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# On $(\alpha, \beta)-$Linear Connectivity

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Abstract. In this paper we introduce $(\alpha, \beta)-$linear connected spaces for nonzero cardinal numbers $\alpha$ and $\beta$. We show that $(\alpha, \beta)-$linear connectivity approach is a tool to classify the class of all linear connected spaces.

**Keywords:** $\alpha-$Arc, $(\alpha, \beta)-$Linear connection degree, $(\alpha, \beta)-$Linear connectivity, Arc, $\beta-$Separated family, Linear connected, Path, Path connected.

**2010 Mathematics subject classification:** 54D05.

## 1. Introduction

Classifying the class of all topological spaces is one of the main problems in topology. For instance in algebraic topology one of the main results of introducing fundamental groups and homological groups is classifying topological spaces.

In this paper we introduce and demonstrate a method to classify the class of all topological spaces, and in particular the class of all linear connected spaces. A topological space $X$ is called path connected (linear connected) if for every $x, y \in X$ there exists a path in $X$ from $x$ to $y$ ( i.e, a continuous map

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$f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$ [4]).
The following questions are asked due to the properties paths from $x$ to $y$.

(1) How much paths from $x$ to $y$ are similar to “one to one” path?
(2) For nonzero cardinal number $\theta$, is it possible to have“ $\theta$ to one” path from $x$ to $y$?
(3) How many “$\theta$ to one” paths are there from $x$ to $y$?
(4) How big could be a collection of “$\theta$ to one” paths from $x$ to $y$ which are “enough separated”?

These questions help to classify the class of all linear connected spaces.
In § 2 main tools, $\alpha-$arcs and $\beta-$separated families are introduced. In § 3 we define $(\alpha, \beta)-$linear connectivity and we have our first and main steps, also § 4 is designed for explanation more details of $(\alpha, \beta)-$linear connectivity. Finally in § 5 the concept of $(\alpha, \beta)-$linear connectivity as a classifying tool for class of all linear connected spaces is shown in detail.
We assume ZFC+CH, moreover $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{R}$ denotes the set of real numbers, $\omega = \text{card}(\mathbb{N})$ is the first infinite cardinal number, and $c = \text{card}(\mathbb{R})$ is the first uncountable cardinal number.

## 2. $\alpha-$Arcs and $\beta-$Separated Families

In this section, $\alpha-$arcs and $\beta-$separated families are introduced. We introduce $\alpha-$arcs in order to work on the second question of Introduction. $\beta-$separated families in accompanying with $\alpha-$arcs work on question (4) of Introduction. These are our main tools for $(\alpha, \beta)-$linear connectivity approach.

**Definition 2.1.** In topological space $X$ for nonzero cardinal number $\alpha$, a continuous map $f : [0, 1] \to X$ is called an $\alpha-$arc (between $a = f(0)$ and $b = f(1)$) if for any $t \in [0, 1]$ we have $\text{card}(f^{-1}(f(t)) - \{t\}) < \alpha$

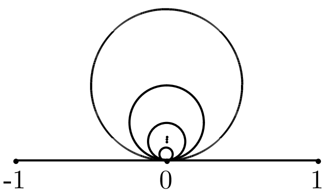
As we see in the following remark, $1-$arcs are well-known. In the literature $1-$arcs are known as arcs [5, page 29].

*Remark* 2.2. A continuous function $f : [0, 1] \to X$ is an $1-$arc if and only if it is one to one. In addition any continuous function $f : [0, 1] \to X$ is an $\alpha-$arc for any $\alpha > c$.

Example 2.3. For $n < \omega$ consider $f : [0, 1] \to \mathbb{C}$ with

$$f(x) = \begin{cases} (n+2)x - 1 & x \in [0, \frac{1}{n+2}] \, , \\ \frac{1}{2^k} e^{2\pi i((n+2)x-k)} + i\frac{1}{2^k} & x \in [\frac{k}{n+2}, \frac{k+1}{n+2}], k \in \{1, \ldots, n\} \, , \\ (n+2)x - (n+1) & x \in [\frac{n+1}{n+2}, 1] \, . \end{cases}$$

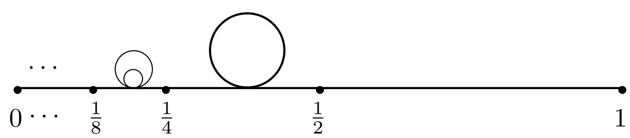
Then $f : [0, 1] \to \mathbb{C}$ is an $\alpha-$arc if and only if $\alpha \geq n + 1$ (In Figure 1 we see $f[0, 1] = \{f(x) : x \in [0, 1]\}$).




(Figure 1)

**Convention** In the rest of this section by $f_n$ we denote constructed $f : [0, 1] \to \mathbb{C}$ in Example 2.3.

Example 2.4. Consider $f : [0, 1] \to \mathbb{C}$ with $f(0) = 0$, $f(x) = \frac{f_n(2^{n+1}x-1)+3}{2^{n+2}}$ for $x \in [\frac{1}{2^{n+1}}, \frac{1}{2^n}]$ and $n \in \omega$. Then $f : [0, 1] \to \mathbb{C}$ is an $\alpha-$arc if and only if $\alpha \geq \omega$ (In Figure 2 we see $f[0, 1] = \{f(x) : x \in [0, 1]\} =: X$).




(Figure 2)

**Definition 2.5.** Let $X$ be a topological space and $\beta \neq 0$ a cardinal number. A collection $\Gamma$ of maps $f : [0, 1] \to X$ with $f(0) = a$ and $f(1) = b$, is called a $\beta-$separated family of maps between $a$ and $b$ if for all $g, h \in \Gamma$ with $g \neq h$ we have $\mathrm{card}(g[0, 1] \cap h[0, 1] - \{a, b\}) < \beta$.

Example 2.6. In Definition 2.5 for $0 < \beta \leq c$:

- If $X = \mathbb{R}$, then every $\beta-$separated family of continuous maps $f : [0, 1] \to \mathbb{R}$ between $-1$ and 1 has at most one element.
- If $X = \mathbb{C}$, then every $\beta-$separated family of continuous maps $f : [0, 1] \to S^1 = \{e^{i\theta} : \theta \in [0, 2\pi]\} (\subseteq \mathbb{C})$ between $-1$ and 1 has at most two element. In addition there exists a $\beta-$separated family continuous maps between $-1$ and 1 with two elements.

