(x_n, x) = 0$. We denote this by $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x (n \to \infty)$.

(ii) (x_n) is Cauchy if and only if $\lim\limits_{n,m\to\infty} d(x_n, x_m) = 0$.

(iii) (X, d) is complete if and only if every Cauchy sequence in X is convergent.

Remark 2.5. [7] In a b-metric space (X, d), the following assertions hold:

(i) A convergent sequence has a unique limit.

(ii) Each convergent sequence is Cauchy.

(iii) In general, a b-metric is not continuous.

The following example shows that a b-metric need not be continuous.

Example 2.6. [18] Let $X = \mathbb{N} \cup \{\infty\}$ and let $d : X \times X \to \mathbb{R}$ be defined by

$$d(m,n) = \begin{cases} 0, \; if \; m = n, \\ \\ \mid \frac{1}{m} - \frac{1}{n} \mid, \; if \; one \; of \; m, n \; is \; even \; and \; the \; other \; is \; even \; or \; \infty, \\ \\ 5, \; if \; one \; of \; m, n \; is \; odd \; and \; the \; other \; is \; odd \, (and \; m \neq n) \; or \; \infty, \\ \\ 2, \; otherwise. \end{cases}$$

Then considering all possible cases, it can be checked that for all $m, n, p \in X$, we have

$$d(m, p) \leq \frac{5}{2}(d(m, n) + d(n, p)).$$

Then, (X, d) is a b-metric space (with $s = \frac{5}{2}$). Let $x_n = 2n$ for each $n \in \mathbb{N}$. Then

$$d(2n, \infty) = \frac{1}{2n} \to 0 \; as \; n \to \infty,$$

that is, $x_n \to \infty$, but $d(x_n, 1) = 2 \not\to 5 = d(\infty, 1)$ as $n \to \infty$.

Theorem 2.7. [4] *Let (X, d) be a b-metric space and suppose that (x_n) and (y_n) converge to $x, y \in X$, respectively. Then, we have*

$$\frac{1}{s^2} d(x, y) \leq \liminf_{n\to\infty} d(x_n, y_n) \leq \limsup_{n\to\infty} d(x_n, y_n) \leq s^2 d(x, y).$$

In particular, if $x = y$, then $\lim\limits_{n\to\infty} d(x_n, y_n) = 0$.

Moreover, for each $z \in X$, we have

$$\frac{1}{s} d(x, z) \leq \liminf_{n\to\infty} d(x_n, z) \leq \limsup_{n\to\infty} d(x_n, z) \leq s d(x, z).$$

Definition 2.8. Let (X, d) be a b-metric space with the coefficient $s \geq 1$ and let $T : X \to X$ be a given mapping. We say that T is continuous at $x_0 \in X$ if for every sequence (x_n) in X, we have $x_n \to x_0$ as $n \to \infty$ then $Tx_n \to Tx_0$ as $n \to \infty$. If T is continuous at each point $x_0 \in X$, then we say that T is continuous on X.

Definition 2.9. Let (X, d) be a b-metric space with the coefficient $s \geq 1$. A mapping $T : X \to X$ is called expansive if there exists a real constant $k > s$ such that

$$d(Tx, Ty) \geq k\, d(x, y)$$

for all $x, y \in X$.

Definition 2.10. [1] Let T and S be self mappings of a set X. If $y = Tx = Sx$ for some x in X, then x is called a coincidence point of T and S and y is called a point of coincidence of T and S.

Definition 2.11. [22] The mappings $T, S : X \to X$ are weakly compatible, if for every $x \in X$, the following holds:

$$T(Sx) = S(Tx) \text{ whenever } Sx = Tx.$$

Proposition 2.12. [1] *Let S and T be weakly compatible selfmaps of a nonempty set X. If S and T have a unique point of coincidence $y = Sx = Tx$, then y is the unique common fixed point of S and T.*

3. Main Results

In this section, we prove some point of coincidence and common fixed point results in b-metric spaces.

