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An Explicit Viscosity Iterative Algorithm for Finding Fixed Points of Two Noncommutative Nonexpansive Mappings

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Abstract. We suggest an explicit viscosity iterative algorithm for finding a common element in the set of solutions of the general equilibrium problem system (GEPS) and the set of all common fixed points of two noncommuting nonexpansive self mappings in the real Hilbert space.

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1. Introduction

Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and norm $||.||$. Recall that a mapping T with domain $D(T)$ and range $R(T)$ in H is called nonexpansive iff for all $x, y \in D(T)$,

$$\|Tx - Ty\| \leq \|x - y\|.$$

$F(T)$ denotes the set of fixed points of T. Moreover, H satisfies the Opial's condition [6], if for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|,$$

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holds for every $y \in H$ with $x \neq y$.
Recall that f is said to be weakly contractive [2] iff for all $x, y \in D(T)$,

$$\|f(x) - f(y)\| \leq \|x - y\| - \phi(\|x - y\|),$$

for some $\phi : [0, \infty) \to [0, \infty)$ is a continuous and strictly increasing function such that ϕ is positive on $(0, \infty)$ and $\phi(0) = 0$. A mapping A is a strongly positive linear bounded operator on H if there exists a constant $\bar{\gamma} > 0$ such that

$$\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2, \text{ for all } x \in H.$$

Moreover, $B : C \to H$ is called $\alpha-$inverse strongly monotone if there exists a positive real number $\alpha > 0$ such that for all $x, y \in C$

$$\langle Bx - By, x - y\rangle \geq \alpha\|Bx - By\|^2.$$

Let C be a nonempty closed convex subset of H. $A : H \to H$ be an inverse strongly monotone mapping and $F : C \times C \to \mathbb{R}$ be a bifunction. The general equilibrium problem is to find $\tilde{x} \in C$ such that for all $y \in C$,

$$F(\tilde{x}, y) + \langle Ax, y - x\rangle \geq 0.$$

There are several other problems, for example, the complementarity problem, fixed point problem and optimization problem, which can also be written in the form of an EP. In other words, the general equilibrium problem system (GEPS) is an unifying model for several problems arising in physics, engineering, science, optimization, economics, etc [1, 4].
To study the generalized equilibrium problem, we assume that the bifunction F satisfies the following conditions:

(A1) $F(x, x) = 0$, for all $x \in C$;
(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;
(A3) for each $x, y, z \in C$, $\limsup_{t\to 0^-} F(tz + (1 - t)x, y) \leq F(x, y)$;
(A4) for each $x \in C$ $y \mapsto F(x, y)$ is convex and weakly lower semi-continuous.

Recently, Yao and Chen [10] introduced a new iteration for two averaged self mappings S and T on a closed convex subset C as follows

$$\begin{cases} x_0 = x \in C; \\ x_{n+1} = \alpha_n x + (1 - \alpha_n)\frac{2}{(n+1)(n+2)} \sum_{i=0}^{n} \sum_{j=0}^{n-i}((ST)^j S^{i-j} \vee (ST)^i T^{j-i})x_n, \end{cases}$$

where $n \geq 0$ and

$$(ST)^j S^{i-j} \vee (ST)^i T^{j-i} = \begin{cases} (ST)^j S^{i-j} & \text{if } i \geq j \\ (ST)^i T^{j-i} & \text{if } i < j. \end{cases} \tag{1.1}$$

By improving this idea, Jankaew et al. [5] considered the following iteration:

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n \frac{2}{(n+1)(n+2)} \sum_{i=0}^{n} \sum_{j=0}^{n-i}((ST)^j S^{i-j} \vee (ST)^i T^{j-i})x_n, \tag{1.2}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset (0,1)$, $\alpha_n+\beta_n+\gamma_n=1$, f is a contraction mapping on C. They proved that the iteration process (1.2) converges strongly to common fixed point of the mapping S and T which solves some variational inequality.

A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mappings on a real Hilbert space H:

$$\min\ \frac{1}{2}\langle Ax, x\rangle - h(x)$$

where A is strongly positive linear bounded operator and h is a potential function for γf, i. e., $h'(x)=\gamma f$, for all $x \in H$. In this paper, we consider and analyze an iterative scheme for finding a common element of the set of solutions of the general equilibrium problem system (GEPS) and the set of all common fixed points of two noncommutative nonexpansive self mapping in the framework of a real Hilbert space. The results in this paper generalize and improve some well known results in Jankaew et al.[5] and others.

In order to prove our main results, we need the following lemmas.

Lemma 1.1. [3] *Let C be a nonempty closed convex subset of H and $F: C\times C\to\mathbb{R}$ be a bifunction satisfying $(A1)-(A4)$. Then for any $r>0$ and $x\in H$ there exists $z\in C$ such that*

$$F(z,y)+\frac{1}{r}\langle y-z, z-x\rangle \geq 0, \forall y \in C.$$

Further, define

$$T_r x = \{z \in C : F(z,y)+\frac{1}{r}\langle y-z, z-x\rangle \geq 0, \forall y \in C\}$$

for all $r>0$ and $x\in H$. Then

(a) *T_r is single-valued;*
(b) *T_r is firmly nonexpansive, i.e., for any $x, y \in H$*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x-y\rangle;$$

(c) *$F(T_r)=GEP(F)$;*
(d) *$\|T_s x - T_r x\| \leq \frac{s-r}{s}\|T_s x - x\|$;*
(e) *$GEP(F)$ is closed and convex.*

Remark 1.2. It is clear that for any $x\in H$ and $r>0$, by Lemma 1.1(a), there exists $z\in H$ such that

$$F(z,y)+\frac{1}{r}\langle y-z, z-x\rangle \geq 0, \forall y \in H. \tag{1.3}$$

Replacing x with $x-r\psi x$ in (1.3), we obtain

$$F(z,y)+\langle \psi x, y-z\rangle+\frac{1}{r}\langle y-z, z-x\rangle \geq 0, \forall y \in H.$$

Lemma 1.3. [9] *Assume $\{a_n\}$ is a sequence of nonnegative numbers such that*

$$a_{n+1} \leq (1-\alpha_n)a_n + \delta_n,$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in real number such that