For $\beta > c$, $\Gamma$ is a $\beta-$separated family of continuous maps $f : [0, 1] \to X$ between $a$ and $b$ if and only if for any $f \in \Gamma$, $f : [0, 1] \to X$ is a continuous map with $f(0) = a$ and $f(1) = b$ (note: the set of all continuous maps $f : [0, 1] \to X$ for $X = \mathbb{R}$ or $X = S^1$ with $f(0) = 1$ and $f(1) = -1$ has cardinal number $c$).

Example 2.7. With the same assumptions as in Example 2.4, let $X = f[0, 1]$, then for each $\beta \leq \omega$, $\{f\}$ is the unique $\beta-$separated family of continuous maps $g : [0, 1] \to X$ between 0 and 1 containing $f$.

## 3. $(\alpha,\beta)-$Linear Connectivity

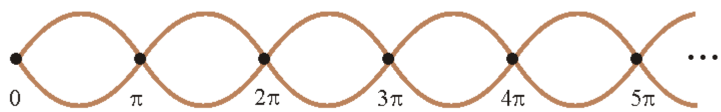
Now we are ready to introduce $(\alpha,\beta)-$linear connectivity. For a nonzero cardinal number $\beta$, and for a $\beta-$separated family of maps between $a$ and $b$ in a topological space $X$ like $\Gamma$, since $\Gamma\subseteq X^{[0,1]}$, $\text{card}(\Gamma)\leq\text{card}(X)^c$, we have the following definition.

**Definition 3.1.** Let $X$ be a linear connected topological space, $a,b\in X$, and $\alpha,\beta$ be nonzero cardinal numbers. Then by $\mathrm{D}^X_{LC(\alpha,\beta)}(a,b)$ or simply $\mathrm{D}_{LC(\alpha,\beta)}(a,b)$ we mean:

$$\sup(\{\text{card}(\Gamma):\Gamma\text{ is a }\beta-\text{separated family of }\alpha-\text{arcs between }a\text{ and }b\})$$

and call it $(\alpha,\beta)-$linear connection degree of $a$ and $b$ [1, Definition 1].

Example 3.2. Suppose $X=\{|x|+i\sin x:x\in\mathbb{R}\}$ (Figure 3) with the induced topology of $\mathbb{C}$.




(Figure 3)

For nonzero cardinal numbers $\alpha,\beta$, also $n,m\in\mathbb{N}\cup\{0\}$ with $n\neq m$ and $k=|n-m|$ we have:

$$\mathrm{D}_{LC(\alpha,\beta)}(n\pi,m\pi)=\mathrm{D}_{LC(\alpha,\beta)}(0,k\pi)=\begin{cases}1 & 0<\beta<k\\ 2 & k\leq\beta\leq c\\ c & \beta>c\end{cases}.$$

Moreover if $x+iy,x'+iy'\in X$ and $(x,y)\neq(x',y')$ (with $x,y,x',y'\in\mathbb{R}$) such that $n\pi\leq x<(n+1)\pi<m\pi<x'\leq(m+1)\pi$, then $\mathrm{D}_{LC(\alpha,\beta)}(x+iy,x'+iy')=\mathrm{D}_{LC(\alpha,\beta)}(n\pi,(m+1)\pi)$.

*Proof.* Since $\mathrm{D}_{LC(\alpha,\beta)}(n\pi,m\pi)=\mathrm{D}_{LC(\alpha,\beta)}(m\pi,n\pi)$, we may assume $n\pi<m\pi$ and $m=n+k$. Note that $f:[0,1]\to X$ such that $f(x)=n\pi+k\pi x+i\sin(n\pi+k\pi x)$ and $g:[0,1]\to X$ such that $g(x)=n\pi+k\pi x+-i\sin(n\pi+k\pi x)$ are two $1-$arcs between $n\pi$ and $m\pi$. Moreover $f[0,1]\cap g[0,1]=\{n\pi,(n+1)\pi,\ldots,m\pi\}$, Thus for $\beta\geq k$, $\{f,g\}$ is a $\beta-$separated family of continuous $1-$arcs (so $\alpha-$arcs) between $n\pi$ and $m\pi$. Therefore:

$$\mathrm{D}_{LC(\alpha,\beta)}(n\pi,m\pi)=\mathrm{D}_{LC(\alpha,\beta)}(0,k\pi)\geq\begin{cases}1 & \beta<k\\ 2 & \beta\geq k\end{cases}\qquad(*).$$

Now we have the following cases:

**Case 1:** $\beta<k$. If $f_1,g_1:[0,1]\to X$ are two continuous maps with $f_1(0)=g_1(0)=n\pi$ and $f_1(1)=g_1(1)=m\pi$, then $f_1[0,1]\cap g_1[0,1]\subseteq\{n\pi,(n+$

$1)\pi, \ldots, m\pi\}$. So $\mathrm{card}((f_1[0,1] \cap g_1[0,1]) - \{m\pi, n\pi\}) \geq k-1 \geq \beta$, which leads to the fact that any $\beta-$separated family of continuous maps between $n\pi$ and $m\pi$ has at most 1 element, i.e. $\mathrm{D}_{LC(\alpha,\beta)}(n\pi, m\pi) \leq 1$. Using (*) we have:

$$\forall \beta < k \quad \mathrm{D}_{LC(\alpha,\beta)}(n\pi, m\pi) = 1.$$

**Case 2:** $\beta > c$. Consider $1-$arc $f : [0,1] \to X$ with $f(x) = n\pi + k\pi x + i\sin(n\pi + k\pi x)$ in the beginning of proof. The set $\Phi := \{f \circ \phi : \phi : [0,1] \to [0,1]$ is a homeomorphism with $\phi(0) = 0$ and $\phi(1) = 1\}$ is a collection of $1-$arcs between $n\pi$ and $m\pi$ moreover it has cardinality $c$. For every $f_1, g_1 \in \Phi$ we have $\mathrm{card}(f_1[0,1] \cap g_1[0,1]) \leq \mathrm{card}(X) = c < \beta$. Therefore $\mathrm{D}_{LC(\alpha,\beta)}(n\pi, m\pi) \geq \mathrm{card}(\Phi) = c$. Since the set of all continuous functions from $[0,1]$ to $X$ has cardinality $c$, $\mathrm{D}_{LC(\alpha,\beta)}(n\pi, m\pi) \leq c$. Thus we have:

$$\forall \beta > c \quad \mathrm{D}_{LC(\alpha,\beta)}(n\pi, m\pi) = c.$$

**Case 3:** $k < \beta \leq c$. If $h_1$, $h_2$ and $h_3$ are distinct elements of a $\beta-$separated family of $\alpha-$arcs between $n\pi$ and $m\pi$, then there exists $x_0 \in (n\pi, m\pi)$ such that for all $j \in \{1,2,3\}$, $\{x + i\sin x : x \in [n\pi, x_0]\} \subseteq h_j[0,1]$ or $\{x - i\sin x : x \in [n\pi, x_0]\} \subseteq h_j[0,1]$. By Pigeonhole Principle there exist $k, j \in \{1,2,3\}$ with $k \neq j$ such that $\{x + i\sin x : x \in [n\pi, x_0]\} \subseteq h_j[0,1] \cap h_k[0,1]$ or $\{x - i\sin x : x \in [n\pi, x_0]\} \subseteq h_j[0,1] \cap h_k[0,1]$. Therefore we have $c \leq \mathrm{card}(h_k[0,1] \cap h_j[0,1] - \{n\pi, m\pi\}) < \beta$, which is a contradiction. Thus every $\beta-$separated family of $\alpha-$arcs between $n\pi$ and $m\pi$ has at most two elements, and $\mathrm{D}_{LC(\alpha,\beta)}(n\pi, m\pi) \leq 2$, using (*) we have $\mathrm{D}_{LC(\alpha,\beta)}(n\pi, m\pi) = 2$. □

**Definition 3.3.** For nonzero cardinal numbers $\alpha, \beta$, a linear connected topological space $X$ is called $(\alpha, \beta)-$linear connected if for any distinct $a, b \in X$ we have $\mathrm{D}^X_{LC(\alpha,\beta)}(a, b) > 1$.

**Lemma 3.4.** *Let $X$ be a topological space and $a \in X$. Then there exists a maximal $(\alpha, \beta)-$linear connected subspace of $X$ containing $a$.*

*Proof.* Let $\Gamma = \{M \subseteq X : a \in M$ and $M$ is $(\alpha, \beta)-$linear connected subspace of $X\}$. We observe that $\{a\} \in \Gamma$ and $\Gamma \neq \varnothing$. Suppose that $(M_\lambda)_{\lambda \in \Lambda}$ is a nonempty chain in $(\Gamma, \subseteq)$. For $c, d \in \bigcup\limits_{\lambda \in \Lambda} M_\lambda$ with $c \neq d$ there exist $\lambda_1, \lambda_2 \in \Lambda$ such that $c \in M_{\lambda_1}$ and $d \in M_{\lambda_2}$ with out any less of generality. We may suppose $M_{\lambda_1} \subseteq M_{\lambda_2}$. Therefore $c, d \in M_{\lambda_2}$. Since $M_{\lambda_2}$ is $(\alpha, \beta)-$linear connected, $\mathrm{D}^{M_{\lambda_2}}_{LC(\alpha,\beta)}(c, d) > 1$. By $\mathrm{D}^{\bigcup_{\lambda \in \Lambda} M_\lambda}_{LC(\alpha,\beta)}(c, d) \geq \mathrm{D}^{M_{\lambda_2}}_{LC(\alpha,\beta)}(c, d)$. We have $\mathrm{D}^{\bigcup_{\lambda \in \Lambda} M_\lambda}_{LC(\alpha,\beta)}(c, d) > 1$, so $\bigcup\limits_{\lambda \in \Lambda} M_\lambda$ is $(\alpha, \beta)-$linear connected and $\bigcup\limits_{\lambda \in \Lambda} M_\lambda \in \Gamma$ is an upper bounded of chain $(M_\lambda)_{\lambda \in \Lambda}$ in $(\Gamma, \subseteq)$. By Zorn's Lemma, $(\Gamma, \subseteq)$ has a maximal element which is a maximal $(\alpha, \beta)-$linear connected subspace of $X$ too. □

Example 3.5. In Example 3.2, for $1 \leq \beta < \omega$, the collection of all maximal $(\alpha,\beta)-$linear connected subspaces of $X$ is $\{\{x+iy \in X : k\pi \leq |x| \leq (k+\beta)\pi\} : k \in \omega\}$. Thus maximal $(\alpha,\beta)-$linear connected subspaces of $X$ are not disjoint. Moreover if $\Xi$ is the collection of all maximal $(\alpha,\beta)-$linear connected subspace of $X$ containing $a+ib(\in X)$, then:

$$\Xi = \{\{x+iy \in X : k\pi \leq |x| \leq (k+\beta)\pi\} : k \geq 0, 0 \leq |a| - k\pi \leq \beta\pi\}.$$

Therefore maximal $(\alpha,\beta)-$linear connected subspace of $X$ containing $0$ is unique (i.e., $\{x+iy \in X : |x| \leq \beta\pi\}$), and there exists two $(\alpha,\beta)-$linear connected subspace of $X$ containing $\pi$ (i.e., $\{x+iy \in X : |x| \leq \beta\pi\}$ and $\{x+iy \in X : \pi \leq |x| \leq (\beta+1)\pi\}$).
In addition $X$ is $(\alpha,\beta)-$linear connected for $\beta \geq \omega$.

Example 3.6. For two point set space $X = \{a,b\}$ with topology $\{X,\varnothing\}$, for nonzero cardinal numbers $\alpha,\beta$ we have:

(1) for $\alpha \leq c$ there is not any $\alpha-$arc between $a$ and $b$ since if $f:[0,1] \to X$ is continuous with $f(0)=a$ and $f(1)=b$, then $f^{-1}(a) \cup f^{-1}(b) = [0,1]$ and therefore $\mathrm{card}((f^{-1}(f(0))-\{0\}) \cup (f^{-1}(f(1))-\{1\})) = \mathrm{card}(0,1) = c$ which shows $\mathrm{card}(f^{-1}(f(0))-\{0\}) = c$ or $\mathrm{card}(f^{-1}(f(1))-\{1\}) = c$ and $f$ is not an $\alpha-$arc.