Theorem 3.1. *Let (X, d) be a b-metric space with the coefficient $s \geq 1$. Suppose the mappings f, $g : X \to X$ satisfy the condition*

$$d(fx, fy) \geq \alpha\, d(gx, gy) + \beta\, d(fx, gx) + \gamma\, d(fy, gy) \tag{3.1}$$

for all $x, y \in X$, where α, β, γ are nonnegative real numbers with $\alpha+\beta+\gamma > s$. Assume the following hypotheses:
(i) $\beta < 1$ and $\alpha \neq 0$, (ii) $g(X) \subseteq f(X)$, (iii) $f(X)$ or $g(X)$ is complete.
Then f and g have a point of coincidence in X. Moreover, if $\alpha > 1$, then the point of coincidence is unique. If f and g are weakly compatible and $\alpha > 1$, then f and g have a unique common fixed point in X.

Proof. Let $x_0 \in X$ and choose $x_1 \in X$ such that $gx_0 = fx_1$. This is possible since $g(X) \subseteq f(X)$. Continuing this process, we can construct a sequence (x_n) in X such that $fx_n = gx_{n-1}$, for all $n \geq 1$.

By (3.1), we have

$$\begin{aligned} d(gx_{n-1}, gx_n) &= d(fx_n, fx_{n+1}) \\ &\geq \alpha d(gx_n, gx_{n+1}) + \beta d(fx_n, gx_n) + \gamma d(fx_{n+1}, gx_{n+1}) \\ &= \alpha d(gx_n, gx_{n+1}) + \beta d(gx_{n-1}, gx_n) + \gamma d(gx_n, gx_{n+1}) \end{aligned}$$

which gives that

$$d(gx_n, gx_{n+1}) \leq \lambda d(gx_{n-1}, gx_n)$$

where $\lambda = \frac{1-\beta}{\alpha+\gamma}$. It is easy to see that $\lambda \in (0, \frac{1}{s})$.

By induction, we get that

$$d(gx_n, gx_{n+1}) \leq \lambda^n d(gx_0, gx_1) \tag{3.2}$$

for all $n \geq 0$.

For $m, n \in \mathbb{N}$ with $m > n$, we have by repeated use of (3.2)

$$\begin{aligned} d(gx_n, gx_m) &\leq s\left[d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_m)\right] \\ &\leq sd(gx_n, gx_{n+1}) + s^2 d(gx_{n+1}, gx_{n+2}) + \cdots \\ &\quad + s^{m-n-1}\left[d(gx_{m-2}, gx_{m-1}) + d(gx_{m-1}, gx_m)\right] \\ &\leq \left[s\lambda^n + s^2\lambda^{n+1} + \cdots + s^{m-n-1}\lambda^{m-2} + s^{m-n-1}\lambda^{m-1}\right] d(gx_0, gx_1) \\ &\leq \left[s\lambda^n + s^2\lambda^{n+1} + \cdots + s^{m-n-1}\lambda^{m-2} + s^{m-n}\lambda^{m-1}\right] d(gx_0, gx_1) \\ &= s\lambda^n\left[1 + s\lambda + (s\lambda)^2 + \cdots + (s\lambda)^{m-n-2} + (s\lambda)^{m-n-1}\right] d(gx_0, gx_1) \\ &\leq \frac{s\lambda^n}{1 - s\lambda}\, d(gx_0, gx_1). \end{aligned}$$

So (gx_n) is a Cauchy sequence in $g(X)$. Suppose that $g(X)$ is a complete subspace of X. Then there exists $y \in g(X) \subseteq f(X)$ such that $gx_n \to y$ and also $fx_n \to y$. In case, $f(X)$ is complete, this holds also with $y \in f(X)$. Let $u \in X$ be such that $fu = y$.
By (3.1), we have

$$\begin{aligned} d(gx_{n-1}, fu) &= d(fx_n, fu) \\ &\geq \alpha d(gx_n, gu) + \beta d(fx_n, gx_n) + \gamma d(fu, gu) \\ &\geq \alpha d(gx_n, gu). \end{aligned}$$

If $\alpha \neq 0$, then

$$d(gx_n, gu) \leq \frac{1}{\alpha} d(gx_{n-1}, fu).$$

Therefore,

$$\begin{aligned} d(y, gu) &\leq s[d(y, gx_n) + d(gx_n, gu)] \\ &\leq s[d(y, gx_n) + \frac{1}{\alpha} d(gx_{n-1}, fu)] \\ &= s[d(y, gx_n) + \frac{1}{\alpha} d(fx_n, fu)]. \end{aligned}$$

Taking limit as $n \to \infty$, we have $d(y, gu) = 0$, i.e., $gu = y$ and hence $fu = gu = y$. Therefore, y is a point of coincidence of f and g.

