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Some Families of Graphs Whose Domination Polynomials Are Unimodal

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Abstract. Let G be a simple graph of order n. The domination polynomial of G is the polynomial $D(G,x)=\sum_{i=\gamma(G)}^{n} d(G,i)x^i$, where $d(G,i)$ is the number of dominating sets of G of size i and $\gamma(G)$ is the domination number of G. In this paper we present some families of graphs whose domination polynomials are unimodal.

Keywords: Domination polynomial, Unimodal, Family.

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1. Introduction

Graph polynomials are a well-developed area useful for analyzing properties of graphs. There are some polynomials such as chromatic polynomial, clique polynomial, characteristic polynomial, Tutte polynomial and domination polynomial associated to graphs. Also there are some graphs polynomials related to a molecular graph (see [11]). In this paper we consider the domination polynomial of a graph. Let $G=(V,E)$ be a simple graph. For any vertex $v\in V(G)$, the *open neighborhood* of v is the set $N(v)=\{u\in V(G)|\{u,v\}\in E(G)\}$ and the *closed neighborhood* of v is the set $N[v]=N(v)\cup\{v\}$. For a set $S\subseteq V(G)$, the open neighborhood of S is $N(S)=\bigcup_{v\in S}N(v)$ and the closed neighborhood of S is $N[S]=N(S)\cup S$. A set $S\subseteq V(G)$ is a *dominating set* if $N[S]=V$

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or equivalently, every vertex in $V(G)\backslash S$ is adjacent to at least one vertex in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set in G. For a detailed treatment of these parameters, the reader is referred to [16]. Let $\mathcal{D}(G,i)$ be the family of dominating sets of a graph G with cardinality i and let $d(G,i) = |\mathcal{D}(G,i)|$. The *domination polynomial* $D(G,x)$ of G is defined as $D(G,x) = \sum_{i=\gamma(G)}^{|V(G)|} d(G,i)x^i$, where $\gamma(G)$ is the domination number of G (see [1, 8]). For example, since each non-empty subset of the vertices of the complete graph K_n, is a dominating set of K_n, its domination polynomial is then $D(K_n,x) = (1+x)^n - 1$. A root of $D(G,x)$ is called a domination root of G. Two graphs G and H are said to be *dominating equivalent*, or simply $\mathcal{D}$-equivalent, written $G \sim H$, if $D(G,x) = D(H,x)$. It is evident that the relation $\sim$ of being $\mathcal{D}$-equivalent is an equivalence relation on the family $\mathcal{G}$ of graphs, and thus $\mathcal{G}$ is partitioned into equivalence classes, called the *$\mathcal{D}$-equivalence classes*. Given $G \in \mathcal{G}$, let

$$[G] = \{H \in \mathcal{G} : H \sim G\}.$$

We call $[G]$ the equivalence class determined by G. A graph G is said to be dominating unique, or simply $\mathcal{D}$-unique, if $[G] = \{G\}$.

For two graphs $G = (V,E)$ and $H = (W,F)$, the corona $G \circ H$ is the graph arising from the disjoint union of G with $|V|$ copies of H, by adding edges between the ith vertex of G and all vertices of ith copy of H [12]. It is easy to see that the corona operation is not commutative.

Let $a_0, a_1, \ldots, a_n$ be a sequence of nonnegative numbers. It is unimodal if there is some m, called a mode of the sequence, such that

$$a_0 \leq a_1 \leq \cdots \leq a_{m-1} \leq a_m \geq a_{m+1} \geq \cdots \geq a_n.$$

