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Generalized Degree Distance of Strong Product of Graphs

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Abstract. In this paper, the exact formulae for the generalized degree distance, degree distance and reciprocal degree distance of strong product of a connected and the complete multipartite graph with partite sets of sizes $m_0, m_1, \ldots, m_{r-1}$ are obtained. Using the results obtained here, the formulae for the degree distance and reciprocal degree distance of the closed and open fence graphs are computed.

Keywords: Generalized degree distance, Degree distance, Reciprocal degree distance, Strong product.

2000 Mathematics subject classification: 05C12, 05C76.

1. Introduction

All the graphs considered in this paper are simple and connected. For vertices $u, v \in V(G)$, the distance between u and v in G, denoted by $d_G(u, v)$, is the length of a shortest (u, v)-path in G and let $d_G(v)$ be the degree of a vertex $v \in V(G)$. The *strong product* of graphs G and H, denoted by $G \boxtimes H$, is the graph with vertex set $V(G) \times V(H) = \{(u, v) : u \in V(G), v \in V(H)\}$ and $(u, x)(v, y)$ is an edge whenever (i) $u = v$ and $xy \in E(H)$, or (ii) $uv \in E(G)$ and $x = y$, or (iii) $uv \in E(G)$ and $xy \in E(H)$, see Fig.1.

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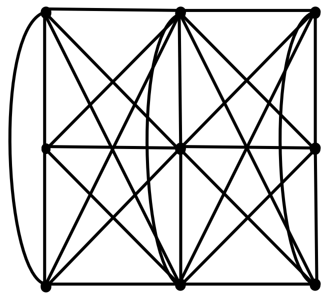

$C_3 \boxtimes P_3$

Fig. 1. Strong product of C_3 and P_3.

A *topological index* of a graph is a real number related to the graph; it does not depend on labeling or pictorial representation of a graph. In theoretical chemistry, molecular structure descriptors (also called topological indices) are used for modelling physicochemical, pharmacologic, toxicologic, biological and other properties of chemical compounds [8]. There exist several types of such indices, especially those based on vertex and edge distances. One of the most intensively studied topological indices is the Weiner index.

Let G be a connected graph. Then $Wiener\ index$ of G is defined as $W(G) = \frac{1}{2} \sum\limits_{u,\, v \,\in\, V(G)} d_G(u,v)$ with the summation going over all pairs of distinct vertices of G. This definition can be further generalized in the following way:
$W_\lambda(G) = \frac{1}{2} \sum\limits_{u,\, v \,\in\, V(G)} d_G^\lambda(u,v)$, where $d_G^\lambda(u,v) = (d_G(u,v))^\lambda$ and λ is a real number [9, 10]. If $\lambda = -1$, then $W_{-1}(G) = H(G)$, where $H(G)$ is Harary index of G. In the chemical literature also $W_{\frac{1}{2}}$ [27] as well as the general case W_λ were examined [6, 11]. Wiener index of 2-dimensional square and comb lattices with open ends is obtained by Graovac et al. in [5]. In [17] the Wiener index of HAC5C7[p, q] and HAC5C6C7[p, q] nanotubes are computed by using GAP program. Dobrynin and Kochetova [4] and Gutman [7] independently proposed a vertex-degree-weighted version of Wiener index called *degree distance* or *Schultz molecular* topological index, which is defined for a connected graph G as
$DD(G) = \frac{1}{2} \sum\limits_{u,v \in V(G)} (d_G(u) + d_G(v)) d_G(u,v)$, where $d_G(u)$ is the degree of the vertex u in G. Note that the degree distance is a degree-weight version of the Wiener index. In the literature, many results on the degree distance $DD(G)$ have been put forward in past decades and they mainly deal with extreme properties of $DD(G)$. Tomescu[24] showed that the star is the unique graph with minimum degree distance within the class on n-vertex connected graphs. Tomescu[25] deduced properties od graphs with minimum degree distance in

the class of n-vertex connected graphs with $m \geq n-1$ edges. For other related results along this line, see [2, 14, 18].

Additively weighted Harary index(H_A) or reciprocal degree distance(RDD) is defined in [1] as $H_A(G) = RDD(G) = \frac{1}{2} \sum\limits_{u,v \in V(G)} \frac{(d_G(u)+d_G(v))}{d_G(u,v)}$. In [12], Hamzeh et. al recently introduced generalized degree distance of graphs. Hua and Zhang [15] have obtained lower and upper bounds for the reciprocal degree distance of graph in terms of other graph invariants including the degree distance, Harary index, the first Zagreb index, the first Zagreb coindex, pendent vertices, independence number, chromatic number and vertex-, and edge-connectivity. Pattabiraman and Vijayaragavan [21, 22] have obtained the reciprocal degree distance of join, tensor product, strong product and wreath product of two connected graphs in terms of other graph invariants. The chemical applications and mathematical properties of the reciprocal degree distance are well studied in [1, 19, 23].