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$,

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 1.4. [5] *Let C be a nonempty bounded closed convex subset of a Hilbert space H, and let S, T be two nonexpansive mappings of C into itself such that $F(ST) = F(S) \bigcap F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined as follows:*

$$\begin{aligned} x_{n+1} &= \alpha_n f(x_n) + \beta_n x_n \\ &\quad + \gamma_n \frac{2}{(n+1)(n+2)} \sum_{i=0}^{n} \sum_{j=0}^{n-i} ((ST)^j S^{i-j} \vee (ST)^i T^{j-i}) x_n, \end{aligned}$$

and put

$$\Lambda_n = \frac{2}{(n+1)(n+2)} \sum_{i=0}^{n} \sum_{j=0}^{n-i} ((ST)^j S^{i-j} \vee (ST)^i T^{j-i}) x_n.$$

Then,

$$\limsup_{n\to\infty} \sup_{x\in C} \|\Lambda_n(x) - ST\Lambda_n(x)\| = 0.$$

2. Explicit Viscosity Iterative Algorithm

In this section, we introduce an explicit viscosity iterative algorithm for finding a common element of the set of solution for an equilibrium problem system involving a bifunction defined on a closed convex subset and the set of fixed points for two noncommutative nonexpansive mappings.

Theorem 2.1. *Let $x_0 \in C$, $\{u_{n,i}\} \subset C$ and C be a nonempty closed convex subset of a real Hilbert space H, $F_1, F_2, \ldots, F_k$ be bifunctions from $C \times C$ to $\mathbb{R}$ satisfying $(A1)-(A4)$, $\Psi_1, \Psi_2, \ldots, \Psi_k$ be μ_i-inverse strongly monotone mapping on C, f be a weakly contractive mapping with a function ϕ on H, A be a strongly positive linear bounded operator with coefficient $\bar{\gamma}$ such that $\bar{\gamma} \leq \|A\| \leq 1$, B be strongly positive linear bounded operator on H with coefficient $\bar{\beta} \in (0,1]$ such that $\|B\| = \bar{\beta}$, S, T be nonexpansive mappings on C, such that $F(ST) = F(TS) = F(T) \bigcap F(S) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated in the* *following manner:*

$$\begin{cases} F_1(u_{n,1},y)+\langle \Psi_1 x_n, y-u_{n,1}\rangle + \frac{1}{r_n}\langle y-u_{n,1}, u_{n,1}-x_n\rangle \geq 0, \ \textit{for all } y\in C \\ F_2(u_{n,2},y)+\langle \Psi_2 x_n, y-u_{n,2}\rangle + \frac{1}{r_n}\langle y-u_{n,2}, u_{n,2}-x_n\rangle \geq 0, \ \textit{for all } y\in C \\ \vdots \\ F_k(u_{n,k},y)+\langle \Psi_k x_n, y-u_{n,k}\rangle + \frac{1}{r_n}\langle y-u_{n,k}, u_{n,k}-x_n\rangle \geq 0, \ \textit{for all } y\in C \\ \\ \omega_n = \frac{1}{k}\sum_{i=1}^{k} u_{n,i}, \\ \\ \Lambda_n = \frac{2}{(n+1)(n+2)}\sum_{i=0}^{n}\sum_{j=0}^{n-i}((ST)^j S^{i-j} \vee (ST)^i T^{j-i})\omega_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n B x_n + ((1-\varepsilon_n)I - \beta_n B - \alpha_n A)\Lambda_n. \end{cases}$$

where $\{\alpha_n\}\subset(0,1)$, $\{\beta_n\}$ *and* $\{\varepsilon_n\}$ *are the sequences in* $[0,1)$ *such that* $\varepsilon_n \leq \alpha_n$ *and* $\{r_n\}\subset(0,\infty)$ *is a real sequence satisfying the following conditions:*

- (C1) $\lim_{n\to\infty}\alpha_n = 0, \sum_{n=1}^{\infty}\alpha_n = \infty,$
- (C2) $\lim_{n\to\infty}\beta_n = 0, \ \sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n| < \infty,$
- (C3) $\sum_{n=1}^{\infty}|r_{n+1}-r_n| < \infty$ *and* $\liminf_{n\to\infty} r_n > 0$ *and* $0<b<r_n<a<2\mu_i$ *for* $1\leq i\leq k,$
- (C4) $\sum_{n=1}^{\infty}|\varepsilon_{n+1}-\varepsilon_n| < \infty,$
- (C5) $\lim_{n\to\infty}\dfrac{\varepsilon_n}{\alpha_n} = 0.$

Then

- (i) *the sequence* $\{x_n\}$ *is bounded.*
- (ii) $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0.$
- (iii) $\lim_{n\to\infty}\|\Psi_i x_n - \Psi_i x^*\| = 0.$ *for* $i\in\{1,2,\dots,k\}$.
- (iv) $\lim_{n\to\infty}\|x_n - \Lambda_n\| = 0.$

Proof. **(i)** Without loss of generality, we assume that $\alpha_n < (1-\varepsilon_n-\beta_n\|B\|)\|A\|^{-1}$. Since A, B are two strongly positive bounded linear operator on H, we have

$$\|A\| = \sup\{|\langle Ax,x\rangle| : x\in H, \|x\|=1\},$$

$$\|B\| = \sup\{|\langle Bx,x\rangle| : x\in H, \|x\|=1\}.$$

Also, $(1-\varepsilon_n)I - \beta_n B - \alpha_n A$ is positive. Indeed,

$$\begin{aligned}\langle((1-\varepsilon_n)I-\beta_n B-\alpha_n A)x,x\rangle &= (1-\varepsilon_n)\langle x,x\rangle - \beta_n\langle Bx,x\rangle - \alpha_n\langle Ax,x\rangle \\ &\geq 1-\varepsilon_n-\beta_n\|B\|-\alpha_n\|A\| > 0.\end{aligned}$$

Notice that

$$\begin{aligned}
\|(1-\varepsilon_n)I &- \beta_n B - \alpha_n A\| \\
&= \sup\{\langle((1-\varepsilon_n)I - \beta_n B - \alpha_n A)x, x\rangle : x \in H, \|x\| = 1\} \\
&= \sup\{(1-\varepsilon_n)\langle x, x\rangle - \beta_n\langle Bx, x\rangle - \alpha_n\langle Ax, x\rangle : x \in H, \|x\| = 1\} \\
&\leq 1 - \varepsilon_n - \beta_n\bar{\beta} - \alpha_n\bar{\gamma} \\
&\leq 1 - \beta_n\bar{\beta} - \alpha_n\bar{\gamma}.
\end{aligned}$$