It is log-concave if $a_k^2 \geq a_{k-1}a_{k+1}$ for all $1 \leq k \leq n-1$. It is symmetric if $a_k = a_{n-k}$ for $0 \leq k \leq n$. A log-concave sequence of positive numbers is unimodal (see, e.g., [9, 10, 19]). We say that a polynomial $\sum_{k=0}^{n} a_k x^k$ is unimodal (log-concave, symmetric, respectively) if the sequence of its coefficients $a_0, a_1, \ldots, a_n$ is unimodal (log-concave, symmetric, respectively). A mode of the sequence $a_0, a_1, \ldots, a_n$ is also called a mode of the polynomial $\sum_{k=0}^{n} a_k x^k$. Unimodality problems of graph polynomials have always been of great interest to researchers in graph theory [7, 18]. There are a number of results concerning the coefficients of independence polynomials, many of which consider graphs formed by applying some sort of operation to simpler graphs. In [21], for instance, Rosenfeld examines the independence polynomials of graphs formed by taking various rooted products of simpler graphs. (See [14] for the definition of the rooted product of two graphs.) In particular, he shows that the property of having all real roots is preserved under forming rooted products. Mandrescu in [20] has shown that the independence polynomial of corona product of any graph with 2 copies of K_1, i.e., $I(G \circ 2K_1, x)$ is unimodal. Recently, Levit and

Mandrescu in [19] generalized this result and have shown that if $H = K_r - e$, $r \geq 2$, then the polynomial $I(G \circ H, x)$ is unimodal and symmetric for every graph G.

Although the unimodality of independence polynomial has been actively studied, almost no attention has been given to the unimodality of domination polynomials. It is conjectured that the domination polynomial of a graph is unimodal (see [8]). This conjecture is still open, even for paths and cycles. Regarding this conjecture, we have the following result:

Theorem 1.1. [8] *Let G be a graph of order n. Then for every $0 \leq i < \frac{n}{2}$, we have $d(G, i) \leq d(G, i+1)$.*

A clique cover of a graph G is a spanning subgraph of G, each component of which is a clique. If $\Omega = \{C_1, C_2, \ldots, C_q\}$ is a clique cover of G, construct a new graph H from G, which is denoted by $H = \Omega\{G\}$ (see[18]), as follows: for each clique $C \in \Omega$, add two new non-adjacent vertices and join them to all the vertices of C. Note that all old edge of G are kept in $\Omega\{G\}$.

In the next section, we consider some specific graphs and show that their domination polynomials are unimodal. In Section 3, we consider graphs of the form $H_n = \Omega\{P_n\}$, where P_n is the path of order n. We study the domination polynomial, domination roots and $\mathcal{D}$-equivalence classes of H_n. As a consequence, we show that $D(H_n, x)$ is unimodal.

2. Unimodality of Domination Polynomials of Some Specific Graphs

In this section, we consider some specific graphs constructed from products with complete graphs and study the unimodality of their domination polynomials. First we consider corona product. We need the following theorem which provides a formula to compute the domination polynomial of corona products of two graphs.

Theorem 2.1. [2, 17] *Let $G = (V, E)$ and $H = (W, F)$ be nonempty graphs of order n and m, respectively. Then*

$$D(G \circ H, x) = (x(1+x)^m + D(H, x))^n.$$

We also need the following results:

Theorem 2.2. [22] *Let $f(x)$ and $g(x)$ be polynomials with positive coefficients. If both $f(x)$ and $g(x)$ are log-concave, then so is their product $f(x)g(x)$.*

The following corollary is an immediate consequence of the above theorem.

Corollary 2.3. *If polynomials $P_i(x)$ for $i = 1, \ldots, k$ with positive coefficients are log-concave, then $\prod_{i=1}^{k} P_i(x)$ is log-concave as well.*

The following theorem gives us many graphs whose domination polynomials are unimodal:

Theorem 2.4. *The domination polynomial of $G \circ K_n$ is unimodal.*

Proof. By theorem 2.1 we can deduce that for each arbitrary graph G,

$$\begin{aligned} D(G \circ K_n, x) &= \Big(x(1+x)^n + (x+1)^n - 1\Big)^{|V(G)|} \\ &= \Big((1+x)^{n+1} - 1\Big)^{|V(G)|}. \end{aligned}$$

By Corollary 2.3, this polynomial is log-concave and therefore unimodal. □

The join $G = G_1 + G_2$ of two graphs G_1 and G_2 with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 is the graph union $G_1 \cup G_2$ together with all the edges joining V_1 and V_2.