The generalized degree distance, denoted by $H_\lambda(G)$, is defined as $H_\lambda(G) = \frac{1}{2} \sum\limits_{u,v \in V(G)} (d_G(u) + d_G(v)) d_G^\lambda(u,v)$, where λ is a any real number. If $\lambda = 1$, then $H_\lambda(G) = DD(G)$ and if $\lambda = -1$, then $H_\lambda(G) = RDD(G)$. The generalized degree distance of unicyclic and bicyclic graphs are studied by Hamzeh et. al [12, 13]. Also they are given the generalized degree distance of Cartesian product, join, symmetric difference, composition and disjunction of two graphs. It is well known that many graphs arise from simpler graphs via various graph operations. Hence it is important to understand how certain invariants of such product graphs are related to the corresponding invariants of the original graphs. In this paper, the exact formulae for the generalized degree distance, degree distance and reciprocal degree distance of strong product $G \boxtimes K_{m_0,\, m_1,\, \ldots,\, m_{r-1}}$, where $K_{m_0,\, m_1,\, \ldots,\, m_{r-1}}$ is the complete multipartite graph with partite sets of sizes $m_0,\, m_1,\, \ldots,\, m_{r-1}$ are obtained.

The *first Zagreb index* is defined as $M_1(G) = \sum\limits_{u \in V(G)} d_G(u)^2$. In fact, one can rewrite the first Zagreb index as $M_1(G) = \sum\limits_{uv \in E(G)} (d_G(u) + d_G(v))$. The Zagreb indices are found to have applications in QSPR and QSAR studies as well, see [3].

If $m_0 = m_1 = \ldots = m_{r-1} = s$ in $K_{m_0,\, m_1,\, \ldots,\, m_{r-1}}$ (the complete multipartite graph with partite sets of sizes $m_0,\, m_1,\, \ldots,\, m_{r-1}$), then we denote it by $K_{r(s)}$. For $S \subseteq V(G)$, $\langle S \rangle$ denotes the subgraph of G induced by S. For two subsets $S, T \subset V(G)$, not necessarily disjoint, by $d_G(S,T)$, we mean the sum of the distances in G from each vertex of S to every vertex of T, that is, $d_G(S,T) = \sum\limits_{s \in S,\, t \in T} d_G(s,t)$.

2. Generalized Degree Distance of Strong Product of Graphs

In this section, we obtain the generalized degree distance of $G\boxtimes K_{m_0,\,m_1,\,\dots,\,m_{r-1}}$. Let G be a simple connected graph with $V(G)=\{v_0,v_1,\dots,v_{n-1}\}$ and let $K_{m_0,\,m_1,\,\dots,\,m_{r-1}}$, $r\geq 2$, be the complete multiparite graph with partite sets $V_0,\,V_1,\,\dots,\,V_{r-1}$ and let $|V_i|=m_i$, $0\leq i\leq r-1$. In the graph $G\boxtimes K_{m_0,\,m_1,\,\dots,\,m_{r-1}}$, let $B_{ij}=v_i\times V_j, v_i\in V(G)$ and $0\leq j\leq r-1$. For our convenience, as in the case of tensor product, the vertex set of $G\boxtimes K_{m_0,\,m_1,\,\dots,\,m_{r-1}}$ is written as $V(G)\times V(K_{m_0,\,m_1,\,\dots,\,m_{r-1}})=\bigcup\limits_{\substack{i=0\\ j=0}}^{\substack{r-1\\ n-1}} B_{ij}$. As in the tensor product of graphs, let $\mathscr{B}=\{B_{ij}\}_{\substack{i=0,1,\dots,n-1\\ j=0,1,\dots,r-1}}$. Let $X_i=\bigcup\limits_{j=0}^{r-1} B_{ij}$ and $Y_j=\bigcup\limits_{i=0}^{n-1} B_{ij}$; we call X_i and Y_j as *layer* and *column* of $G\boxtimes K_{m_0,\,m_1,\,\dots,\,m_{r-1}}$, respectively, see Figures 2 and 3. If we denote $V(B_{ij})=\{x_{i1},x_{i2},\dots,x_{im_j}\}$ and $V(B_{kp})=\{x_{k1},x_{k2},\dots,x_{k\,m_p}\}$, then $x_{i\ell}$ and $x_{k\ell}, 1\leq \ell\leq j$, are called the *corresponding vertices* of B_{ij} and B_{kp}. Further, if $v_iv_k\in E(G)$, then the induced subgraph $\langle B_{ij}\bigcup B_{kp}\rangle$ of $G\boxtimes K_{m_0,\,m_1,\,\dots,\,m_{r-1}}$ is isomorphic to $K_{|V_j||V_p|}$ or, m_p independent edges joining the corresponding vertices of B_{ij} and B_{kj} according as $j\neq p$ or $j=p$, respectively.