(2) for $\alpha > c$, $f_1, f_2 : [0,1] \to X$ with:

$$f_1(x) = \begin{cases} a & x=0 \\ b & 0<x\leq 1 \end{cases} \quad , \quad f_2(x) = \begin{cases} a & 0 \leq x < 1 \\ b & x=1 \end{cases} \quad ,$$

are two $\alpha-$arcs between $a$ and $b$ moreover: $\mathrm{card}((f_1[0,1] \cap f_2[0,1]) - \{a,b\}) = 0 < \beta$ and $\{f_1,f_2\}$ is a $\beta-$separated family of $\alpha-$arcs between $a$ and $b$.

Thus:

$$\mathrm{D}_{LC(\alpha,\beta)}(a,b) = \begin{cases} 0 & \alpha \leq c \\ \geq 2 & \alpha > c \end{cases}$$

which shows:

(1) for $\alpha \leq c$, maximal $(\alpha,\beta)-$linear connected subspaces of $X$ are $\{a\}$ and $\{b\}$.

(2) for $\alpha > c$, $X$ is $(\alpha,\beta)-$linear connected.

**Note 3.7.** $(2^c, 2^c)-$linear connected maximal subspaces of $X$ are linear connected component of $X$.

Now we study the product of two $(\alpha,\beta)-$linear conectivity.

**Lemma 3.8.** *In linear connected topological spaces $X, Y$, for nonzero cardinal numbers $\alpha_1,\alpha_2,\beta_1,\beta_2$ and $a,c \in X$, $b,d \in Y$ with $(a,b) \neq (c,d)$ we have:*
**1.** *If $f:[0,1] \to X$ is an $\alpha_1-$arc and $g:[0,1] \to X$ is an $\alpha_2-$arc, then $f \times g : [0,1] \to X \times Y$ with $f \times g(x,y) = (f(x),g(y))$ $((x,y) \in X \times Y)$ is a*

$\min(\alpha_1, \alpha_2)-$*arc (for this item see* [1, Theorem 5], *here we present a similar but more clear proof).*

**2.** *If* $\Gamma$ *is a* $\beta_1-$*separated family of maps between* $a$, $c$ *and* $\Lambda$ *is a* $\beta_2-$*separated family of maps between* $b$, $d$, *such that* $\varphi : \Gamma \to \Lambda$ *is an injection, then* $\{f \times \varphi(f) : f \in \Gamma\}$ *is a* $(\beta_1\beta_2 + 2\beta_1 + 2\beta_2 + 2)-$*separated family of maps between* $(a, b)$ *and* $(c, d)$.

**3.** *If* $\Gamma$ *is a* $\beta_1-$*separated family of maps between* $a$, $c$ *and* $\Lambda$ *is a* $\beta_2-$*separated family of maps between* $b$, $d$, *such that* $\psi : \Lambda \to \Gamma$ *is an injection, then* $\{\psi(h) \times h : h \in \Lambda\}$ *is a* $(\beta_1\beta_2 + 2\beta_1 + 2\beta_2 + 2)-$*separated family of maps between* $(a, b)$ *and* $(c, d)$.

**4.** *The following inequality holds:*

$$\mathrm{D}^X_{LC(\alpha_1,\beta_1)}(a, c)\mathrm{D}^Y_{LC(\alpha_2,\beta_2)}(b, d) \leq \mathrm{D}^{X\times Y}_{LC(\min(\alpha_1,\alpha_2),(\beta_1\beta_2+2\beta_1+2\beta_2+2))}((a, b), (c, d)).$$

*In particular for infinite* $\beta_1, \beta_2$ *the inequality*

$$\mathrm{D}^X_{LC(\alpha_1,\beta_1)}(a, c)\mathrm{D}^Y_{LC(\alpha_2,\beta_2)}(b, d) \leq \mathrm{D}^{X\times Y}_{LC(\min(\alpha_1,\alpha_2),\max(\beta_1,\beta_2))}((a, b), (c, d))$$

*holds.*

*Proof.* **1.** Suppose $f : [0, 1] \to X$ is an $\alpha_1-$arc and $g : [0, 1] \to X$ is an $\alpha_2-$arc. For every $t \in [0, 1]$ we have

$$(f \times g)^{-1}((f \times g)(t)) = (f \times g)^{-1}(f(t), g(t)) \subseteq f^{-1}(f(t)) \cap g^{-1}(g(t))$$

which leads to

$$\begin{aligned} &\mathrm{card}((f \times g)^{-1}((f \times g)(t) - \{t\})) \\ &\quad \leq \mathrm{card}((f^{-1}(f(t)) - \{t\}) \cap (g^{-1}(g(t)) - \{t\})) \\ &\quad \leq \min(\mathrm{card}(f^{-1}(f(t)) - \{t\}), \mathrm{card}(g^{-1}(g(t)) - \{t\})) \\ &\quad < \min(\alpha_1, \alpha_2) \end{aligned}$$

which leads to the desired result.