Now we suppose that $\alpha > 1$. Let v be another point of coincidence of f and g. So $fx = gx = v$ for some $x \in X$. Then

$$d(y, v) = d(fu, fx) \geq \alpha d(gu, gx) + \beta d(fu, gu) + \gamma d(fx, gx) = \alpha d(y, v),$$

which implies that

$$d(y,v) \leq \frac{1}{\alpha} d(y,v).$$

Since $\alpha > 1$, we have $d(v,y) = 0$ i.e., $v = y$. Therefore, f and g have a unique point of coincidence in X.

If f and g are weakly compatible, then by Proposition 2.12 , f and g have a unique common fixed point in X. □

Corollary 3.2. *Let (X,d) be a b-metric space with the coefficient $s \geq 1$. Suppose the mappings $f, g : X \to X$ satisfy the condition*

$$d(fx, fy) \geq \alpha d(gx, gy)$$

for all $x, y \in X$, where $\alpha > s$ is a constant. If $g(X) \subseteq f(X)$ and $f(X)$ or $g(X)$ is complete, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in X.

Proof. It follows by taking $\beta = \gamma = 0$ in Theorem 3.1.

□

The following Corollary is the b-metric version of Banach's contraction principle.

Corollary 3.3. *Let (X,d) be a complete b-metric space with the coefficient $s \geq 1$. Suppose the mapping $g : X \to X$ satisfies the contractive condition*

$$d(gx, gy) \leq \lambda\, d(x,y)$$

for all $x, y \in X$, where $\lambda \in (0, \frac{1}{s})$ is a constant. Then g has a unique fixed point in X. Furthermore, the iterative sequence $(g^n x)$ converges to the fixed point.

Proof. It follows by taking $\beta = \gamma = 0$ and $f = I$, the identity mapping on X, in Theorem 3.1.

□

Corollary 3.4. *Let (X,d) be a complete b-metric space with the coefficient $s \geq 1$. Suppose the mapping $f : X \to X$ is onto and satisfies*

$$d(fx, fy) \geq \alpha d(x,y)$$

for all $x, y \in X$, where $\alpha > s$ is a constant. Then f has a unique fixed point in X.

Proof. Taking $g = I$ and $\beta = \gamma = 0$ in Theorem 3.1, we obtain the desired result.

□

Corollary 3.5. *Let (X,d) be a complete b-metric space with the coefficient $s \geq 1$. Suppose the mapping $f : X \to X$ is onto and satisfies the condition*

$$d(fx,fy) \geq \alpha\, d(x,y) + \beta\, d(fx,x) + \gamma\, d(fy,y)$$

for all $x,y \in X$, where α, β, γ are nonnegative real numbers with $\alpha \neq 0$, $\beta < 1$, $\alpha+\beta+\gamma > s$. Then f has a fixed point in X. Moreover, if $\alpha > 1$, then the fixed point of f is unique.

Proof. It follows by taking $g = I$ in Theorem 3.1.

□

Theorem 3.6. *Let (X,d) be a complete b-metric space with the coefficient $s \geq 1$. Suppose the mappings S, $T : X \to X$ satisfy the following conditions:*

$$d(T(Sx),Sx) + \frac{k}{s} d(T(Sx),x) \geq \alpha d(Sx,x) \tag{3.3}$$

and

$$d(S(Tx),Tx) + \frac{k}{s} d(S(Tx),x) \geq \beta d(Tx,x) \tag{3.4}$$

for all $x \in X$, where α, β, k are nonnegative real numbers with $\alpha > s + (1+s)k$ and $\beta > s + (1+s)k$. If S and T are continuous and surjective, then S and T have a common fixed point in X.

Proof. Let $x_0 \in X$ be arbitrary and choose $x_1 \in X$ such that $x_0 = Tx_1$. This is possible since T is surjective. Since S is also surjective, there exists $x_2 \in X$ such that $x_1 = Sx_2$. Continuing this process, we can construct a sequence (x_n) in X such that $x_{2n} = Tx_{2n+1}$ and $x_{2n-1} = Sx_{2n}$ for all $n \in \mathbb{N}$.

