On Tensor Product of Graphs, Girth and Triangles

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Abstract. The purpose of this paper is to obtain a necessary and sufficient condition for the tensor product of two or more graphs to be connected, bipartite or eulerian. Also, we present a characterization of the duplicate graph $G \oplus K_2$ to be unicyclic. Finally, the girth and the formula for computing the number of triangles in the tensor product of graphs are worked out.

Keywords: Tensor product, Bipartite graph, Connected graph, Eulerian graph, Girth, Cycle, Path.


1. Introduction

We shall consider only finite, undirected graphs without loops or multiple edges. We follow the terminology of [1]. For a graph $G$, $V(G)$ and $E(G)$ denote the vertex set and edge set of $G$, respectively. For a connected graph $G$, $nG$ is the graph with $n$ components, each being isomorphic to $G$. It is well-known that a graph is bipartite if and only if it contains no odd cycle. We now define the tensor product of two graphs [8] as follows: The tensor product of two graphs $G_1$ and $G_2$ is the graph, denoted by $G_1 \oplus G_2$, with vertex set $V(G_1 \oplus G_2) = V(G_1) \times V(G_2)$, and any two of its vertices $(u_1, v_1)$ and $(u_2, v_2)$ are adjacent, whenever $u_1$ is adjacent to $u_2$ in $G_1$ and $v_1$ is adjacent to $v_2$ in $G_2$.

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The graphs $G_1$ and $G_2$ are called factors of the product $G_1 \oplus G_2$. Other popular names for tensor product that have appeared in the literature are Kronecker product, cross product, direct product, conjunction product. Sampathkumar [6] defines the tensor product of a graph $G$ by $K_2$ as the duplicate graph of $G$, and studied its properties and a characterization in great detail. This product is also studied in [5]. Now, we define the two special type of tensor products: $G \oplus nK_2$ and $G \oplus [\oplus_{i=1}^n K_2]$ as the generalized duplicate graphs of a graph $G$, for any integer $n \geq 2$, and study their structural properties for our later use.

2. STRUCTURAL PROPERTIES OF THE GENERALIZED DUPLICATE GRAPHS

The following theorem of Weichsel [8] will be useful in the proof of our results.

**Theorem 2.1.** If the connected graphs $G$ and $H$ are bipartite, then $G \oplus H$ has exactly two components.

Next, we present some elementary results of the generalized duplicate graphs for our immediate use.

**Theorem 2.2.** For any connected, bipartite graph $G$, $G \oplus nK_2 = 2nG$ for $n \geq 1$.

*Proof.* For $n = 1$, Theorem 2.1 implies that $G \oplus K_2$ has exactly two components. Furthermore, by using the definition of tensor product, we see that each component of $G \oplus K_2$ is isomorphic to $G$. Therefore, $G \oplus K_2 = 2G$. Moreover, corresponding to $n \geq 1$ copies of $K_2$, $G \oplus nK_2$ certainly contains exactly $2n$ copies of $G$. Thus, $G \oplus nK_2 = 2nG$. □

**Theorem 2.3.** For any connected graph $G$, $G \oplus nK_2$ for $n \geq 1$, is bipartite.

*Proof.* We discuss two cases depending on $G$.

**Case 1.** Suppose $G$ is bipartite. By Theorem 2.2, $G \oplus nK_2 = 2nG$. Since $G$ is bipartite, it follows immediately that $G \oplus nK_2$ is bipartite.

**Case 2.** Suppose $G$ is non-bipartite. Certainly, $G$ contains a cycle $C_m$ for odd $m \geq 3$. Corresponding to each copy of $K_2$ in $G \oplus nK_2$, there are exactly $n$ distinct subgraphs in $G \oplus nK_2$, each isomorphic to $C_m \oplus K_2$. It is shown in [2] that $C_m \oplus K_2$ is isomorphic to $C_{2m}$. For even $m \geq 4$, it is also shown in [2] that $C_m \oplus K_2 = C_m \cup C_m$. This proves that $G \oplus nK_2$ has no odd cycles. Hence, $G \oplus nK_2$ is bipartite. □

**Theorem 2.4.** Let $G$ be a connected, bipartite graph and let $H = \oplus_{i=1}^n K_2$. Then $G \oplus H = 2^nG$ for $n \geq 1$.

*Proof.* We proceed by induction on $n$. If $n = 1$, then by Theorem 2.2, $G \oplus H = 2G$. Assume the result holds with at most $n-1$. Consider $G \oplus H = G \oplus [\oplus_{i=1}^n K_2]$
= \bigoplus [\bigoplus_{i=1}^{n-1} K_2 \oplus K_2] \oplus K_2 \]. By induction hypothesis, we have \( G \oplus \bigoplus_{i=1}^{n-1} K_2 \) = \( 2^{n-1} G \). Hence,
\[
G \oplus \bigoplus_{i=1}^{n} K_2 = 2^n G. \quad (2.1)
\]

In view of Theorem 2.2 (with \( n = 1 \)), \( G \oplus K_2 = 2G \). Using this in (2.1), we get \( G \oplus H = 2^n G \).

\[\square\]

3. Characterization of Connected Tensor Product of Graphs

Now, we obtain a characterization of connected tensor product of arbitrarily many graphs. We see that Weichsel [6] studied the connectedness of the tensor product of two graphs as follows:

**Theorem 3.1.** Let \( G \) and \( H \) be connected graphs. Then \( G \oplus H \) is connected if and only if either \( G \) or \( H \) contains an odd cycle.