We need the following Theorems:

Theorem 2.5. [1] *Let G_1 and G_2 be graphs of orders n_1 and n_2, respectively. Then*

$$D(G_1 + G_2, x) = \Big((1+x)^{n_1} - 1\Big)\Big((1+x)^{n_2} - 1\Big) + D(G_1, x) + D(G_2, x).$$

Theorem 2.6. [8] *If a graph G consists of k connected components $G_1, \ldots, G_k$, then $D(G, x) = \prod_{i=1}^{k} D(G_i, x)$.*

The vertex contraction G/u of a graph G by a vertex u is the operation under which all vertices in $N(u)$ are joined to each other and then u is deleted (see[23]). Note that $G - v$ denotes the graph obtained from G by removing the vertex v and all edges incident to v and $G - N[v]$ denotes the graph obtained by deleting all of the vertices in the closed neighborhood of v and the edges incident to them.

The following theorem is useful for finding a recurrence relation for the domination polynomials of graphs.

Theorem 2.7. [3, 17] *Let G be a graph. For any vertex u in G we have*

$$D(G, x) = xD(G/u, x) + D(G - u, x) + xD(G - N[u], x) - (1+x)p_u(G, x),$$

where $p_u(G, x)$ is the polynomial counting the dominating sets of $G - u$ which do not contain any vertex of $N(u)$ in G.

Theorem 2.7 can be used to give a recurrence relation which removes triangles. Analogously to [3, 17], we denote with $G \odot u$ the graph obtained from G by removing all edges between each pair of neighbors of u. Note that u is not removed when forming $G \odot u$. The following recurrence relation is useful for graphs which have many triangles.

Theorem 2.8. [3, 17] *Let G be a graph and $u \in V$. Then*

$$D(G, x) = D(G - u, x) + D(G \odot u, x) - D(G \odot u - u, x).$$

Consider the friendship graphs F_n obtained by selecting one vertex in each of n triangles and identifying all of them (Figure 1). They are sometimes called Dutch-Windmill graphs [13, 24].

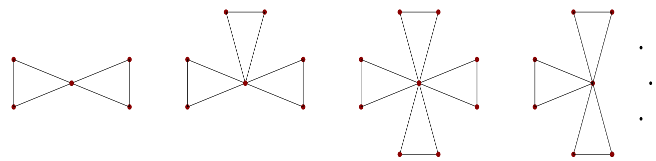

FIGURE 1. Friendship graphs F_2, F_3, F_4 and F_n, respectively.

Since the friendship graph F_n is the join of K_1 and n copies of K_2, nK_2, by Theorem 2.5 we have the following theorem:

Theorem 2.9. [6] *For every* $n \in \mathbb{N}$, $D(F_n, x) = (2x + x^2)^n + x(1 + x)^{2n}$.

Corollary 2.10. *The domination polynomials of* F_n *are unimodal.*

Proof. By Theorem 2.9 we have, $D(F_n, x) = x[(1+x)^{2n} + x^{n-1}(x+2)^n]$. If $n = 1$, then $D(F_1, x) = x^3 + 3x^2 + 3x$, which is unimodal. Now assume that $n \geq 2$. Let $p(x) = \sum_{i=0}^{2n} p_i x^i = (1 + x)^{2n} = \sum_{i=0}^{2n} \binom{2n}{i} x^i$ and $q(x) = \sum_{i=0}^{2n-1} q_i x^i = x^{n-1}(x+2)^n = \sum_{i=0}^{2n-1} \binom{n}{i-n+1} 2^{2n-i-1} x^i$. Note that $p(x)$ and $q(x)$ have $2n+1$ and $2n$ terms, respectively and $q_i = 0$, for $0 \leq i \leq n-2$. The polynomial $p(x)$ has an odd number of terms and is symmetric and unimodal. Also the sequence of its coefficient has mode n. The polynomial $q(x)$ is unimodal, and the sequence of its coefficients has mode

$$k = \begin{cases} n - 1 + \lfloor \frac{n}{3} \rfloor; & n \equiv 0 \text{ or } n \equiv 1 (mod\, 3) \\ \\ n - 1 + \lfloor \frac{n}{3} \rfloor \text{ and } n + \lfloor \frac{n}{3} \rfloor; & n \equiv 2 (mod\, 3). \end{cases}$$