Structure of shortest paths in $G\boxtimes K_{m_0,m_1,\dots,m_{r-1}}$ **corresponding to an edge in** G.

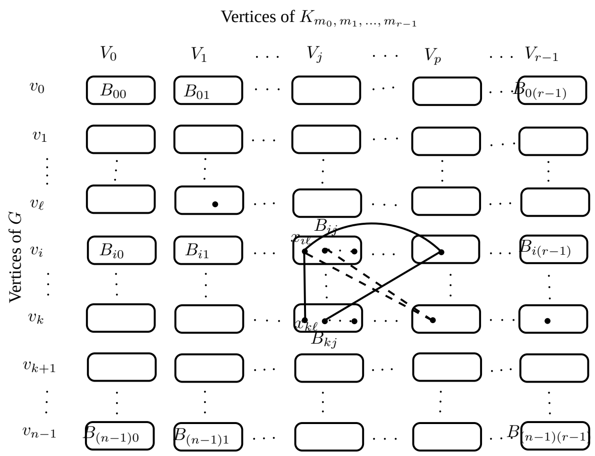

Fig. 2.

If $v_iv_k\in E(G)$, then shortest paths of length 1 and 2 from B_{ij} to B_{kj} are shown in solid edges, where the vertical line between B_{ij} and B_{kj} denotes the edge joining the corresponding vertices of B_{ij} and B_{kj}. The broken edges denote a shortest path of length 2 from a vertex of B_{ij} to a vertex of B_{ij}.

The proof of the following lemma follows easily from the properties and structure of $G\boxtimes K_{m_0,\,m_1,\,\dots,\,m_{r-1}}$, see Figs. 2 and 3.

Lemma 2.1. *Let G be a connected graph and let $B_{ij},\ B_{kp} \in \mathscr{B}$ of the graph $G' = G \boxtimes K_{m_0, m_1, \ldots, m_{r-1}}$, where $r \geq 2$.*
(i) If $v_iv_k \in E(G)$ and $x_{it} \in B_{ij},\ x_{k\ell} \in B_{kj}$, then

$$d_{G'}(x_{it}, x_{k\ell}) = \begin{cases} 1, & \text{if } t = \ell, \\ 2, & \text{if } t \neq \ell, \end{cases}$$

and if $x_{it} \in B_{ij},\ x_{k\ell} \in B_{kp},\ j \neq p$, then $d_{G'}(x_{it}, x_{k\ell}) = 1$.
(ii) If $v_iv_k \notin E(G)$, then for any two vertices $x_{it} \in B_{ij},\ x_{k\ell} \in B_{kp}$, $d_{G'}(x_{it}, x_{k\ell}) = d_G(v_i, v_k)$.
(iii) For any two distinct vertices in B_{ij}, their distance is 2.

The proof of the following lemma follows easily from Lemma 2.1. The lemma is used in the proof of the main theorems of this section.

Lemma 2.2. *Let G be a connected graph and let $B_{ij},\ B_{kp} \in \mathscr{B}$ of the graph $G' = G \boxtimes K_{m_0, m_1, \ldots, m_{r-1}}$, where $r \geq 2$.*
(i) If $v_iv_k \in E(G)$, then

$$d^{\lambda}_{G'}(B_{ij}, B_{kp}) = \begin{cases} m_jm_p, & \text{if } j \neq p, \\ (1 - 2^{\lambda}(m_j - 1))m_j, & \text{if } j = p. \end{cases}$$

(ii) If $v_iv_k \notin E(G)$, then $d^{\lambda}_{G'}(B_{ij}, B_{kp}) = \begin{cases} m_jm_pd^{\lambda}_G(v_i, v_k), & \text{if } j \neq p, \\ m_j^2\, d^{\lambda}_G(v_i, v_k), & \text{if } j = p. \end{cases}$