Using (3.3), we have for $n \in \mathbb{N} \cup \{0\}$

$$d(T(Sx_{2n+2}),Sx_{2n+2}) + \frac{k}{s} d(T(Sx_{2n+2}),x_{2n+2}) \geq \alpha d(Sx_{2n+2},x_{2n+2})$$

which implies that

$$d(x_{2n},x_{2n+1}) + \frac{k}{s} d(x_{2n},x_{2n+2}) \geq \alpha d(x_{2n+1},x_{2n+2}).$$

Hence, we have

$$\alpha d(x_{2n+1},x_{2n+2}) \leq d(x_{2n},x_{2n+1}) + kd(x_{2n},x_{2n+1}) + kd(x_{2n+1},x_{2n+2}).$$

Therefore,

$$d(x_{2n+1},x_{2n+2}) \leq \frac{1+k}{\alpha-k} d(x_{2n},x_{2n+1}). \tag{3.5}$$

Using (3.4) and by an argument similar to that used above, we obtain that

$$d(x_{2n},x_{2n+1}) \leq \frac{1+k}{\beta-k} d(x_{2n-1},x_{2n}). \tag{3.6}$$

Let $\lambda = max\left\{\frac{1+k}{\alpha-k}, \frac{1+k}{\beta-k}\right\}$. It is easy to see that $\lambda \in (0, \frac{1}{s})$.
Combining (3.5) and (3.6), we get

$$d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_n) \tag{3.7}$$

for all $n \geq 1$. By repeated application of (3.7), we obtain

$$d(x_n, x_{n+1}) \leq \lambda^n d(x_0, x_1).$$

By an argument similar to that used in Theorem 3.1, it follows that (x_n) is a Cauchy sequence in X. Since X is complete, there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$. Now, $x_{2n+1} \to u$ and $x_{2n} \to u$ as $n \to \infty$. The continuity of S and T imply that $Tx_{2n+1} \to Tu$ and $Sx_{2n} \to Su$ as $n \to \infty$ i.e., $x_{2n} \to Tu$ and $x_{2n-1} \to Su$ as $n \to \infty$. The uniqueness of limit yields that $u = Su = Tu$. Hence, u is a common fixed point of S and T. □

Corollary 3.7. *Let (X, d) be a complete b-metric space with the coefficient $s \geq 1$. Let $T : X \to X$ be a continuous surjective mapping such that*

$$d(T^2x, Tx) + \frac{k}{s} d(T^2x, x) \geq \alpha d(Tx, x)$$

for all $x \in X$, where α, k are nonnegative real numbers with $\alpha > s + (1+s)k$. Then T has a fixed point in X.

Proof. It follows from Theorem 3.6 by taking $S = T$ and $\beta = \alpha$.

□

Corollary 3.8. *Let (X, d) be a complete b-metric space with the coefficient $s \geq 1$. Let $T : X \to X$ be a continuous surjective mapping such that*

$$d(T^2x, Tx) \geq \alpha d(Tx, x)$$

for all $x \in X$, where $\alpha > s$ is a constant. Then T has a fixed point in X.

Proof. It follows from Theorem 3.6 by taking $S = T$ and $\beta = \alpha,\ k = 0$.

□

We conclude with some examples.

Example 3.9. Let $X = [0, 1]$ and $p > 1$ be a constant. We define $d : X \times X \to \mathbb{R}^+$ as

$$d(x, y) =\mid x - y \mid^p \ \ for\ all\ x, y \in X.$$

Then (X, d) is a b-metric space with the coefficient $s = 2^{p-1}$. Let us define $f, g : X \to X$ as $fx = \frac{x}{3}$ and $gx = \frac{x}{9} - \frac{x^2}{27}$ for all $x \in X$. Then, for every $x, y \in X$ one has $d(fx, fy) \geq 3^p d(gx, gy)$ i.e., the condition (3.1) holds for $\alpha = 3^p, \beta = \gamma = 0$. Thus, we have all the conditions of Theorem 3.1 and $0 \in X$ is the unique common fixed point of f and g.

EXAMPLE 3.10. Let $X = [0, \infty)$. We define $d : X \times X \to \mathbb{R}^+$ as

$$d(x, y) = \mid x - y \mid^2 \;\; for\ all\ x, y \in X.$$

Then (X, d) is a complete b-metric space with the coefficient $s = 2$. Let us define $S, T : X \to X$ as $Sx = 3x$ and $Tx = 4x$ for all $x \in X$. Then, the conditions (3.3) and (3.4) hold for $\alpha = \beta = 3 + 3k > s + (1 + s)k$, where k is a nonnegative real number. We see that all the conditions of Theorem 3.6 are satisfied and $0 \in X$ is a common fixed point of S and T.