We have $p(x) + q(x) = \sum_{i=0}^{2n} (p_i + q_i) x^i$. Since $q_0 = q_1 = \ldots = q_{n-2} = 0$, the inequality $p_i + q_i \leq p_{i+1} + q_{i+1}$ is true for $0 \leq i \leq n-2$, because of the unimodality of $\{p_j\}$. Also the inequality $p_i + q_i \geq p_{i+1} + q_{i+1}$ is true for $k \leq i \leq 2n-1$. Because from the unimodality of coefficients of $p(x)$ and $q(x)$, for $i \geq k$, we have $q_i \geq q_{i+1}$ and $p_i \geq p_{i+1}$. To complete the proof of the unimodality of $p(x) + q(x)$, we shall prove the following inequalities:

(i) $p_{n-1} + q_{n-1} \leq p_n + q_n$

(ii) $p_i + q_i \geq p_{i+1} + q_{i+1}$, for all $n \leq i \leq k-1$.

Since $\binom{2n}{n-1} - \binom{2n}{n} \leq (n-2)2^{n-1}$, we have $\binom{2n}{n-1} + 2^n \leq \binom{2n}{n} + n2^{n-1}$, and so the inequality (i) is true. Now, we prove the inequality (ii). We shall prove that for $n \leq i \leq k-1$, we have

$$\binom{2n}{i} + \binom{n}{i-n+1} 2^{2n-i-1} \geq \binom{2n}{i+1} + \binom{n}{i+2-n} 2^{2n-i-2}.$$

By an easy computation, we observe that, we have to prove the following inequality for $n \leq i \leq k-1$:

$$\frac{(2i-2n+1)(2n)!}{(2n-i)(i+1)!} \geq \frac{(4n-3i-5)n!}{(i-n+2)!}2^{2n-i-2}. \tag{2.1}$$

From the inequality (2.1), we obtain $k-n$ inequality with one variable n. All of these inequalities can be proved using induction on n. For example, here we prove the inequalities (2.1) for $i=n$, i.e., we prove

$$a_n := \frac{(2n)!}{n!(n+1)!} \geq 2^{n-3}(n^2-5n).$$

This inequality is true for $n = 1, ..., 7$. Since $a_{n+1} = \frac{2(2n+1)}{n+2}a_n$, it suffices to prove the following inequality:

$$\frac{2n+1}{n+2}a_n \geq 2^{n-3}(n^2-3n-4).$$

By induction hypothesis we have $\frac{2n+1}{n+2}a_n \geq \frac{2n+1}{n+2}2^{n-3}(n^2-5n)$, and hence we shall prove $\frac{2n+1}{n+2}(n^2-5n) \geq (n^2-3n-4)$. Since $n^3-8n^2+5n+8 \geq 0$ is true for every natural number $n \geq 7$, so the proof is complete. □

Here we prove that the friendship graphs with an extra edge are unimodal.

Corollary 2.11. *The graphs $H = F_n + uv$ are unimodal, where u is the center vertex of F_n (degree $2n$) so that uv is a pendant edge.*

Proof. We use Theorem 2.8 to obtain the domination polynomial of this kind of graphs:

$$\begin{aligned} D(H,x) &= D(H-u,x) + D(H \odot u,x) - D(H \odot u - u,x) \\ &= D(nK_2 \cup K_1,x) + D(K_{1,2n+1},x) - D((2n+1)K_1,x) \\ &= x(2x+x^2)^n + x^{2n+1} + x(1+x)^{2n+1} - x^{2n+1} \\ &= x\big((x^2+2x)^n + (x+1)^{2n+1}\big). \end{aligned}$$