(iii) $d^{\lambda}_{G'}(B_{ij}, B_{ip}) = \begin{cases} m_jm_p, & \text{if } j \neq p, \\ 2^{\lambda}m_j(m_j - 1), & \text{if } j = p. \end{cases}$

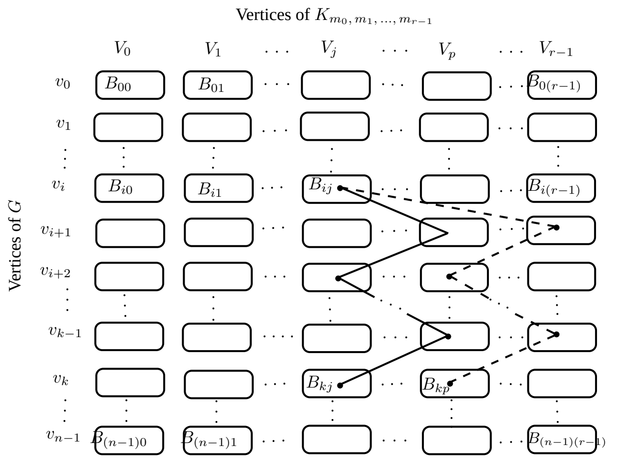

Fig. 3.

Corresponding to a shortest path of length $k > 1$ in G, the shortest path from any vertex of B_{ij} to any vertex of B_{kj} (resp. any vertex of B_{ij} to any

vertex of
B_{kp}, $p \neq j$) of lenght k is shown in solid edges (resp. broken edges).

Lemma 2.3. *Let G be a connected graph and let B_{ij} in $G' = G \boxtimes K_{m_0, m_1, \ldots, m_{r-1}}$. Then the degree of a vertex $(v_i, u_j) \in B_{ij}$ in G' is $d_{G'}((v_i, u_j)) = d_G(v_i) + (n_0 - m_j) + d_G(v_i)(n_0 - m_j)$, where $n_0 = \sum\limits_{j=0}^{r-1} m_j$.*

Remark 2.4. The sums $\sum_{\substack{j,\,p=0 \\ j \neq p}}^{r-1} m_j m_p = 2q$, $\sum\limits_{j=0}^{r-1} m_j^2 = n_0^2 - 2q$, $\sum_{\substack{j,\,p=0 \\ j \neq p}}^{r-1} m_j^2 m_p = n_0^3 - 2n_0 q - \sum\limits_{j=0}^{r-1} m_j^3 = \sum_{\substack{j,\,p=0 \\ j \neq p}}^{r-1} m_j m_p^2$ and $\sum_{\substack{j,\,p=0 \\ j \neq p}}^{r-1} m_j^3 m_p = n_0 \sum\limits_{j=0}^{r-1} m_j^3 - \sum\limits_{j=0}^{r-1} m_j^4 = \sum_{\substack{j,\,p=0 \\ j \neq p}}^{r-1} m_j m_p^3$, where $n_0 = \sum\limits_{j=0}^{r-1} m_j$ and q is the number of edges of $K_{m_0, m_1, \ldots, m_{r-1}}$.

Theorem 2.5. *Let G be a connected graph with n vertices and m edges. Then* $H_\lambda(G \boxtimes K_{m_0, m_1, \ldots, m_{r-1}}) = (n_0^2 + 2n_0 q)H_\lambda(G) + 4n_0 q W_\lambda(G) + M_1(G)(1 - 2^\lambda)\Big(2qn_0 - n_0^3 - n_0^2 + n_0 + 4q + \sum\limits_{j=0}^{r-1} m_j^3\Big) + m\Big(4qn_0(3 - 2^{\lambda+1}) - 2n_0^3(2 - 2^{\lambda+1}) + 4q(2 - 2^\lambda - 2^{\lambda+1}) + n_0(n_0 - 1)2^{\lambda+1} + 2(2 - 2^{\lambda+1}) \sum\limits_{j=0}^{r-1} m_j^3\Big) + n\Big(2n_0 q(2 - 2^\lambda) + n_0^3(2^\lambda - 1) - 2^{\lambda+1} q + (1 - 2^\lambda) \sum\limits_{j=0}^{r-1} m_j^3\Big), r \geq 2.$