Let $Q = P_{F(ST)\bigcap GEP(F_i,\Psi_i)}$. It is clear that $Q(I - A + \gamma f)$ is a contraction. Hence, there exists a unique element $z \in H$ such that $z = Q(I - A + \gamma f)z$.
Let $x^* \in \bigcap_{i=1}^k F(ST)\bigcap GEP(F_i, \Psi_i)$. For any $i = 1, 2, \ldots, k$, $I - r_n\Psi_i$ is a nonexpansive mapping and $\|u_{n,i} - x^*\| \leq \|x_n - x^*\|$. Also $\|\omega_n - x^*\| \leq \|x_n - x^*\|$. Thus

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|\alpha_n\gamma f(x_n) + \beta_n Bx_n + ((1-\varepsilon_n)I - \beta_n B - \alpha_n A)\Lambda_n - x^*\| \\
&\leq \alpha_n\|\gamma f(x_n) - Ax^*\| + \beta_n\|B\|\|x_n - x^*\| \\
&\quad + \|((1-\varepsilon_n)I - \beta_n B - \alpha_n A)\|\|\Lambda_n - x^*\| + \varepsilon_n\|x^*\| \\
&\leq \alpha_n\gamma\|f(x_n) - f(x^*)\| + \alpha_n\|\gamma f(x^*) - Ax^*\| + \beta_n\bar{\beta}\|x_n - x^*\| \\
&\quad + (1 - \beta_n\bar{\beta} - \alpha_n\bar{\gamma})\|x_n - x^*\| + \alpha_n\|x^*\| \\
&\leq \alpha_n\gamma\|x_n - x^*\| - \phi(\|x_n - x^*) + \alpha_n\|\gamma f(x^*) - Ax^*\| \\
&\quad + \beta_n\bar{\beta}\|x_n - x^*\| + (1 - \beta_n\bar{\beta} - \alpha_n\bar{\gamma})\|x_n - x^*\| + \alpha_n\|x^*\| \\
&\leq (1 - (\bar{\gamma} - \gamma)\alpha_n)\|x_n - x^*\| + \alpha_n(\|\gamma f(x^*) - Ax^*\| + \|x^*\|) \\
&\leq \max\{\|x_n - x^*\|, \frac{\|\gamma f(x^*) - Ax^*\|}{\bar{\gamma} - \gamma}\}.
\end{aligned}$$

By induction

$$\|x_n - x^*\| \leq \max\{\|x_1 - x^*\|, \frac{\|\gamma f(x^*) - Ax^*\|}{\bar{\gamma} - \gamma}\}.$$

and the sequence $\{x_n\}$ is bounded and also $\{f(x_n)\}$, $\{\omega_n\}$ and $\{\Lambda_n\}$ are bounded.

(ii) Note that $u_{n,i}$ can be written as $u_{n,i} = T_{r_n,i}(x_n - r_n\psi_i x_n)$. It follows from Lemma 1.1 that

$$\|u_{n+1,i} - u_{n,i}\| \leq \|x_{n+1} - x_n\| + 2M_i|r_{n+1} - r_n|, \tag{2.1}$$

where

$$M_i = \max\{\sup\{\frac{\|T_{r_{n+1},i}(I - r_n\Psi_i)x_n - T_{r_n,i}(I - r_n\Psi_i)x_n\|}{r_{n+1}}\}, \sup\{\|\Psi_i x_n\|\}\}.$$

Let $M = \frac{1}{k}\sum_{i=1}^{k} 2M_i < \infty$. Next, we estimate $\|\omega_{n+1} - \omega_n\|$,

$$\|\omega_{n+1} - \omega_n\| \leq \frac{1}{k}\sum_{i=1}^{k}\|u_{n+1,i} - u_{n,i}\| \leq \|x_{n+1} - x_n\| + M|r_{n+1} - r_n|. \quad (2.2)$$

Now, we prove that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. We observe that

$$\begin{aligned}
\|x_{n+2} - x_{n+1}\| &= \|\alpha_{n+1}\gamma f(x_{n+1}) + \beta_{n+1}Bx_{n+1} \\
&\quad +((1-\varepsilon_{n+1})I - \beta_{n+1}B - \alpha_{n+1}A)\Lambda_{n+1} - \alpha_{n+1}\gamma f(x_n) \\
&\quad -\beta_n Bx_n - ((1-\varepsilon_n)I - \beta_n B - \alpha_n A)\Lambda_n\| \\
&= \|((1-\varepsilon_{n+1})I - \beta_{n+1}B - \alpha_{n+1}A)(\Lambda_{n+1} - \Lambda_n) \\
&\quad +\{(\varepsilon_n - \varepsilon_{n+1})\Lambda_n + (\beta_n - \beta_{n+1})B\Lambda_n + (\alpha_n - \alpha_{n+1})A\Lambda_n\} \\
&\quad +\alpha_{n+1}\gamma(f(x_{n+1}) - f(x_n)) + (\alpha_{n+1} - \alpha_n)\gamma f(x_n) \\
&\quad +\beta_{n+1}B(x_{n+1} - x_n) + (\beta_{n+1} - \beta_n)Bx_n\| \\
&\leq \|(1-\varepsilon_{n+1})I - \beta_{n+1}B - \alpha_{n+1}A\|\|\Lambda_{n+1} - \Lambda_n\| \\
&\quad +\|\varepsilon_n - \varepsilon_{n+1}\|\|\Lambda_n\| + |\beta_n - \beta_{n+1}|\|B\|\Lambda_n\| \\
&\quad +|\alpha_n - \alpha_{n+1}|\|A\Lambda_n\| + \alpha_{n+1}\gamma\|f(x_{n+1}) - f(x_n)\| \\
&\quad +|\alpha_{n+1} - \alpha_n|\gamma\|f(x_n)\| + \beta_{n+1}\|B\|\|x_{n+1} - x_n\| \\
&\quad +|\beta_{n+1} - \beta_n|\|B\|\|x_n\| \\
&\leq (1 - \beta_{n+1}\bar{\beta} - \alpha_{n+1}\bar{\gamma})\|\Lambda_{n+1} - \Lambda_n\| + |\varepsilon_n - \varepsilon_{n+1}|\|\Lambda_n\| \\
&\quad +|\beta_n - \beta_{n+1}|\bar{\beta}\|\Lambda_n\| + |\alpha_n - \alpha_{n+1}|\|A\Lambda_n\| \\
&\quad +\alpha_{n+1}\gamma\|x_{n+1} - x_n\| - \alpha_{n+1}\gamma\phi(\|x_{n+1} - x_n\|) \\
&\quad +|\alpha_{n+1} - \alpha_n|\gamma\|f(x_n)\| + \beta_{n+1}\bar{\beta}\|x_{n+1} - x_n\| \\
&\quad +|\beta_{n+1} - \beta_n|\bar{\beta}\|x_n\| \\
&\leq (1 - \beta_{n+1}\bar{\beta} - \alpha_{n+1}\bar{\gamma})\|\Lambda_{n+1} - \Lambda_n\| + |\varepsilon_n - \varepsilon_{n+1}|\|\Lambda_n\| \\
&\quad +|\beta_n - \beta_{n+1}|\bar{\beta}K + K|\alpha_n - \alpha_{n+1}| \\
&\quad +(\alpha_{n+1}\gamma + \beta_{n+1}\bar{\beta})\|x_{n+1} - x_n\| - \alpha_{n+1}\gamma\phi(\|x_{n+1} - x_n\|)
\end{aligned}$$