Therefore

$$D(H,x) = \begin{cases} x(x^3+4x^2+5x+1); & n=1, \\ x(x^5+6x^4+14x^3+14x^2+5x+1); & n=2. \end{cases}$$

For $n \geq 3$, in analogy with the proof of Corollary 2.10, suppose that $p(x) = \sum_{i=0}^{2n+1} p_i x^i = (1+x)^{2n+1} = \sum_{i=0}^{2n+1}\binom{2n+1}{i}x^i$ and $q(x) = x^n(x+2)^n = \sum_{i=0}^{2n} q_i x^i = \sum_{i=0}^{2n}\binom{n}{i-n}2^{2n-i}x^i$. We know that $p(x)$ and $q(x)$ have $2n+2$ and $2n+1$ terms, respectively. Also $p(x)$ is symmetric and unimodal, the sequence of its coefficient has two modes n and $n+1$. The polynomial $q(x)$ is unimodal with $q_i = 0$ for $0 \leq i \leq n-1$ and the sequence of its coefficients has mode

$$k = \begin{cases} n + \lfloor \frac{n}{3} \rfloor; & n \equiv 0 \ or \ n \equiv 1 (mod\, 3), \\ \\ n + \lfloor \frac{n}{3} \rfloor \ and \ n + 1 + \lfloor \frac{n}{3} \rfloor; & n \equiv 2 (mod\, 3). \end{cases}$$

We have

$$\begin{aligned} p(x) + q(x) &= p_0 + p_1 x + \cdots + p_{n-1} x^{n-1} + (p_n + q_n) x^n + (p_n + q_{n+1}) x^{n+1} \\ &+ (p_{n-1} + q_{n+2}) x^{n+2} + \cdots + (p_1 + q_{2n}) x^{2n} + p_0 x^{2n+1}. \end{aligned}$$

The proof of unimodality of $p(x) + q(x)$ is exactly the same of proof of unimodality of $p(x) + q(x)$ in Corollary 2.10. □

The reader is able to see the sequence of coefficients of $D(F_n + uv, x)$ in the site "The on-line encyclopedia of integer sequences" [25] as A213658.

3. Unimodality of Domination Polynomial of a Family of Graphs

In this section, we investigate the domination polynomial of a family of graphs. First we recall the definition of clique cover of a graph. A clique cover of a graph G is a spanning subgraph of G, each component of which is a clique. If $\Omega = \{C_1, C_2, \ldots, C_q\}$ is a clique cover of G, construct a new graph H from G, which is denoted by $H = \Omega\{G\}$ (see[18]), as follows: for each clique $C \in \Omega$, add two new non-adjacent vertices and join them to all the vertices of C. Note that all old edge of G are kept in $\Omega\{G\}$. We consider graphs of the form $H_n = \Omega\{P_n\}$, which constructed from the path P_n with vertex set $\{1, ..., n\}$, by the clique cover construction. Note that in $H_n = \Omega\{P_n\}$ (Figure 2), for even n, $\Omega = \{\{1, 2\}, \{3, 4\}, ..., \{n-1, n\}\}$, and for odd n, $\Omega = \{\{1, 2\}, \{3, 4\}, ..., \{n-2, n-1\}, \{n\}\}$. By H_0 we mean the null graph. We shall study the domination polynomial of H_n.

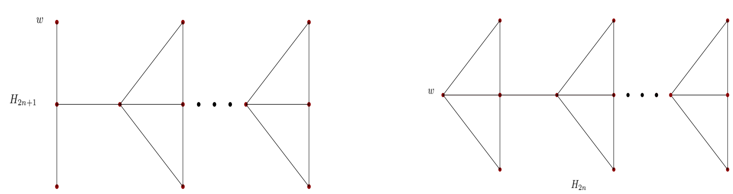

Figure 2. Graphs H_{2n+1} and H_{2n}, respectively.