**2.** Let $f, g \in \Gamma$ and $f \neq g$, then:

$$(f \times g)[0, 1] \cap (\varphi(f) \times \varphi(g))[0, 1] \subseteq (f[0, 1] \cap g[0, 1]) \times (\varphi(f)[0, 1] \cap \varphi(g)[0, 1]).$$

Since $\varphi$ is injective and $f \neq g$, then $\varphi(f) \neq \varphi(g)$. Now we have

$$\begin{aligned} &\mathrm{card}((f \times g)[0, 1] \cap (\varphi(f) \times \varphi(g))[0, 1]) \\ &\quad \leq \mathrm{card}((f[0, 1] \cap g[0, 1]) \times (\varphi(f)[0, 1] \cap \varphi(g)[0, 1])) \\ &\quad < (\beta_1 + 2)(\beta_2 + 2) \end{aligned}$$

which leads to

$$\mathrm{card}(((f \times g)[0, 1] \cap (\varphi(f) \times \varphi(g))[0, 1]) - \{(a, b), (c, d)\}) \leq \beta_1\beta_2 + 2(\beta_1 + \beta_2) + 2.$$

**3.** Use a similar method described in (2).

**4.** Use (2) and (3) and the fact that there exists an injection $\varphi : \Gamma \to \Lambda$ or an injection $\psi : \Lambda \to \Gamma$. $\square$

**Theorem 3.9.** *For nonzero cardinal numbers $\alpha_1, \alpha_2, \beta_1, \beta_2$ if $X$ is $(\alpha_1, \beta_1)-$linear connected and $Y$ is $(\alpha_2, \beta_2)-$linear connected, then $X \times Y$ is $(\min(\alpha_1, \alpha_2), (\beta_1\beta_2 + 2\beta_1 + 2\beta_2 + 2))-$linear connected, in particular if at least one of $\beta_1$ or $\beta_2$ is infinite, then $X \times Y$ is $(\min(\alpha_1, \alpha_2), \max(\beta_1, \beta_2))-$linear connected.*

*Proof.* Use Lemma 3.8. □

## 4. Acceptability with Respect to $(\alpha, \beta)$

It is well-known that the collection of all maximal linear connected subspaces of a topological space $X$ is a partition of $X$ and every point $a \in X$ belongs to a unique maximal linear connected subspace of $X$. Regarding Lemma 3.4 in the topological space $X$ every $a \in X$ belongs to a maximal $(\alpha, \beta)-$linear connected subspace of $X$. By Example 3.5 we see that there are examples in which the maximal $(\alpha, \beta)-$linear connected subspace of $X$ containing $a$ is not unique. In this section we want to have a glance to the topological spaces in which maximal $(\alpha, \beta)-$linear connected subspaces are unique.

**Lemma 4.1.** *For nonzero cardinal numbers $\alpha$ and $\beta$ in the topological $X$ the following assertions are equivalent:*

- *for every $a \in X$ there exists a unique maximal $(\alpha, \beta)-$linear connected subspace of $X$.*
- *The collection of all maximal $(\alpha, \beta)-$linear connected subspaces of $X$ is a partition of $X$.*

*Proof.* Use Lemma 3.4. □

**Definition 4.2.** The topological space $X$ is called acceptable with respect to $(\alpha, \beta)$, if maximal $(\alpha, \beta)-$linear connected subspaces of $X$ make a partition of $X$.
If $X$ is acceptable with respect to $(\alpha, \beta)$, then we call its maximal $(\alpha, \beta)-$linear connected subspaces, the $(\alpha, \beta)-$linear connected components of $X$.

*Remark* 4.3.

(1) In Example 3.2 for $1 \leq \beta < \omega$ and nonzero $\alpha$, $X$ is not acceptable with respect to $(\alpha, \beta)$ (use notes in Example 3.5).
(2) In Example 3.6 for any nonzero $\alpha$ and $\beta$, $X$ is acceptable with respect to $(\alpha, \beta)$.

*Remark* 4.4. Using [1, Theorem 6], in linear connected topological spaces $X$ for nonzero cardinal numbers $\alpha, \alpha_1$ for all $a, b, d \in X$ we have:

$$\mathrm{D}_{Lc(\alpha,2^c)}(a, b)\mathrm{D}_{Lc(\alpha_1,2^c)}(b, d) \leq \mathrm{D}_{Lc(\alpha+\alpha_1+1,2^c)}(a, d),$$

and therefore for infinite cardinal number $\alpha$ we have:

$$\mathrm{D}_{Lc(\alpha,2^c)}(a,b)\mathrm{D}_{Lc(\alpha,2^c)}(b,d) \leq \mathrm{D}_{Lc(\alpha,2^c)}(a,d).$$

*Remark* 4.5. Using [1, Theorem 2] in linear connected topological space $X$ if $Y$ is a linear connected subspace of $X$ and $a, b \in Y$, then for nonzero cardinal numbers $\alpha, \beta$ we have:

$$\mathrm{D}^Y_{LC(\alpha,\beta)}(a,b) \leq \mathrm{D}^X_{LC(\alpha,\beta)}(a,b).$$

**Lemma 4.6.** *For infinite cardinal number $\alpha$ every topological space $X$ is acceptable with respect to $(\alpha, 2^c)$.*

*Proof.* Let $A$, $B$ be maximal $(\alpha, 2^c)-$linear connected subspaces of $X$ with $d \in A \cap B$ using Remark 4.4 and Remark 4.5 for all $a \in A$ and $b \in B$ we have:

$$\begin{aligned} \mathrm{D}^{A\cup B}_{Lc(\alpha,2^c)}(a,b) &\geq \mathrm{D}^{A\cup B}_{Lc(\alpha,2^c)}(a,d)\mathrm{D}^{A\cup B}_{Lc(\alpha,2^c)}(d,b) \\ &\geq \mathrm{D}^{A}_{Lc(\alpha,2^c)}(a,d)\mathrm{D}^{B}_{Lc(\alpha,2^c)}(d,b) \geq 4. \end{aligned}$$

Thus $A \cup B$ is $(\alpha, 2^c)-$linear connected, which leads to $A = B$ (since $A$ and $B$ are maximal $(\alpha, 2^c)-$linear connected subspaces of $X$). $\square$

**Definition 4.7.** For nonzero cardinal numbers $\alpha$, $\beta$ we call topological space $X$ locally $(\alpha, \beta)-$linear connected in $a \in X$ if for every open neighborhood $U$ of $a$ there exists an $(\alpha, \beta)-$linear connected open subset $V$ of $X$ such that $a \in V \subseteq U$. We call $X$ locally $(\alpha, \beta)-$linear connected if it is locally $(\alpha, \beta)-$linear connected in every $x \in X$.