where $K = \sup\{max\{\|\Lambda_n\| + \|x_n\|, \gamma\|f(x_n)\| + \|A\Lambda_n\|\}, \forall n \geq 0\} < \infty$.
Let $\Delta_n = \|\beta_n - \beta_{n+1}\|\bar{\beta}K + K|\alpha_n - \alpha_{n+1}| + |\varepsilon_n - \varepsilon_{n+1}|\|\Lambda_n\|$, then

$$\begin{aligned}
\|x_{n+2} - x_{n+1}\| \leq\ &(1 - \beta_{n+1}\bar{\beta} - \alpha_{n+1}\bar{\gamma})\|\Lambda_{n+1} - \Lambda_n\| \\
&+(\alpha_{n+1}\gamma + \beta_{n+1}\bar{\beta})\|x_{n+1} - x_n\| \\
&-\alpha_{n+1}\gamma\phi(\|x_{n+1} - x_n\|) + \Delta_n,
\end{aligned} \quad (2.3)$$

From [5], we conclude

$$\|\Lambda_{n+1} - \Lambda_n\| \leq \|\omega_{n+1} - \omega_n\| + \frac{4}{n+3}\|\omega_{n+1} - x^*\| + \frac{4}{n+3}\|x^*\|. \quad (2.4)$$

Substituting (2.2) and (2.4) into (2.3), thus

$$\begin{aligned}\|x_{n+2}-x_{n+1}\| \quad \leq \quad & (1-\beta_{n+1}\bar{\beta}-\alpha_{n+1}\bar{\gamma})\{\|x_{n+1}-x_n\|+M|r_{n+1}-r_n| \\ & +\frac{4}{n+3}\|\omega_{n+1}-x^*\|+\frac{4}{n+3}\|x^*\|\} \\ & +(\alpha_{n+1}\gamma+\beta_{n+1}\bar{\beta})\|x_{n+1}-x_n\|-\alpha_{n+1}\gamma\phi(\|x_{n+1}-x_n\|) \\ & +\Delta_n,\end{aligned}$$

for some positive constant M. It follows that

$$\begin{aligned}\|x_{n+2}-x_{n+1}\| \quad \leq \quad & (1-\alpha_{n+1}(\bar{\gamma}-\gamma\rho))\|x_{n+1}-x_n\| \\ & +M(1-\beta_{n+1}\bar{\beta}-\alpha_{n+1}\bar{\gamma})|r_{n+1}-r_n| \\ & +(1-\beta_{n+1}\bar{\beta}-\alpha_{n+1}\bar{\gamma})\frac{4}{n+3}\|\omega_{n+1}-x^*\| \\ & +(1-\beta_{n+1}\bar{\beta}-\alpha_{n+1}\bar{\gamma})\frac{4}{n+3}\|x^*\|+\Delta_n.\end{aligned}$$

By Lemma 1.3,

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{2.5}$$

(iii) For any $i\in\{1,2,\dots,k\}$,

$$\begin{aligned}\|u_{n,i}-x^*\|^2 \quad & \leq \quad \|(x_n-x^*)-r_n(\Psi_i x_n-\Psi_i x^*)\|^2 \\ & = \quad \|x_n-x^*\|^2-2r_n\langle x_n-x^*,\Psi_i x_n-\Psi_i x^*\rangle+r_n^2\|\Psi_i x_n-\Psi_i x^*\|^2 \\ & \leq \quad \|x_n-x^*\|^2-r_n(2\mu_i-r_n)\|\Psi_i x_n-\Psi_i x^*\|^2,\end{aligned}$$

thus

$$\begin{aligned}\|\omega_n-x^*\|^2 = \quad & \|\sum_{i=1}^{k}\frac{1}{k}(u_{n,i}-x^*)\|^2 \\ \leq \quad & \tfrac{1}{k}\sum_{i=1}^{k}\|u_{n,i}-x^*\|^2 \\ \leq \quad & \|x_n-x^*\|^2-\tfrac{1}{k}\sum_{i=1}^{k}r_n(2\mu_i-r_n)\|\Psi_i x_n-\Psi_i x^*)\|^2.\end{aligned} \tag{2.6}$$