The domination polynomial of an arbitrary graph satisfies a recurrence relation which is based on the edge and vertex elimination operations. As usual $G - e$ denotes the graph obtained from G by removal of edge e. The following recurrence uses composite operations, e.g. $G - e/u$, which stands for $(G - e)/u$.

Theorem 3.1. [17] *Let G be a graph. For every edge $e = \{u, v\} \in E$,*

$$\begin{aligned} D(G,x) &= D(G-e,x) + \frac{x}{x-1}\Big[D(G-e/u,x) + D(G-e/v,x) \\ &- D(G/u,x) - D(G/v,x) - D(G-N[u],x) - D(G-N[v],x) \\ &+ D(G-e-N[u],x) + D(G-e-N[v],x)\Big]. \end{aligned}$$

The following theorem gives recurrence relations for the domination polynomial of H_n.

Theorem 3.2. *For every $n \in \mathbb{N}$,*

(i) $D(H_{2n+1}, x) = (x^3 + 3x^2 + x)D(H_{2n}, x)$,

(ii) $D(H_{2n}, x) = (x + 1)D(H_{2n-1}, x) + (2x^2 + x)D(H_{2n-2}, x)$,

where $D(H_0, x) = 1$, $D(H_1, x) = x^3+3x^2+x$ *and* $D(H_2, x) = x^4+4x^3+6x^2+2x$.

Proof. (i) Consider graph H_{2n+1} as shown in Figure 2. By Theorem 2.7 we have:

$D(H_{2n+1}, x) = xD(H_{2n+1}/w, x)+D(H_{2n+1}-w, x)+xD(H_{2n+1}-N[w], x)-(1 + x)p_w(H_{2n+1}, x)$

$= xD(H_{2n+1}/w, x)+D(H_{2n+1}-w, x)+xD(H_{2n}\cup K_1, x)-(1+x)p_w(H_{2n+1}, x)$.

Since H_{2n+1}/w is isomorphic to $H_{2n+1} - w$, we have:

$D(H_{2n+1}, x) = (x+1)D(H_{2n+1}/w, x)+x^2D(H_{2n}, x)-(1+x)p_w(H_{2n+1}, x)$.

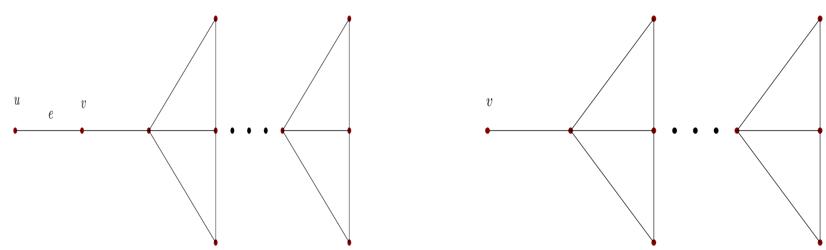

FIGURE 3. Graphs H_{2n+1}/w and $(H_{2n+1}/w) - u$, respectively.

Now we use Theorems 2.6 and 3.1 to obtain the domination polynomial of the graph H_{2n+1}/w (see Figure 3). We have $D(H_{2n+1}/w, x) = xD(H', x) + x[D(H_{2n}, x) + D(H_{2n-1}, x)]$, where $H' = (H_{2n+1}/w) - u$, as shown in Figure 3. Using Theorem 2.7 we deduce that, $D(H', x) = (x + 1)D(H_{2n}, x) - D(H_{2n-1}, x)$. Note that in this case $p_v(H', x) = D(H_{2n-1}, x)$. Also it is clear that $p_w(H_{2n+1}, x) = xD(H_{2n}, x)$. Consequently,

$D(H_{2n+1}, x) = (x+1)\Big[x[(x+1)D(H_{2n}, x) - D(H_{2n-1}, x)] + x[D(H_{2n}, x) + D(H_{2n-1}, x)]\Big] + x^2D(H_{2n}, x) - (1+x)xD(H_{2n}, x)$

$= (x^3 + 3x^2 + x)D(H_{2n}, x)$.