Proof. Let $G' = G \boxtimes K_{m_0, m_1, \ldots, m_{r-1}}$. Clearly,

$$\begin{aligned}
H_\lambda(G') &= \frac{1}{2} \sum_{B_{ij},\, B_{kp} \in \mathscr{B}} \Big(d_{G'}(B_{ij}) + d_{G'}(B_{kp})\Big) d^\lambda_{G'}(B_{ij}, B_{kp}) \\
&= \frac{1}{2} \Bigg(\sum_{i\,=\,0}^{n-1} \sum_{\substack{j,\,p\,=\,0 \\ j\,\neq\,p}}^{r-1} \Big(d_{G'}(B_{ij}) + d_{G'}(B_{ip})\Big) d^\lambda_{G'}(B_{ij}, B_{ip}) \\
&\quad + \sum_{\substack{i,\,k\,=\,0 \\ i\,\neq\,k}}^{n-1} \sum_{j\,=\,0}^{r-1} \Big(d_{G'}(B_{ij}) + d_{G'}(B_{kj})\Big) d^\lambda_{G'}(B_{ij}, B_{kj}) \\
&\quad + \sum_{\substack{i,\,k\,=\,0 \\ i\,\neq\,k}}^{n-1} \sum_{\substack{j,\,p\,=\,0 \\ j\,\neq\,p}}^{r-1} \Big(d_{G'}(B_{ij}) + d_{G'}(B_{kp})\Big) d^\lambda_{G'}(B_{ij}, B_{kp}) \\
&\quad + \sum_{i\,=\,0}^{n-1} \sum_{j\,=\,0}^{r-1} \Big(d_{G'}(B_{ij}) + d_{G'}(B_{ij})\Big) d^\lambda_{G'}(B_{ij}, B_{ij}) \Bigg) \\
&= \frac{1}{2}\{A_1 + A_2 + A_3 + A_4\}, \qquad (2.1)
\end{aligned}$$

where A_1, A_2, A_3 and A_4 are the sums of the terms of the above expression, in order.

We shall obtain A_1 to A_4 of (2.1), separately.

$$
\begin{aligned}
A_1 &= \sum_{i=0}^{n-1}\sum_{\substack{j,\,p=0\\ j\neq p}}^{r-1}\Big(d_{G'}(B_{ij})+d_{G'}(B_{ip})\Big)d^{\lambda}_{G'}(B_{ij},B_{ip})\\
&= \sum_{i=0}^{n-1}\sum_{\substack{j,\,p=0\\ j\neq p}}^{r-1}\Big(2d_G(v_i)+d_G(v_i)(2n_0-m_j-m_p)+(2n_0-m_j-m_p)\Big)m_jm_p,\\
&\quad\text{by Lemmas 2.2 and 2.3}\\
&= 8mq+2m\Big(4n_0q-2(n_0^3-2n_0q-\sum_{j=0}^{r-1}m_j^3)\Big)+n\Big(4n_0q-2(n_0^3-2n_0q-\sum_{j=0}^{r-1}m_j^3)\Big),\\
&\quad\text{by Remark 2.4}\\
&= 2m\Big(4q+8n_0q-2n_0^3+2\sum_{j=0}^{r-1}m_j^3\Big)+n\Big(8n_0q-2n_0^3+2\sum_{j=0}^{r-1}m_j^3\Big).
\end{aligned}
$$

$$
\begin{aligned}
A_2 &= \sum_{j=0}^{r-1}\sum_{\substack{i,\,k=0\\ i\neq k}}^{n-1}\Big(d_{G'}(B_{ij})+d_{G'}(B_{kj})\Big)d^{\lambda}_{G'}(B_{ij},B_{kj})\\
&= \sum_{j=0}^{r-1}\sum_{\substack{i,\,k=0\\ i\neq k\\ v_iv_k\in E(G)}}^{n-1}\Big((d_G(v_i)+d_G(v_k))+2(n_0-m_j)+(n_0-m_j)(d_G(v_i)+d_G(v_k))\Big)\\
&\quad\times\Big(1-2^{\lambda}+2^{\lambda}m_j\Big)m_j\\
&\quad+\sum_{j=0}^{r-1}\sum_{\substack{i,\,k=0\\ i\neq k\\ v_iv_k\notin E(G)}}^{n-1}\Big((d_G(v_i)+d_G(v_k))+2(n_0-m_j)+(n_0-m_j)(d_G(v_i)+d_G(v_k))\Big)\\
&\quad\times m_j^2\ d^{\lambda}_G(v_i,v_k),\\
&= \sum_{j=0}^{r-1}\sum_{\substack{i,\,k=0\\ i\neq k\\ v_iv_k\in E(G)}}^{n-1}\Big((d_G(v_i)+d_G(v_k))+2(n_0-m_j)+(n_0-m_j)(d_G(v_i)+d_G(v_k))\Big)\\
&\quad\times\Big((1-2^{\lambda})m_j+(2^{\lambda}-1)m_j^2\Big)\\
&\quad+\sum_{j=0}^{r-1}\sum_{\substack{i,\,k=0\\ i\neq k}}^{n-1}\Big((d_G(v_i)+d_G(v_k))+2(n_0-m_j)+(n_0-m_j)(d_G(v_i)+d_G(v_k))\Big)\\
&\quad\times m_j^2\ d^{\lambda}_G(v_i,v_k)\\
&= 2H_{\lambda}(G)\Big(n_0^3+n_0^2-2q-2n_0q-\sum_{j=0}^{r-1}m_j^3\Big)+4W_{\lambda}(G)\Big(n_0^3-2n_0q-\sum_{j=0}^{r-1}m_j^3\Big)\\
&\quad+2M_1(G)(1-2^{\lambda})\Big(2qn_0-n_0^3-n_0^2+n_0+4q+\sum_{j=0}^{r-1}m_j^3\Big)\\
&\quad+4m(1-2^{\lambda})\Big(2qn_0-n_0^3+2q+\sum_{j=0}^{r-1}m_j^3\Big),
\end{aligned}
$$