From (2.6),

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \|\alpha_n(\gamma f(x_n)-Ax^*)+\beta_n B(x_n-x^*)\\
&\quad +((1-\varepsilon_n)I-\beta_n B-\alpha_n A)(\Lambda_n-x^*)-\varepsilon_n x^*\|^2\\
&\leq \|\alpha_n(\gamma f(x_n)-Ax^*)+\beta_n B(x_n-x^*)\\
&\quad +((1-\varepsilon_n)I-\beta_n B-\alpha_n A)(\omega_n-x^*)+\varepsilon_n x^*\|^2\\
&\leq \alpha_n\|\gamma f(x_n)-Ax^*\|^2+\beta_n\|B\|^2\|x_n-x^*\|^2\\
&\quad +(1-\beta_n\bar{\beta}-\alpha_n\bar{\gamma})\|\Lambda_n-x^*\|^2+\varepsilon_n^2\|x^*\|^2\\
&\leq \alpha_n\|\gamma f(x_n)-Ax^*\|^2+\beta_n\bar{\beta}^2\|x_n-x^*\|^2\\
&\quad +(1-\beta_n\bar{\beta}-\alpha_n\bar{\gamma})\|\omega_n-x^*\|^2+\varepsilon_n^2\|x^*\|^2\\
&\leq \alpha_n\|\gamma f(x_n)-Ax^*\|^2+\beta_n\bar{\beta}\|x_n-x^*\|^2\\
&\quad +(1-\beta_n\bar{\beta}-\alpha_n\bar{\gamma})\{\|x_n-x^*\|^2\\
&\quad -\frac{1}{k}\sum_{i=1}^{k} r_n(2\mu_i-r_n)\|\Psi_i x_n-\Psi_i x^*)\|^2\}+\varepsilon_n^2\|x^*\|^2\\
&\leq \alpha_n\|\gamma f(x_n)-Ax^*\|^2+\|x_n-x^*\|^2\\
&\quad -(1-\beta_n\bar{\beta}-\alpha_n\bar{\gamma})\frac{1}{k}\sum_{i=1}^{k} r_n(2\mu_i-r_n)\|\Psi_i x_n-\Psi_i x^*)\|^2\\
&\quad +\varepsilon_n^2\|x^*\|^2
\end{aligned}$$

and hence

$$\begin{aligned}
(1\quad &-\beta_n\bar{\beta}-\alpha_n\bar{\gamma})\tfrac{1}{k}\sum_{i=1}^{k} b(2\mu_i-a)\|\Psi_i x_n-\Psi_i x^*\|^2\\
&\leq \alpha_n\|\gamma f(x_n)-Ax^*\|^2+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+\varepsilon_n^2\|x^*\|^2\\
&\leq \alpha_n\|\gamma f(x_n)-Ax^*\|^2+\|x_{n+1}-x_n\|(\|x_{n+1}-x^*\|-\|x_n-x^*\|)\\
&\quad +\varepsilon_n^2\|x^*\|^2.
\end{aligned}$$

Since $\alpha_n\to 0$ and $\varepsilon_n\leq\alpha_n$ then $\varepsilon_n\to 0$ as $n\to\infty$. The inequality (2.5) implies that

$$\lim_{n\to\infty}\|\Psi_i x_n-\Psi_i x^*\|=0, \forall i=1,2,\ldots,k. \tag{2.7}$$

(iv) By Lemma 1.1

$$\|u_{n,i}-x^*\|^2\leq\|x_n-x^*\|^2-\|x_n-u_{n,i}\|^2+2r_n\|x_n-u_{n,i}\|\|\Psi_i x_n-\Psi_i x^*\| \tag{2.8}$$

and hence

$$\begin{aligned}
\|\omega_n-x^*\|^2 = &\ \|\textstyle\sum_{i=1}^{k}\frac{1}{k}(u_{n,i}-x^*)\|^2\\
\leq &\ \tfrac{1}{k}\textstyle\sum_{i=1}^{k}\|u_{n,i}-x^*\|^2\\
\leq &\ \|x_n-x^*\|^2-\tfrac{1}{k}\textstyle\sum_{i=1}^{k}\|u_{n,i}-x_n\|^2\\
&\ +\tfrac{1}{k}\textstyle\sum_{i=1}^{k}2r_n\|x_n-u_{n,i}\|\|\Psi_i x_n-\Psi_i x^*\|.
\end{aligned} \tag{2.9}$$

From (2.9),

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\leq \alpha_n\|\gamma f(x_n)-Ax^*\|^2+\beta_n\bar{\beta}\|x_n-x^*\|^2\\ &\quad+(1-\beta_n\bar{\beta}-\alpha_n\bar{\gamma})\|\omega_n-x^*\|^2+\varepsilon_n^2\|x^*\|^2\\ &\leq \alpha_n\|\gamma f(x_n)-Ax^*\|^2+\beta_n\bar{\beta}\|x_n-x^*\|^2\\ &\quad+(1-\beta_n\bar{\beta}-\alpha_n\bar{\gamma})\{\|x_n-x^*\|^2-\frac{1}{k}\sum_{i=1}^{k}\|u_{n,i}-x_n\|^2\\ &\quad+\frac{1}{k}\sum_{i=1}^{k}2r_n\|x_n-u_{n,i}\|\|\Psi_i x_n-\Psi_i x^*\|\}+\varepsilon_n^2\|x^*\|^2\end{aligned}$$

thus

$$(1-\beta_n\bar{\beta}-\alpha_n\bar{\gamma})\frac{1}{k}\sum_{i=1}^{k}\|u_{n,i}-x_n\|^2$$

$$\begin{aligned}&\leq \alpha_n\|\gamma f(x_n)-Ax^*\|^2+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2\\ &\quad+(1-\beta_n\bar{\beta}-\alpha_n\bar{\gamma})\frac{1}{k}\sum_{i=1}^{k}2r_n\|x_n-u_{n,i}\|\|\Psi_i x_n-\Psi_i x^*\|+\varepsilon_n^2\|x^*\|^2\\ &\leq \alpha_n\|\gamma f(x_n)-Ax^*\|^2+\|x_{n+1}-x_n\|(\|x_{n+1}-x^*\|-\|x_n-x^*\|)\\ &\quad+(1-\beta_n\bar{\beta}-\alpha_n\bar{\gamma})\frac{1}{k}\sum_{i=1}^{k}2r_n\|x_n-u_{n,i}\|\|\Psi_i x_n-\Psi_i x^*\|+\varepsilon_n^2\|x^*\|^2\end{aligned}$$

From the condition $(C1)$, (2.5) and (2.7), we get

$$\lim_{n\to\infty}\|u_{n,i}-x_n\|=0. \tag{2.10}$$

It is easy to prove

$$\lim_{n\to\infty}\|\omega_n-x_n\|=0. \tag{2.11}$$

By definition of the sequence $\{x_n\}$, we obtain

$$\begin{aligned}\|x_n-\Lambda_n\| &\leq \|x_{n+1}-x_n\|+\|x_{n+1}-\Lambda_n\|\\ &\leq \|x_{n+1}-x_n\|+\|\alpha_n\gamma f(x_n)+\beta_n Bx_n\\ &\quad+((1-\varepsilon_n)I-\beta_n B-\alpha_n A)\Lambda_n-\Lambda_n\|\\ &\leq \|x_{n+1}-x_n\|+\alpha_n\|\gamma f(x_n)-A\Lambda_n\|+\beta_n\bar{\beta}\|x_n-\Lambda_n\|+\varepsilon_n\|\Lambda_n\|.\end{aligned}$$