(ii) Now consider graph H_{2n} as shown in Figure 2. We use Theorem 2.7 to compute $D(H_{2n}, x)$. Note that in this case $p_w(H_{2n}, x) = 0$. $D(H_{2n}, x) =$

$xD(H_{2n}/w,x) + D(H_{2n} - w,x) + xD(H_{2n} - N[w],x) - (1+x)p_w(H_{2n},x) =$
$xD(H_{2n}/w,x) + D(H_{2n-1},x) + xD(H_{2n-2},x)$
$= x[D(H_{2n-1},x) + 2xD(H_{2n-2},x)] + D(H_{2n-1},x) + xD(H_{2n-2},x)$
$= (x+1)D(H_{2n-1},x) + (2x^2+x)D(H_{2n-2},x)$.

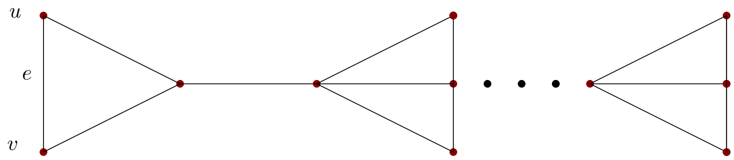

FIGURE 4. Graph H_{2n}/w.

Note that in the third equality, since $(H_{2n}/w) - e = H_{2n-1}$, $D((H_{2n}/w) - e/u,x) = D((H_{2n}/w)/u,x)$, $D((H_{2n}/w) - e/v,x) = D((H_{2n}/w)/v,x)$, and $D((H_{2n}/w) - e - N[u],x) = D((H_{2n}/w) - e - N[v],x) = D(H_{2n-2} \cup K_1,x)$ (see Figure 4), using Theorems 2.6 and 3.1, we have $D(H_{2n}/w,x) = D(H_{2n-1},x) + 2xD(H_{2n-2},x)$. □

By Theorem 3.2 we have the following corollary which gives formula for the domination polynomials of H_n graphs:

Corollary 3.3. *Suppose that H_n are graphs in the Figure 2. we have:*
(i) For every $n \in \mathbb{N}$, $D(H_{2n},x) = (x^4+4x^3+6x^2+2x)^n$.
(ii) For every $n \in \mathbb{N}$, $D(H_{2n+1},x) = (x^3+3x^2+x)(x^4+4x^3+6x^2+2x)^n$.

Proof. (i) By Parts (i) and (ii) of Theorem 3.2:

$$\begin{aligned}
D(H_{2n},x) &= (1+x)(x^3+3x^2+x)D(H_{2n-2},x) + (2x^2+x)D(H_{2n-2},x)\\
&= \Big[(1+x)(x^3+3x^2+x) + (2x^2+x)\Big]D(H_{2n-2},x)\\
&= (x^4+4x^3+6x^2+2x)D(H_{2n-2},x)\\
&= D(H_2,x)D(H_{2n-2},x)\\
&= (D(H_2,x))^n\\
&= (x^4+4x^3+6x^2+2x)^n.
\end{aligned}$$

(ii) It follows from Part (i) and Theorem 3.2(i). □

Corollary 3.4. *For every $n \in \mathbb{N}$, the domination polynomials of H_n are unimodal.*

Proof. It is clear that $D(H_1,x)$ and $D(H_2,x)$ are both log-concave. So we have the result by Corollaries 2.3 and 3.3. □

In the rest of paper, we will consider the domination roots and $\mathcal{D}$-equivalence classes of H_n .

Theorem 3.5. (i) *For each natural number n, the graph H_{2n} is not $\mathcal{D}$-unique.*
(ii) *For each natural number n, the graph H_{2n+1} is not $\mathcal{D}$-unique.*

Proof. (i) Let G be the graph of order 4 in Figure 5. It is easy to see that $D(G,x) = x^4 + 4x^3 + 6x^2 + 2x$. If $H = \cup_{i=1}^{n} G$, then $D(H,x) = (x^4 + 4x^3 + 6x^2 + 2x)^n = D(H_{2n}, x)$.
(ii) Let H be the graph in the proof of Part (i). Here we consider the graph $P_3 \cup H$. We see that $D(H_{2n+1}, x) = D(P_3 \cup H, x)$. Therefore we have the result.