**Theorem 4.8.** *If $X$ is acceptable with respect to $(\alpha, \beta)$ and it is locally $(\alpha, \beta)-$linear connected, then $(\alpha, \beta)-$linear connected components of $X$ are open.*

*Proof.* Suppose $a \in X$ and $M$ is $(\alpha, \beta)-$linear connected components of $X$ containing $a$. Since $X$ is locally $(\alpha, \beta)-$linear connected, there exists an open $(\alpha, \beta)-$linear connected neighborhood of $a$ like $U(\subseteq X)$. We set

$$\Gamma := \{L \subseteq X : L \text{ is } (\alpha, \beta) - \text{linear connected and } U \subseteq L \subseteq X\}.$$

Using Zorn's Lemma $(\Gamma, \subseteq)$ has a maximal element like $L$. $L$ is a maximal $(\alpha, \beta)-$linear connected subspace of $X$. Since $a \in M \cap L$, $L$ and $M$ are maximal $(\alpha, \beta)-$linear connected subspaces of $X$, and $X$ is acceptable with respect to $(\alpha, \beta)$, $L = M$. By $a \in U \subseteq M$, $a$ is an interior point of $M$. $\square$

## 5. A Table

In this section we bring a table which shows how $(\alpha, \beta)-$linear connectivity approach classify the class of all linear connected spaces.

**Table 5.1.** For nonzero cardinal numbers $\alpha$, $\beta$ if $\mathcal{L}C(\alpha,\beta)$ denotes the class of all $(\alpha,\beta)-$linear connected spaces., then we have the following table:

$$
\begin{array}{ccccc}
\mathcal{L}C(1,1)\subset & \cdots\subset\mathcal{L}C(1,n)\subset\cdots & \subset\ \mathcal{L}C(1,\omega) & \subset\mathcal{L}C(1,c)\subset & \mathcal{L}C(1,2^c) \\
|\cap & |\cap & |\cap & |\cap & |\cap \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
|\cap & |\cap & |\cap & |\cap & |\cap \\
\mathcal{L}C(m,1)\subset & \cdots\subset\mathcal{L}C(m,n)\subset\cdots & \subset\ \mathcal{L}C(m,\omega) & \subset\mathcal{L}C(m,c)\subset & \mathcal{L}C(m,2^c) \\
|\cap & |\cap & |\cap & |\cap & |\cap \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
|\cap & |\cap & |\cap & |\cap & |\cap \\
\mathcal{L}C(\omega,1)\subset & \cdots\subset\mathcal{L}C(\omega,n)\subset\cdots & \subset\ \mathcal{L}C(\omega,\omega) & \subset\mathcal{L}C(\omega,c)\subset & \mathcal{L}C(\omega,2^c) \\
|\cap & |\cap & |\cap & |\cap & |\cap \\
\mathcal{L}C(c,1)\subset & \cdots\subset\mathcal{L}C(c,n)\subset\cdots & \subset\ \mathcal{L}C(c,\omega) & \subset\mathcal{L}C(c,c)\subset & \mathcal{L}C(c,2^c) \\
\cap & \cap & \cap & \cap & \cap \\
\mathcal{L}C(2^c,1)\subset & \cdots\subset\mathcal{L}C(2^c,n)\subset\cdots & \subset\ \mathcal{L}C(2^c,\omega) & \subset\mathcal{L}C(2^c,c)\subset & \mathcal{L}C(2^c,2^c) \\
 & & & & || \\
\end{array}
$$

The class of all linear connected spaces

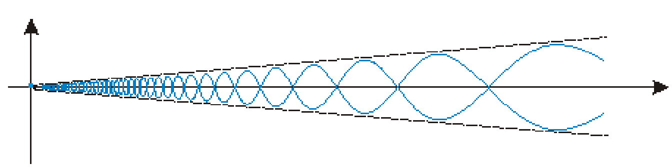
Where "$\subset$" means strict inclusion.

*Proof.* Consider the following counterexamples:

**Counterexample 1.** Let $\alpha>0$, $0<n<\omega$ and $X=\{|x|+i\sin x:0\leq x\leq(n+1)\pi\}$, then $X$ is an $(\alpha,n+1)-$linear connected space, but isn't an $(\alpha,n)-$linear connected space.

**Counterexample 2.** In the Example 3.2 for $\alpha>0$, $X$ is an $(\alpha,\omega)-$linear connected space but isn't an $(\alpha,n)-$linear connected space.

**Counterexample 3.** Let $X=\{|x|+i|x|\sin(\frac{1}{x}):-1\leq x\leq 1,x\neq 0\}\cup\{0\}$ (a schema has been presented in Figure 4) with the induced topology of $\mathbb{C}$.

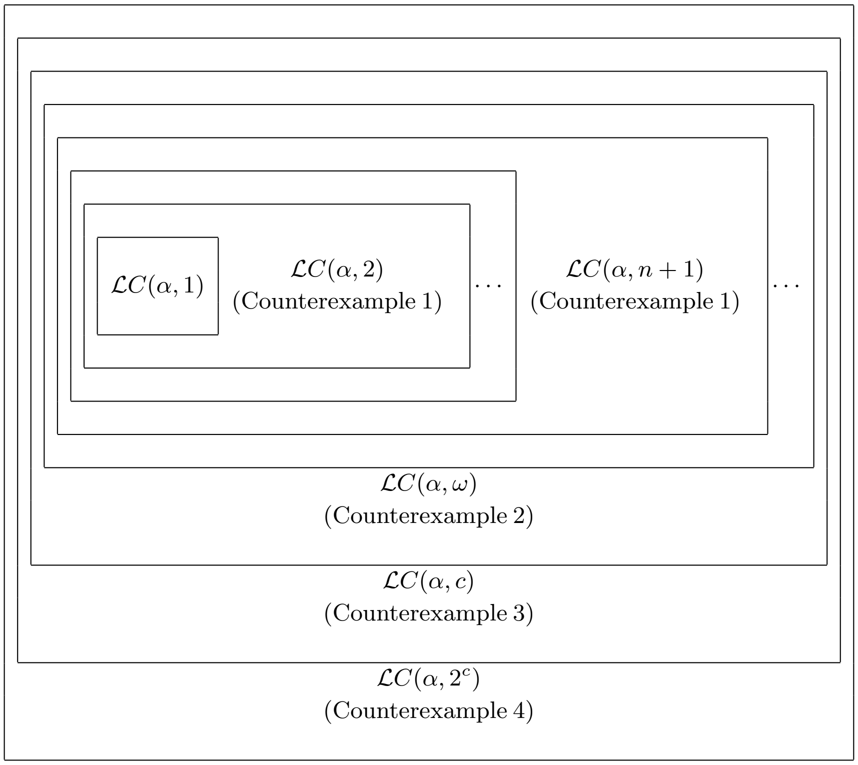


(Figure 4)

The topological space $X$ is an $(\alpha,c)-$linear connected space, but that isn't an $(\alpha,\omega)-$linear connected space.