Then

$$\begin{aligned}\|x_n-\Lambda_n\| &\leq \frac{1}{1-\beta_n\bar{\beta}}\|x_{n+1}-x_n\|+\frac{\alpha_n}{1-\beta_n\bar{\beta}_n}\|\gamma f(x_n)-A\Lambda_n\|\\ &\quad+\frac{\varepsilon_n}{1-\beta_n\bar{\beta}}\|\Lambda_n\|.\end{aligned}$$

Thanks to the conditions $(C1)-(C2)$ and (2.5), we conclude that

$$\lim_{n\to\infty}\|x_n-\Lambda_n\|=0. \tag{2.12}$$

Also

$$\|\omega_n - \Lambda_n\| \leq \|\omega_n - x_n\| + \|x_n - \Lambda_n\|,$$

hence

$$\lim_{n\to\infty} \|\omega_n - \Lambda_n\| = 0. \tag{2.13}$$

□

Theorem 2.2. *Suppose all assumptions of Theorem 2.1 are holds. Then the sequence $\{x_n\}$ is strongly convergent to a point $\bar{x}$, where $\bar{x} \in \bigcap_{i=1}^{k} F(ST) \bigcap GEP(F_i, \Psi_i)$ solves the variational inequality*

$$\langle (A - \gamma f)\bar{x}, \bar{x} - x\rangle \leq 0.$$

Equivalently, $\bar{x} = P_{\bigcap_{i=1}^{k} F(ST) \bigcap GEP(F_i,\Psi_i)}(I - A + \gamma f)(\bar{x})$.

Proof. Observe that $P_{\bigcap_{i=1}^{k} F(ST) \bigcap GEP(F_i,\Psi_i)}(I - A + \gamma f)$ is a contraction of H into itself. Since H is complete, there exists a unique element $\bar{x} \in H$ such that

$$\bar{x} = P_{\bigcap_{i=1}^{k} F(ST) \bigcap GEP(F_i,\Psi_i)}(I - A + \gamma f)(\bar{x}).$$

Next, we prove

$$\limsup_{n\to\infty} \langle (A - \gamma f)\bar{x}, \bar{x} - \Lambda_n\rangle \leq 0$$

Let $\tilde{x} = P_{\bigcap_{i=1}^{k} F(ST) \bigcap GEP(F_i,\Psi_i)} x_1$, set

$$E = \{\bar{y} \in H : \|\bar{y} - \tilde{x}\| \leq \|x_1 - \tilde{x}\| + \frac{\|\gamma f(\tilde{x}) - A\tilde{x}\|}{\bar{\gamma} - \gamma\rho}\} \bigcap C.$$

It is clear, E is nonempty closed bounded convex subset of C and $S(E) \subset E$, $T(E) \subset E$. Without loss of generality, we may assume S and T are mappings of E into itself. Since $\{\Lambda_n\} \subset E$ is bounded, there is a subsequence $\{\Lambda_{n_j}\}$ of $\{\Lambda_n\}$ such that

$$\limsup_{n\to\infty} \langle (A - \gamma f)\bar{x}, \bar{x} - \Lambda_n\rangle = \lim_{j\to\infty} \langle (A - \gamma f)\bar{x}, \bar{x} - \Lambda_{n_j}\rangle. \tag{2.14}$$

As $\{\Lambda_{n_j}\}$ is also bounded, there exists a subsequence $\{\Lambda_{n_{j_l}}\}$ of $\{\Lambda_{n_j}\}$ such that $\Lambda_{n_{j_l}} \rightharpoonup \xi$. Without loss of generality, let $\Lambda_{n_j} \rightharpoonup \xi$. Now, we prove the following items:

(i): $\xi \in F(ST) = F(T) \bigcap F(S)$.

Assume $\xi \notin F(ST)$. By Lemma 1.4 and Opial's condition,

$$\begin{aligned}
\liminf_{j\to\infty} \|\Lambda_{n_j} - \xi\| &< \liminf_{j\to\infty} \|\Lambda_{n_j} - ST(\xi)\| \\
&\leq \liminf_{j\to\infty} (\|\Lambda_{n_j} - ST(\Lambda_{n_j})\| + \|ST(\Lambda_{n_j}) - ST(\xi)\|) \\
&\leq \liminf_{j\to\infty} \|\Lambda_{n_j} - \xi\|.
\end{aligned}$$

That is a contradiction. Hence $\xi = F(ST)\xi$.

(ii): By the same argument as in the proof of [7, Theorem 3.2], we conclude that $\xi \in GEP(F_i, \Psi_i)$, for all $i = 1, 2, \ldots, k$. Then $\xi \in \bigcap_{i=1}^{k} GEP(F_i, \Psi_i)$. Now, in view of (2.14), we see

$$\limsup_{n\to\infty}\langle (A-\gamma f)\bar{x}, \bar{x}-\Lambda_n\rangle = \langle (A-\gamma f)\bar{x}, \bar{x}-\xi\rangle \leq 0,$$

Finally, we show that $\{x_n\}$ is strongly convergent to $\bar{x}$. As a matter of fact,