by Remark 2.4.

$$\begin{aligned}
A_3 &= \sum_{\substack{i,\,k=0\\ i\neq k}}^{n-1}\sum_{\substack{j,\,p=0,\\ j\neq p}}^{r-1}\Big(d_{G'}(B_{ij})+d_{G'}(B_{kp})\Big)d^{\lambda}_{G'}(B_{ij},B_{kp})\\
&= \sum_{\substack{i,\,k=0\\ i\neq k}}^{n-1}\sum_{\substack{j,\,p=0,\\ j\neq p}}^{r-1}\Big(d_G(v_i)+(n_0-m_j)+d_G(v_i)(n_0-m_j)+d_G(v_k)+(n_0-m_p)\\
&\quad +d_G(v_k)(n_0-m_p)\Big)m_jm_p\ d^{\lambda}_G(v_i,v_k),
\end{aligned}$$

by Lemmas 2.2 and Lemma 2.3

$$\begin{aligned}
&= \sum_{\substack{i,\,k=0\\ i\neq k}}^{n-1}\sum_{\substack{j,\,p=0,\\ j\neq p}}^{r-1}\Bigg((d_G(v_i)+d_G(v_k))d^{\lambda}_G(v_i,v_k)m_jm_p+d_G(v_i)\ d^{\lambda}_G(v_i,v_k)(n_0-m_j)m_jm_p\\
&\quad +(2n_0-m_j-m_p)m_jm_p\ d^{\lambda}_G(v_i,v_k)+d_G(v_k)\ d^{\lambda}_G(v_i,v_k)(n_0-m_p)m_jm_p\Bigg)\\
&= 2H_\lambda(G)\Big(2q+4n_0q-n_0^3+\sum_{j=0}^{r-1}m_j^3\Big)+2W_\lambda(G)\Big(8n_0q-2n_0^3+2\sum_{j=0}^{r-1}m_j^3\Big),
\end{aligned}$$

by Remark 2.4.

$$\begin{aligned}
A_4 &= \sum_{i=0}^{n-1}\Bigg(\sum_{j=0}^{r-1}\Big(d_{G'}(B_{ij})+d_{G'}(B_{ij})\Big)d^{\lambda}_{G'}(B_{ij},B_{ij})\Bigg)\\
&= \sum_{i=0}^{n-1}\sum_{j=0}^{r-1}2^{\lambda+1}\Big(d_G(v_i)+(n_0-m_j)+d_G(v_i)(n_0-m_j)\Big)m_j(m_j-1),\ \text{by Lemmas 2.2 and 2.3}\\
&= 2^{\lambda+1}\sum_{i=0}^{n-1}\sum_{j=0}^{r-1}\Big(d_G(v_i)(m_j^2-m_j)+(n_0-m_j)(m_j^2-m_j)+d_G(v_i)(n_0-m_j)(m_j^2-m_j)\Big)\\
&= 2^{\lambda+1}\Bigg(2m\Big(n_0^2-2q-n_0\Big)+n\Big(n_0^3-2qn_0-n_0^2-\sum_{j=0}^{r-1}m_j^3+n_0^2-2q\Big)\\
&\quad +2m\Big(n_0^3-2qn_0-n_0^2-\sum_{j=0}^{r-1}m_j^3+n_0^2-2q\Big)\Bigg),\ \text{by Remark 2.4}\\
&= 2^{\lambda+2}m\Big(n_0^3+n_0^2-4q-n_0-2qn_0-\sum_{j=0}^{r-1}m_j^3\Big)+2^{\lambda+1}n\Big(n_0^3-2qn_0-2q-\sum_{j=0}^{r-1}m_j^3\Big).
\end{aligned}$$