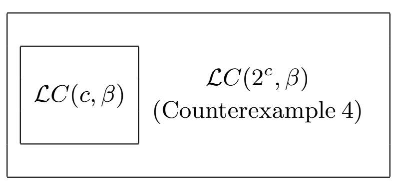
**Counterexample 4.** In the Example 2.4, let $\alpha>0$, then $X$ is an $(\alpha,2^c)-$linear connected space, moreover $X$ is not an $(\alpha,c)-$linear connected space.

Using Counterexamples 1, 2, 3, 4 we have the following diagrams which complete the proof.




In the above diagram for nonzero cardinal number $\alpha$ regarding mentioned Counterexample we may find a corresponding topological space.

And the following diagram:

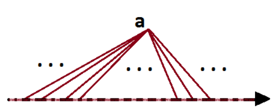



which indicates that regarding Counterexample 4, there exists a topological space $X$ such that $X \in \mathcal{L}C(2^c, \beta) - \mathcal{L}C(c, \beta)$. $\square$

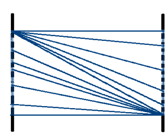
## 6. Final Note

6.1. **Why should we deal with** $(\alpha, \beta)-$**linear connectivity?** As it has been mentioned in the first paragraph of the abstract of [3] "Similarity concept, finding the resemblance or classifying some groups of objects and study their common properties has been the interest of many researchers.", also one may try to know exactly the objects of a category (like [2]). In this text we tried to introduce "$(\alpha, \beta)-$linear connectivity concept" as a tool to classify the class of all linear connected topological spaces. Here we want to show how we reach to $(\alpha, \beta)-$linear connectivity approach. The following is a real story (about one of the authors).

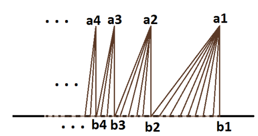
When I was an undergraduate student, for the first time saw infinite bloom (Figure 5)): that subspace of the Euclidean space $\mathbb{R}^2$ which consists of all closed line segments joining $a = (2, 1)$ to a rational point in $x-$axis. This infinite bloom is linear connected and it is not locally connected in any point except $a$, in fact it is locally linear connected in $a$. One of the most well known exercises in this approach is this: Find a linear connected set $X$ which is not locally linear connected in any point (a schema of my answer is in Figure 6). So, I decided to classify linear connected spaces which are locally linear connected in just one point, and they are not locally connected in any other points. The result was a a collection of examples Figures 8, 9, 10, and 11.



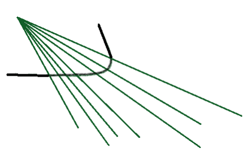

(Figure 5)



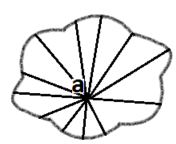
(Figure 6)



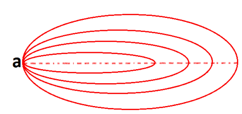

(Figure 7)



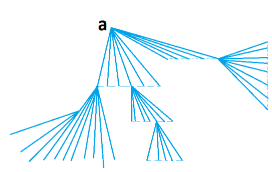
(Figure 8)




(Figure 9)




(Figure 10)

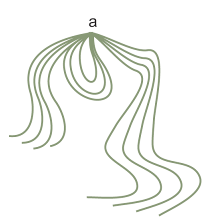



(Figure 11)

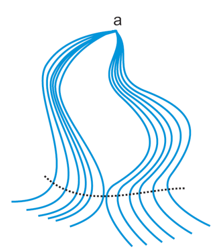
Where:

- Figure 5 is a presentation of infinite bloom.
- Figure 6 is a presentation of my answer to: Find a linear connected set $X$ which is not locally linear connected in any point.
- Figure 7 is a presentation of a subset of $\mathbb{R}^2$ which is linear connected and locally connected in just countable points.
- Figure 8 proposes to select a dense subset of the graph of a "good function" with dense complement, instead of choosing rational (irrational) numbers of $x-$axis in Figure 5.
- Figure 9 is a more generalization of Figure 8.
- Figure 10 uses arcs instead of lines in previous Figures.
- Figure 11 presents a generalization of Figure 5.

However these spaces may be more complicated (Figure 12). Now the main question is: "Suppose $X$ is a linear connected subspace of plane with more than one point which is locally linear connected in just one point like $u$ and $X$ is not locally connected in every $x \in X \setminus \{u\}$.Does $X$ has a subspace homeomorph with infinite bloom?" The result of working in this question was a join lecture in undergraduate math students' seminar. But still the question was unsolved, however a new concept has been introduced, I was hopeful that this concept help us to find the answer. This concept was "$(\alpha, \beta)-$linear connection degree". Our idea about the relation between the concept of $(\alpha, \beta)-$linear connection degree and our question was true, one may find this relation in next subsection. As a matter of fact Theorem 6.2 guaranties the existance of a subspace similar to Figure 13 in our target subspaces of $\mathbb{R}^2$.



(Figure 12)



(Figure 13)

6.2. **A Theorem.** Let's generalize the notion of $\mathrm{D}^X_{LC(\alpha,\beta)}(a, b)$ from linear connected space $X$ to arbitrary topological space $X$.

**Definition 6.1.** In topological space $X$, for $a, b \in X$ and nonzero cardinal numbers $\alpha, \beta$, suppose $L_a$ is linear connected component of $X$ which contains

$a$, and $C_a$ is connected component of $X$ containing $a$. Define:

$$\mathrm{D}^{X}_{LC(\alpha,\beta)}(a,b) := \begin{cases} \mathrm{D}^{L_a}_{LC(\alpha,\beta)}(a,b) & b \in L_a\ , \\ \\ -1 & b \in C_a \setminus L_a\ , \\ \\ -2 & \text{otherwise}\ . \end{cases}$$

$\mathrm{D}^{X}_{LC(\alpha,\beta)}(a,b)$ (or simply $\mathrm{D}_{LC(\alpha,\beta)}(a,b)$) is called $(\alpha,\beta)-$linear connection degree of $a$ and $b$.