$$\begin{aligned}
\| \quad & x_{n+1}-\bar{x}\|^2 = \|\alpha_n\gamma f(x_n)+\beta_n Bx_n+((1-\varepsilon_n)I-\beta_n B-\alpha_n A)\Lambda_n-\bar{x}\|^2\\
= \quad & \|\alpha_n(\gamma f(x_n)-A\bar{x})+\beta_n B(x_n-\bar{x})\\
& +((1-\varepsilon_n)I-\beta_n B-\alpha_n A)(\Lambda_n-\bar{x})-\varepsilon_n\bar{x}\|^2\\
\leq \quad & \|\alpha_n(\gamma f(x_n)-A\bar{x})+\beta_n B(x_n-\bar{x})\\
& +((1-\varepsilon_n)I-\beta_n B-\alpha_n A)(\Lambda_n-\bar{x})+\varepsilon_n\bar{x}\|^2\\
= \quad & \|\alpha_n(\gamma f(x_n)-A\bar{x})+\beta_n B(x_n-\bar{x})\\
& +((1-\varepsilon_n)I-\beta_n B-\alpha_n A)(\Lambda_n-\bar{x})\|^2\\
& +2\varepsilon_n\|\alpha_n(\gamma f(x_n)-A\bar{x})+\beta_n B(x_n-\bar{x})\\
& +((1-\varepsilon_n)I-\beta_n B-\alpha_n A)(\Lambda_n-\bar{x})\|\|\bar{x}\|+\varepsilon_n^2\|\bar{x}\|^2\\
= \quad & \|\beta_n B(x_n-\bar{x})+((1-\varepsilon_n)I-\beta_n B-\alpha_n A)(\Lambda_n-\bar{x})\|^2\\
& +2\alpha_n\langle((1-\varepsilon_n)I-\beta_n B-\alpha_n A)(\Lambda_n-\bar{x}),\gamma f(x_n)-A\bar{x}\rangle\\
& +2\varepsilon_n\|\alpha_n(\gamma f(x_n)-A\bar{x})+\beta_n B(x_n-\bar{x})\\
& +((1-\varepsilon_n)I-\beta_n B-\alpha_n A)(\Lambda_n-\bar{x})\|\|\bar{x}\|\\
& +\varepsilon_n^2\|\bar{x}\|^2+\alpha_n^2\|\gamma f(x_n)-A\bar{x}\|^2+2\alpha_n\beta_n\langle B(x_n-\bar{x}),\gamma f(x_n)-A\bar{x}\rangle\\
\leq \quad & (\beta_n\|B\|\|x_n-\bar{x}\|+\|(1-\varepsilon_n)I-\beta_n B-\alpha_n A\|\|\Lambda_n-\bar{x}\|)^2\\
& +2\alpha_n\gamma\langle\Lambda_n-\bar{x},f(x_n)-f(\bar{x})\rangle+2\varepsilon_n\|\alpha_n(\gamma f(x_n)-A\bar{x})+\beta_n B(x_n-\bar{x})\\
& +((1-\varepsilon_n)I-\beta_n B-\alpha_n A)(\Lambda_n-\bar{x})\|\|\bar{x}\|+\varepsilon_n^2\|\bar{x}\|^2+\alpha_n^2\|\gamma f(x_n)-A\bar{x}\|^2\\
& +2\alpha_n\beta_n\langle B(x_n-\bar{x}),\gamma f(x_n)-A\bar{x}\rangle+2\alpha_n\langle\Lambda_n-\bar{x},\gamma f(x_n)-A\bar{x}\rangle\\
& -2\alpha_n\langle(\varepsilon_n I+\beta_n B+\alpha_n A)(\Lambda_n-\bar{x}),\gamma f(x_n)-A\bar{x}\rangle\\
\leq \quad & (\beta_n\bar{\beta}\|x_n-\bar{x}\|+(1-\beta_n\bar{\beta}-\alpha_n\bar{\gamma})\|x_n-\bar{x}\|)^2+2\alpha_n\gamma\|x_n-\bar{x}\|^2\\
& +2\varepsilon_n\|\alpha_n(\gamma f(x_n)-A\bar{x})+\beta_n B(x_n-\bar{x})\\
& +((1-\varepsilon_n)I-\beta_n B-\alpha_n A)(\Lambda_n-\bar{x})\|\|\bar{x}\|+\varepsilon_n^2\|\bar{x}\|^2\\
& +\alpha_n^2\|\gamma f(x_n)-A\bar{x}\|^2+2\alpha_n\beta_n\langle B(x_n-\bar{x}),\gamma f(x_n)-A\bar{x}\rangle\\
& +2\alpha_n\langle\Lambda_n-\bar{x},\gamma f(x_n)-A\bar{x}\rangle\\
& -2\alpha_n\langle(\varepsilon_n I+\beta_n B+\alpha_n A)(\Lambda_n-\bar{x}),\gamma f(x_n)-A\bar{x}\rangle\\
= \quad & (1-2(\bar{\gamma}-\gamma)\alpha_n)\|x_n-\bar{x}\|^2+\alpha_n^2\bar{\gamma}^2\|x_n-\bar{x}\|^2+2\varepsilon_n\|\alpha_n(\gamma f(x_n)-A\bar{x})\\
& +\beta_n B(x_n-\bar{x})+((1-\varepsilon_n)I-\beta_n B-\alpha_n A)(\Lambda_n-\bar{x})\|\|\bar{x}\|+\varepsilon_n^2\|\bar{x}\|^2\\
& +\alpha_n^2\|\gamma f(x_n)-A\bar{x}\|^2+2\alpha_n\beta_n\langle B(x_n-\bar{x}),\gamma f(x_n)-A\bar{x}\rangle\\
& +2\alpha_n\langle\Lambda_n-\bar{x},\gamma f(x_n)-A\bar{x}\rangle\\
& -2\alpha_n\langle(\varepsilon_n I+\beta_n B+\alpha_n A)(\Lambda_n-\bar{x}),\gamma f(x_n)-A\bar{x}\rangle\\
\leq \quad & (1-2(\bar{\gamma}-\gamma)\alpha_n)\|x_n-\bar{x}\|^2+\alpha_n\vartheta_n,
\end{aligned}$$

where

$$\begin{aligned}
\zeta_n &= (\bar{\gamma} - \gamma\rho)\alpha_n, \\
\vartheta_n &= \alpha_n \bar{\gamma}^2 \|x_n - \bar{x}\|^2 + 2\|\alpha_n(\gamma f(x_n) - A\bar{x}) \\
&\quad + \beta_n B(x_n - \bar{x}) + ((1-\varepsilon_n)I - \beta_n B - \alpha_n A)(\Lambda_n - \bar{x})\|\|\bar{x}\| + \varepsilon_n \|\bar{x}\|^2 \\
&\quad + \alpha_n \|\gamma f(x_n) - A\bar{x}\|^2 + 2\beta_n \langle B(x_n - \bar{x}), \gamma f(x_n) - A\bar{x}\rangle \\
&\quad + 2\alpha_n \langle \Lambda_n - \bar{x}, \gamma f(x_n) - A\bar{x}\rangle \\
&\quad - 2\alpha_n \langle (\varepsilon_n I + \beta_n B + \alpha_n A)(\Lambda_n - \bar{x}), \gamma f(x_n) - A\bar{x}\rangle.
\end{aligned}$$

From conditions $(C1)-(C2)$ and $(C5)$ and Lemma 1.3, we obtain the sequence $\{x_n\}$ strongly convergence to $\bar{x}$. □

Using Theorems 2.1 and 2.2, we obtain the following corollaries.