□

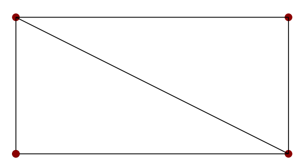

FIGURE 5. Graph H in the proof of Theorem 3.5.

The characterization of graphs for which certain polynomials have few distinct roots has been of interest to researchers in graph theory [5, 15]. By Corollary 3.3, we see that the domination polynomial of H_n has few roots.

Corollary 3.6. *For every $n \in \mathbb{N}$,*
(i) $Z(D(H_{2n}, x)) = Z(D(H_2, x))$.
(ii) $Z(D(H_{2n+1}, x)) = Z(D(H_1, x)) \cup Z(D(H_2, x))$.

Remark 3.7. Since $Z(D(H_1,x)) = \left\{0, \frac{-3\pm\sqrt{5}}{2}\right\}$ and $Z(D(H_2,x)) = \left\{0, \frac{1}{3}(17 + 3\sqrt{33})^{\frac{1}{3}} - \frac{2}{3(17+3\sqrt{33})^{\frac{1}{3}}} - \frac{4}{3}, -\frac{1}{6}(17+3\sqrt{33})^{\frac{1}{3}} + \frac{1}{3(17+3\sqrt{33})^{\frac{1}{3}}} - \frac{4}{3} \pm i\frac{\sqrt{3}}{2}\left(\frac{1}{3}(17 + 3\sqrt{33})^{\frac{1}{3}} + \frac{2}{3(17+3\sqrt{33})^{\frac{1}{3}}}\right)\right\}$, the graphs H_{2n} and H_{2n+1} have four and six distinct roots, respectively. We denote the four domination roots of H_2 by $\{0, \alpha, \beta \pm \gamma i\}$.

Characterization of graphs with exactly one, two and three distinct domination roots has been done in [1]. Also, in [4, 5] the authors characterized graphs with exactly four distinct domination roots $\{-2, 0, \frac{-3\pm\sqrt{5}}{2}\}$. There are two interesting open problems in this area:

(i) Which numbers can be roots of graphs with exactly four distinct domination roots?
(ii) Characterize all graphs with exactly four distinct domination roots [4, 5].

We have proved that H_{2n} has four distinct domination roots. We now construct a sequence of graphs with having exactly four distinct domination roots.

Theorem 3.8. *Every graph H in the family $\{G \circ P_3, (G \circ P_3) \circ P_3, ((G \circ P_3) \circ P_3) \circ P_3, \cdots\}$ have four distinct domination roots, which are $\{0, \alpha, \beta \pm \gamma i\}$.*

Proof. By theorem 2.1 we can deduce that for each arbitrary graph G,

$$\begin{aligned} D(G \circ P_3, x) &= \Big(x(1+x)^3 + D(P_3, x)\Big)^{|V(G)|} \\ &= (x(1+x)^3 + x^3 + 3x^2 + x)^{|V(G)|} \\ &= \Big(x(x^3 + 4x^2 + 6x + 2)\Big)^{|V(G)|} \\ &= \Big(D(H_2, x)\Big)^{|V(G)|}. \end{aligned}$$

So the set of domination roots of $G \circ P_3$ is $Z(D(H_2, x))$. Therefore the result follows. □

The following corollary is an immediate consequence of the above theorem.

Corollary 3.9. *For every graph G, $G \circ P_3$ is not $\mathcal{D}$-unique.*

Proof. If $H = \cup_{i=1}^{|V(G)|} H_2$, then $D(H, x) = (x^4 + 4x^3 + 6x^2 + 2x)^{|V(G)|} = D(G \circ P_3, x)$. Therefore we have the result. □

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