Using Definition 6.1 we recall the following theorem.

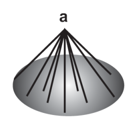
**Theorem 6.2** ([1, Theorem 4]). *Let $\mathcal{S} \subseteq \mathbb{R}^2$, $\beta$ be a nonzero cardinal number, $a \in \mathcal{S}$ and $f : [0,1] \to \mathbb{R}^2$ be a continuous 1-1 map such that $a \notin f[0,1]$, $f(0), f(1) \in \mathcal{S}$ and:*

- *for each $b, d \in f[0,1] \cap \mathcal{S}$ with $b \neq d$ we have $\mathrm{D}^{\mathcal{S}}_{LC(1,\beta)}(b,d) \geq 1$ and $\mathrm{D}^{\mathcal{S}\setminus\{a\}}_{LC(1,\beta)}(b,d) < 1$,*
- *for each countable subset $K$ of $[0,1]$, $f([0,1]\setminus K)\cap\mathcal{S}$ is dense in $f[0,1]$,*
- *if for each $b \in f[0,1] \cap \mathcal{S}$, $g_b : [0,1] \to \mathcal{S}$ be a continuous 1-1 map such that $g_b(1) = b$ and $g_b(0) = a$, then $\bigcup\{g_b(0,1) : b \in f(0,1) \cap \mathcal{S}\}$ is a subset of the enterior of the simple closed curve $f[0,1] \cup g_{f(0)}[0,1] \cup g_{f(1)}[0,1]$.*

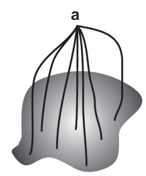
*Then there exists $\mathcal{L} \subseteq \mathcal{S}$ such that:*

- *$a \in \mathcal{L}$ and $\mathcal{L}$ is linear connected,*
- *for each $b \in \mathcal{L} \setminus \{a\}$, $\mathcal{L}$ is not locally connected in $b$,*
- *$(\mathcal{S} \setminus \mathcal{L}) \cap f[0,1]$ is countable and for each countable subset $K$ of $[0,1]$, $f([0,1] \setminus K) \cap \mathcal{L}$ is dense in $f[0,1]$.*

6.3. **Some arising problems.** Now let's generalize Theorem 6.2 through the following example. Here we bring a subset of $\mathbb{R}^3$ which is a generalization of infinite bloom. Let $X$ be that subspace of $\mathbb{R}^3$ consisting of closed line segments joining $a = (0,0,1)$ to an element of $\{(x,y,0) : x^2+y^2 \leq 1, (x,y) \in \mathbb{Q}\times\mathbb{Q}\}$. Then $X$ is a linear connected space it is not locally linear connected in any point but $a$ (Figure 14). However one may consider more complicated examples (Figure 15).




(Figure 14)




(Figure 15)

In the following for $n \in \mathbb{N}$ and linear normed space $E$ let $D_E^n := \{x \in E^n : ||x|| \leq 1\}$ and $S_E^n := \{x \in D_E^{n+1} : ||x|| = 1\}$.
Regarding Theorem 6.2, and above descriptions the following problem occurs:

**Problem 6.3.** Find all normed linear space $E$, nonzero cardinal numbers $\alpha, \beta$ and $n \in \mathbb{N}$ such that the following statement is valid:
Let $\mathcal{S} \subseteq E^{n+1}$, and $f : D_E^n \to E^{n+1}$ be a continuous 1-1 map such that $a \notin f(D_E^n)$, $f(S_E^{n-1}) \subseteq \mathcal{S}$, and:

- for each $b, d \in f(D_E^n) \cap \mathcal{S}$ with $b \neq d$ we have $\mathrm{D}^{\mathcal{S}}_{LC(\alpha,\beta)}(b, d) \geq 1$ and $\mathrm{D}^{\mathcal{S}\setminus\{a\}}_{LC(\alpha,\beta)}(b, d) < 1$,
- for each countable subset $K$ of $D_E^n$, $f(D^n \setminus K) \cap \mathcal{S}$ is dense in $f(D_E^n)$,
- if for each $b \in f(D_E^n) \cap \mathcal{S}$, $g_b : [0, 1] \to \mathcal{S}$ be a continuous $\alpha-$arc such that $g_b(1) = b$ and $g_b(0) = a$, then $M := \overline{\bigcup\{g_b[0, 1] : b \in f(S_E^{n-1})\}} \cup f(D_E^n)$ is a nowhere-dense subset of $E^{n+1}$ such that $E^{n+1} \setminus M$ has exactly two connected component, one bounded and the other unbounded, $\bigcup\{g_b(0, 1) : b \in f(D_E^n \setminus S_E^{n-1}) \cap \mathcal{S}\}$ is a subset of bounded component of $E^{n+1} \setminus M$.

Then there exists $\mathcal{L} \subseteq \mathcal{S}$ ($\mathcal{L} \neq \{a\}$) such that:

- $a \in \mathcal{L}$ and $\mathcal{L}$ is linear connected,
- for each $b \in \mathcal{L} \setminus \{a\}$, $\mathcal{L}$ is not locally connected in $b$.

Using Theorem 6.2, $E = \mathbb{R}$, $n = 2$ and $\alpha = 1$ is one of the answers of Problem 6.3.
Considering Tabel 5.1, we have the following problem:

**Problem 6.4.** Suppose $0 < \alpha \leq \omega$ and $\beta > 0$. Find an $(\alpha^+, \beta)-$linear connected space $X$ which is not $(\alpha, \beta)-$linear connected space.

However solving the following problem may be useful to find answers for Problems 6.3 and 6.4.

**Problem 6.5.** Let $f : [0, 1] \to X$ be an $\alpha^+-$arc. When there exists $\alpha-$arc $g : [0, 1] \to X$ with $f(0) = g(0)$, $f(1) = g(1)$ and $g[0, 1] \subseteq f[0, 1]$?

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