Corollary 2.3. [5, Theorem 3.1] *Let C be a nonempty closed convex subset of H. Suppose S and T are nonexpansive mappings of C into itself, such that $F(ST) = F(TS) = F(S) \bigcap F(T) \neq \emptyset$. Let f be a contraction mapping from C to C and $\{x_n\}$ be a sequence generated by $x_0 = x \in C$ and*

$$\begin{aligned}
x_{n+1} &= \alpha_n f(x_n) + \beta_n \\
&\quad + (1-\alpha_n-\beta_n)\frac{2}{(n+1)(n+2)} \sum_{i=0}^{n} \sum_{j=0}^{n-i} ((ST)^j S^{i-j} \vee (ST)^i T^{j-i}) x_n
\end{aligned}$$

for all $n \in N \bigcup \{0\}$, where $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ satisfy

$$\lim_{n\to\infty} \alpha_n = 0, \sum_{n=0}^{\infty} \alpha_n = \infty.$$

If $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$, then $\{x_n\}$ converges strongly to $z \in F(S) \bigcap F(T)$, where $z = P_{F(S)\bigcap F(T)} f(z)$ is the unique solution of the variational inequality

$$\langle (I-f)z, z-x\rangle \geq 0, \forall x \in F(S) \bigcap F(T).$$

Proof. Setting $F_i = \Psi_i \equiv 0, \forall i \in \{1, 2, \ldots, k\}, A = B \equiv I, \gamma = \bar{\gamma} = 1$, $\varepsilon_n = 0$, and $\omega_n = x_n, \phi(t) = (1-\rho)t$ in Theorems 2.1 and 2.2. Thus the proof is straightforward. □

Corollary 2.4. [5, Corollary 3.4] *Let C be a nonempty closed convex subset of H. Suppose S and T are averaged mappings of C into itself, such that $F(S) \bigcap F(T) \neq \emptyset$. Let f be a contraction mapping from C to C and $\{x_n\}$ be a* *sequence generated by* $x_0 = x \in C$ *and*

$$\begin{aligned} x_{n+1} &= \alpha_n f(x_n) + \beta_n \\ &\quad +(1-\alpha_n-\beta_n)\frac{2}{(n+1)(n+2)}\sum_{i=0}^{n}\sum_{j=0}^{n-i}((ST)^j S^{i-j} \vee (ST)^i T^{j-i})x_n \end{aligned}$$

for all $n \in N \bigcup \{0\}$, *where* $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ *satisfy*

$$\lim_{n\to\infty} \alpha_n = 0, \lim_{n\to\infty} \beta_n = 0, \sum_{n=0}^{\infty} \alpha_n = \infty.$$

If $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$, *then* $\{x_n\}$ *is strongly convergent to* $z \in F(S) \bigcap F(T)$, *where* $z = P_{F(S)\bigcap F(T)} f(z)$ *is the unique solution of the variational inequality*

$$\langle (I-f)z, z-x\rangle \geq 0, \forall x \in F(S) \bigcap F(T).$$

Proof. Set S and T averaged mappings of C into itself, $F_i = \Psi_i \equiv 0$, for all $i \in \{1, 2, \ldots, k\}$, $A = B \equiv I, \gamma = \bar{\gamma} = 1$, $\varepsilon_n = 0$ and $\omega_n = x_n, \phi(t) = (1-\rho)t$ in Theorems 2.1 and 2.2. Thus the proof is straightforward. □

Corollary 2.5. [8, Theorem 1] *Let* C *be a nonempty closed convex subset of* H. *Suppose* S *and* T *are nonexpansive mappings of* C *into itself, such that* $ST = TS$ *and* $F(S) \bigcap F(T) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence generated by* $x_0 = x \in C$ *and*

$$x_{n+1} = \alpha_n x + (1-\alpha_n)\frac{2}{(n+1)(n+2)}\sum_{i=0}^{n}\sum_{j=0}^{n-i} SiT^j x_n.$$

for all $n \in N$, *where* $\{\alpha_n\} \subset [0,1]$ *satisfy*

$$\lim_{n\to\infty} \alpha_n = 0, \sum_{n=0}^{\infty} \alpha_n = \infty.$$

Then $\{x_n\}$ *is strongly convergent to* $z \in F(S) \bigcap F(T)$, *where* $P_{F(S)\bigcap F(T)}$ *is a metric projection of* H *onto* $F(S) \bigcap F(T)$.

Proof. Setting $ST = TS$, $F_i = \Psi_i \equiv 0$, for all $i \in \{1, 2, \ldots, k\}, A = B \equiv I, \gamma = \bar{\gamma} = 1$, $f(y) = x$, for all $y \in C$, $\varepsilon_n = \beta_n = 0$ and $\omega_n = x_n, \phi(t) = (1-\rho)t$ in Theorems 2.1 and 2.2. Thus the proof is straightforward. □

Corollary 2.6. [10, Theorem 4] *Let* H *be a Hilbert space and* C *a nonempty closed convex subset of* H. *Suppose* S *and* T *are averaged mappings of* C *into itself such that* $F(S) \bigcap F(T)$ *is nonempty. Suppose that* $\{\alpha_n\}$ *satisfies*

$$\lim_{n\to\infty} \alpha_n = 0, \sum_{n=0}^{\infty} \alpha_n = \infty.$$

For an arbitrary $x \in C$, the sequence $\{x_n\}$ generated by

$$\begin{cases} x_0 = x, \\ x_{n+1} = \alpha_n x + (1-\alpha_n)\frac{2}{(n+1)(n+2)}\sum_{i=0}^{n}\sum_{j=0}^{n-i}((ST)^j S^{i-j} \vee (ST)^i T^{j-i})x_n. \end{cases}$$

is strongly convergent to a common fixed point Px of S and T, where P is the metric projection of H onto $F(S)\bigcap F(T)$.

Proof. Setting S and T be averaged mappings of C into itself and $F_i = \Psi_i \equiv 0$, for all $i \in \{1, 2, \ldots, k\}, A = B \equiv I, \gamma = \bar{\gamma} = 1$, $f(y) = x, \forall y \in C$, $\varepsilon_n = \beta_n = 0$ and $\omega_n = x_n, \phi(t) = (1-\rho)t$ in Theorems 2.1 and 2.2. Thus the proof is straightforward. □

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