4. Scalarization Functions and Fixed Points

Let E be a real Banach space and θ denote the zero element in E. A cone P is a subset of E such that

$$\begin{aligned} &(i) && P\ is\ closed,\ nonempty\ and\ P \neq \{\theta\}; \\ &(ii) && a, b \in \mathbb{R},\ a, b \geq 0,\ x, y \in P\ \Rightarrow\ ax + by \in P; \\ &(iii) && P \cap (-P) = \{\theta\}. \end{aligned}$$

For any cone $P \subseteq E$, we can define a partial ordering $\preceq$ on E with respect to P by $x \preceq y$(equivalently, $y \succeq x$) if and only if $y - x \in P$. We shall write $x \prec y$ (equivalently, $y \succ x$) if $x \preceq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in int(P)$, where $int(P)$ denotes the interior of P. Throughout this section, we suppose that E is a real Banach space, P is a cone in E with $int(P) \neq \emptyset$ and $\preceq$ is a partial ordering on E with respect to P.

Definition 4.1. [15] Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies

(i) $\theta \preceq d(x, y)$ $for\ all\ x, y \in X\ and\ d(x, y) = \theta\ if\ and\ only\ if\ x = y$;

(ii) $d(x, y) = d(y, x)$ $for\ all\ x, y \in X$;

(iii) $d(x, y) \preceq d(x, z) + d(z, y)$ $for\ all\ x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space.

Definition 4.2. [16] Let X be a nonempty set and E a real Banach space with cone P. A vector valued function $p : X \times X \to E$ is said to be a cone b-metric function on X with the constant $s \geq 1$ if the following conditions are satisfied:

(i) $\theta \preceq p(x, y)$ $for\ all\ x, y \in X\ and\ p(x, y) = \theta\ if\ and\ only\ if\ x = y$;

(ii) $p(x, y) = p(y, x)$ $for\ all\ x, y \in X$;

(iii) $p(x, y) \preceq s\,(p(x, z) + p(z, y))$ $for\ all\ x, y, z \in X$.

The pair (X, p) is called a cone b-metric space.

Definition 4.3. [11, 12, 13] The nonlinear scalarization function $\xi_e : E \to \mathbb{R}$, where $e \in int\,(P)$ is defined as follows:

$$\xi_e(y) = inf\{r \in \mathbb{R} : y \in re - P\} \; for \; all \; y \in E.$$

Lemma 4.4. [11, 12, 13] *For each $r \in \mathbb{R}$ and $y \in E$, the following statements are satisfied:*

(i) $\xi_e(y) \leq r \iff y \in re - P,$
(ii) $\xi_e(y) > r \iff y \notin re - P,$
(iii) $\xi_e(y) \geq r \iff y \notin re - int\,(P),$
(iv) $\xi_e(y) < r \iff y \in re - int\,(P),$
(v) $\xi_e(\cdot)$ *is positively homogeneous and continuous on* E,
(vi) *if* $y_1 \in y_2 + P\,(i.e.\,y_2 \preceq y_1)$, *then* $\xi_e(y_2) \leq \xi_e(y_1)$,
(vii) $\xi_e(y_1 + y_2) \leq \xi_e(y_1) + \xi_e(y_2)$ *for all* $y_1,\, y_2 \in E$.

Remark 4.5. [13]

(a) Clearly $\xi_e(\theta) = 0$.
(b) It is worth mentioning that the reverse statement of (vi) in Lemma 4.4 does not hold in general.

Theorem 4.6. [13] *Let (X, p) be a cone b-metric space. Then, $d_p : X \times X \to [0, \infty)$ defined by $d_p = \xi_e \circ p$ is a b-metric.*

Definition 4.7. [16] Let (X, p) be a cone b-metric space, $x \in X$ and (x_n) be a sequence in X. Then

(i) (x_n) converges to x whenever, for every $c \in E$ with $\theta \ll c$, there is a natural number n_0 such that for all $n > n_0$, $p(x_n, x) \ll c$. We denote this by $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$ $(n \to \infty)$;
(ii) (x_n) is a Cauchy sequence whenever, for every $c \in E$ with $\theta \ll c$, there is a natural number n_0 such that for all $n, m > n_0$, $p(x_n, x_m) \ll c$;
(iii) (X, p) is a complete cone b-metric space if every Cauchy sequence is convergent.