Using (2.2), (2), (2.2) and (2.2) in (2.1), we have

$$\begin{aligned}
H_\lambda(G') &= (n_0^2+2n_0q)H_\lambda(G)+4n_0qW_\lambda(G)\\
&\quad +M_1(G)(1-2^{\lambda})\Big(2qn_0-n_0^3-n_0^2+n_0+4q+\sum_{j=0}^{r-1}m_j^3\Big)\\
&\quad +m\Big(4qn_0(3-2^{\lambda+1})-2n_0^3(2-2^{\lambda+1})+4q(2-2^{\lambda}-2^{\lambda+1})+n_0(n_0-1)2^{\lambda+1}\\
&\quad +2(2-2^{\lambda+1})\sum_{j=0}^{r-1}m_j^3\Big)+n\Big(2n_0q(2-2^{\lambda})+n_0^3(2^{\lambda}-1)-2^{\lambda+1}q+(1-2^{\lambda})\sum_{j=0}^{r-1}m_j^3\Big).
\end{aligned}$$

□

Using $\lambda=1$ in Theorem 2.5, we have the following corollary, which is the degree distance of the strong product of graphs.

Corollary 2.6. *Let G be a connected graph with n vertices. Then* $DD(G \boxtimes K_{m_0, m_1, \ldots, m_{r-1}}) = (n_0^2+2n_0q)DD(G)+4n_0q\,W(G)+M_1(G)\Big(n_0^3+n_0^2-2qn_0-n_0-4q-\sum_{j=0}^{r-1} m_j^3\Big)+4m\Big(n_0^3-4q-n_0q+n_0^2-n_0-\sum_{j=0}^{r-1} m_j^3\Big)+n\Big(n_0^3-4q-\sum_{j=0}^{r-1} m_j^3\Big),\ r \geq 2.$

If $m_i = s,\ 0 \leq i \leq r-1$, in Corollary 2.6, we have the following

Corollary 2.7. *Let G be a connected graph with n vertices and m edges. Then* $DD(G\boxtimes K_{r(s)}) = r^2s^2(rs-s+1)DD(G)+2r^2s^3(r-1)W(G)+M_1(G)rs\Big(rs^2-rs+2s-s^2-1\Big)+2mrs\Big(r^2s^2-2rs+rs^2+4s-2s^2-2\Big)+nrs^2\Big(r^2s-2r-s+2\Big),\ r \geq 2.$

As $K_r = K_{r(1)}$, the above corollary gives the following

Corollary 2.8. *Let G be a connected graph with n vertices and m edges. Then* $DD(G\boxtimes K_r) = r^3DD(G)+2r^2(r-1)W(G)+2rm(r-1)^2+rn(r-1)^2,\ r \geq 2.$

Using $\lambda = -1$ in Theorem 2.5, we obtain the reciprocal degree distance of strong product of graphs.

Corollary 2.9. *Let G be a connected graph with n vertices. Then* $RDD(G \boxtimes K_{m_0, m_1, \ldots, m_{r-1}})$
$= (n_0^2+2n_0q)RDD(G)+4n_0qH(G)+\frac{M_1(G)}{2}\Big(n_0(n_0+1)-(n_0+2)(n_0^2-2q)+\sum_{j=0}^{r-1} m_j^3\Big)+m\Big(8n_0q-2n_0^3+n_0^2-n_0+2q+2\sum_{j=0}^{r-1} m_j^3\Big)+\frac{n}{2}\Big(6n_0q-n_0^3-2q+\sum_{j=0}^{r-1} m_j^3\Big), r \geq 2.$

If $m_i = s,\ 0 \leq i \leq r-1$, in Corollary 2.9, we have the following

Corollary 2.10. *Let G be a connected graph with n vertices and m edges. Then* $RDD(G\boxtimes K_{r(s)}) = r^2s^2(rs-s+1)RDD(G)+2r^2s^3(r-1)H(G)+\frac{M_1(G)rs}{2}(rs-rs^2-2s+s^2+1)+mrs\Big(2r^2s^2-4rs^2+2rs+2s^2-s-1\Big)+\frac{nrs^2}{2}(2r^2s-3rs+s-r+1).$

As $K_r = K_{r(1)}$, the above corollary gives the following

Corollary 2.11. *Let G be a connected graph with n vertices and m edges. Then* $RDD(G \boxtimes K_r) = r\Big(r^2RDD(G)+2r(r-1)H(G)+2r(r-1)m+n(r-1)^2\Big).$

As an application we present formulae for degree distance and reciprocal degree distance of open and closed fences, $P_n \boxtimes K_2$ and $C_n \boxtimes K_2$, see Fig.4.