Definition 4.8. Let (X, p) be a cone b-metric space and let $T : X \to X$ be a given mapping. We say that T is continuous at $x_0 \in X$ if for every sequence (x_n) in X, we have $x_n \to x_0$ as $n \to \infty$ then $Tx_n \to Tx_0$ as $n \to \infty$. If T is continuous at each point $x_0 \in X$, then we say that T is continuous on X.

Theorem 4.9. [13] *Let (X, p) be a cone b-metric space, $x \in X$ and (x_n) be a sequence in X. Set $d_p = \xi_e \circ p$. Then the following statements hold:*

(i) (x_n) *converges to x in cone b-metric space (X, p) if and only if $d_p(x_n, x) \to 0$ as $n \to \infty$,*
(ii) (x_n) *is a Cauchy sequence in cone b-metric space (X, p) if and only if (x_n) is a Cauchy sequence in (X, d_p),*

(iii) *(X,p) is a complete cone b-metric space if and only if (X,d_p) is a complete b-metric space.*

Theorem 4.10. *Let (X,p) be a complete cone b-metric space with the coefficient $s \geq 1$. Suppose the mappings $S,T : X \to X$ satisfy the following conditions:*

$$p(T(Sx),Sx) + \frac{k}{s}p(T(Sx),x) \succeq \alpha p(Sx,x) \tag{4.1}$$

and

$$p(S(Tx),Tx) + \frac{k}{s}p(S(Tx),x) \succeq \beta p(Tx,x) \tag{4.2}$$

for all $x \in X$, where α,β,k are nonnegative real numbers with $\alpha > s+(1+s)k$ and $\beta > s+(1+s)k$. If S and T are continuous and surjective, then S and T have a common fixed point in X.

Proof. Taking $d_p = \xi_e \circ p$, it follows that d_p is a b-metric on X. Using Theorem 4.9, we conclude that (X,d_p) is a complete b-metric space and S, T are continuous on (X,d_p). By applying Lemma 4.4, we obtain from (4.1) and (4.2) that

$$d_p(T(Sx),Sx) + \frac{k}{s}d_p(T(Sx),x) \geq \alpha d_p(Sx,x)$$

and

$$d_p(S(Tx),Tx) + \frac{k}{s}d_p(S(Tx),x) \geq \beta d_p(Tx,x)$$

for all $x \in X$, where α,β,k are nonnegative real numbers with $\alpha > s+(1+s)k$ and $\beta > s+(1+s)k$.
Now, Theorem 3.6 applies to obtain the desired result. □

Following a similar argument as in Theorem 3.1, we can derive the following theorem.

Theorem 4.11. *Let (X,p) be a cone b-metric space with the coefficient $s \geq 1$. Suppose the mappings f, $g : X \to X$ satisfy*

$$p(fx,fy) \succeq \alpha\, p(gx,gy) + \beta\, p(fx,gx) + \gamma\, p(fy,gy)$$

for all $x,y \in X$, where α, β, γ are nonnegative real numbers with $\alpha+\beta+\gamma > s$. Assume the following hypotheses:
(i) $\beta < 1$ and $\alpha \neq 0$, (ii) $g(X) \subseteq f(X)$, (iii) $f(X)$ or $g(X)$ is complete.
Then f and g have a point of coincidence in X. Moreover, if $\alpha > 1$, then the point of coincidence is unique. If f and g are weakly compatible and $\alpha > 1$, then f and g have a unique common fixed point in X.

Remark 4.12. We observe that the last theorem cannot be derived by the techniques using scalarization function. In fact, Lemma 4.4 does not imply that

$$\xi_e(p(fx,fy)) \geq \alpha\, \xi_e(p(gx,gy)) + \beta\, \xi_e(p(fx,gx)) + \gamma\, \xi_e(p(fy,gy))$$

for all $x, y \in X$, where α, β, γ are nonnegative real numbers with $\alpha+\beta+\gamma > s$. or, equivalently,

$$d_p(fx, fy) \geq \alpha\, d_p(gx, gy) + \beta\, d_p(fx, gx) + \gamma\, d_p(fy, gy).$$

for all $x, y \in X$, where α, β, γ are nonnegative real numbers with $\alpha+\beta+\gamma > s$.

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