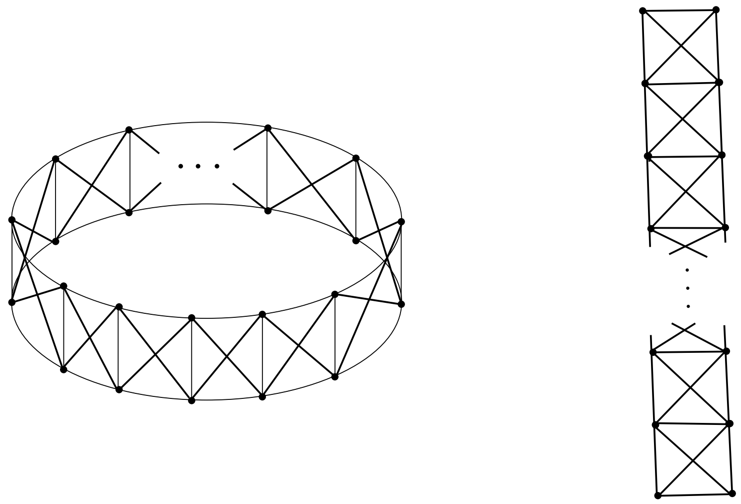

Fig. 4. Closed and open Fence graphs.

One can easily check that $W(P_n) = \frac{n(n^2-1)}{6}$ and $W(C_n) = \begin{cases} \frac{n^3}{8} \text{ n is even} \\ \frac{n(n^2-1)}{8} \text{ n is odd.} \end{cases}$ Similarly, we have $DD(P_n) = \frac{1}{3}n(n-1)(2n-1)$ and $DD(C_n) = 4W(C_n)$.

One can observe that $M_1(C_n) = 4n,\ n \geq 3$, $M_1(P_1) = 0$, and $M_1(P_n) = 4n-6,\ n > 1$. By direct calculations we obtain the Harary indices of P_n and C_n as follows. $H(P_n) = n\Big(\sum\limits_{i=1}^{n} \frac{1}{i}\Big) - n$ and $H(C_n) = \begin{cases} n\Big(\sum\limits_{i=1}^{\frac{n}{2}} \frac{1}{i}\Big) - 1, \text{ if n is even} \\ n\Big(\sum\limits_{i=1}^{\frac{n-1}{2}} \frac{1}{i}\Big), \text{ if n is odd.} \end{cases}$

The following are the reciprocal degree distance of path and cycle on n vertices. $RDD(P_n) = H(P_n) + 4\Big(\sum\limits_{i=1}^{n-1} \frac{1}{i}\Big) - \frac{3}{n-1}$ and $RDD(C_n) = 4H(C_n)$.

By using Corollaries 2.8 and 2.11, we obtain the exact formulae for degree distance and reciprocal degree distance of the following graphs.

Example 2.12. (i) $DD(P_n \boxtimes K_2) = \frac{4}{3}\Big(5n^3 - 6n^2 + 31n - 24\Big)$.

(ii) $DD(C_n \boxtimes K_2) = \begin{cases} 5n(n^2+2) \text{ n is even} \\ 5n(n^2+1) \text{ n is odd.} \end{cases}$

(iii) $RDD(P_n \boxtimes K_2) = 16\Big(\sum\limits_{i=1}^{n} \frac{1}{i}\Big) + 32\Big(\sum\limits_{i=1}^{n-1} \frac{1}{i}\Big) - 6n - \frac{24}{n-1} - 8$.

(iv) $RDD(C_n \boxtimes K_2) = \begin{cases} 10n\Big(1 + 4\sum\limits_{i=1}^{\frac{n}{2}} \frac{1}{i}\Big) - 40 \text{ n is even} \\ 10n\Big(1 + 4\sum\limits_{i=1}^{\frac{n-1}{2}} \frac{1}{i}\Big) \text{ n is odd.} \end{cases}$

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