Now, we present the natural finite extension of Weichsel’s Theorem as follows:

**Theorem 3.2.** Let \( G_k \) \((1 \leq k \leq n; n \geq 2)\) be connected graph, and let \( G = \bigoplus_{k=1}^{n} G_k \). Then \( G \) is connected if and only if at most one of \( G_k \)'s is bipartite.

**Proof.** Assume that \( G \) is connected. We prove by contradiction. If possible, assume that there are at least two distinct graphs \( G_i \) and \( G_j \) \((1 \leq i, j \leq n)\), which are bipartite. By Theorem 2.1, \( G_i \oplus G_j \) contains exactly two components say, \( F \) and \( H \). Now, we have \( G = \bigoplus_{k=1}^{n} G_k = (F \oplus M) \cup (H \oplus M) \), where \( M = \bigoplus_{k=1}^{n} G_k \) \((k \neq i, j)\). This shows that \( G \) is certainly disconnected, and hence we immediately arrive at a contradiction. Thus, it proves that at most one of \( G_k \)'s is bipartite.

Conversely, assume that at most one of \( G_k \)'s is bipartite.

We discuss two cases.

**Case 1.** None of \( G_k \)'s is bipartite. Immediately, it follows that each \( G_k \) contains an odd cycle.

**Case 2.** Exactly one of \( G_k \)'s is bipartite. Without loss of generality, we assume that \( G_1 \) is bipartite. The remaining \( G_i \) \((2 \leq i \leq n)\) is non-bipartite, and hence each such \( G_i \) contains an odd cycle.

In either case, by applying Theorem 3.1 and the mathematical induction on the number of factors, the result follows. \[\square\]

4. Characterization of Bipartite Tensor Product of Graphs

Now, we shall obtain a necessary and sufficient condition for the tensor product of two or more graphs to be bipartite, (which is proposed in [3]).

**Theorem 4.1.** Let \( G_1 \) and \( G_2 \) be two connected graphs. Then \( G_1 \oplus G_2 \) is bipartite if and only if at least one of \( G_1 \) and \( G_2 \) is bipartite.
Proof. Suppose $G_1 \oplus G_2$ is bipartite. We claim that at least one of $G_1$ and $G_2$ is bipartite. If this is not so, then both $G_1$ and $G_2$ are non-bipartite. Consequently, there exist two odd cycles $C_m$ (for $m \geq 3$) and $C_n$ (for $n \geq 3$) in $G_1$ and $G_2$, respectively. Without loss of generality, we consider $m \leq n$. Let $C_m : u_1, u_2, \ldots, u_m, u_1$ and let $C_n : v_1, v_2, \ldots, v_m, v_{m+1}, \ldots, v_n, v_1$. Then $C_m \oplus C_n$ contains the cycle $Z$ of length $n$ as follows:

$$Z : (u_1, v_1), (u_2, v_2), \ldots, (u_m, v_m), (u_{m-1}, v_{m+1}), (u_m, v_{m+2}), (u_{m-1}, v_{m+3}), (u_m, v_{m+4}), \ldots, (u_{m-1}, v_{n-1}), (u_m, v_n), (u_1, v_1).$$

So, $G_1 \oplus G_2$ contains the odd cycle $Z$. Hence, $G_1 \oplus G_2$ is non-bipartite. This is a contradiction.

Conversely, assume that at least one of $G_1$ and $G_2$ is bipartite. We discuss two cases.

**Case 1.** Suppose both $G_1$ and $G_2$ are bipartite. Then by Theorem 2.1, $G_1 \oplus G_2$ contains exactly two components, say $H_1$ and $H_2$. Now, we claim that both $H_1$ and $H_2$ are bipartite. On contrary, if one of $H_i$’s is non-bipartite. Without loss of generality, we assume that $H_1$ is non-bipartite. Then $H_1$ contains an odd cycle, say $C : (u_1, v_1), (u_2, v_2), \ldots, (u_n, v_n), (u_1, v_1)$ in $G_1 \oplus G_2$, where $u_i \in V(G_1)$ ($1 \leq i \leq n$), $v_j \in V(G_2)$ ($1 \leq j \leq n$). Certainly, the first co-ordinate vertices $u_1, u_2, \ldots, u_n, u_1$ of the cycle $C$ forms a closed odd walk $W$ in $G_1$. Since every closed odd walk in a graph contains an odd cycle, it follows that the walk $W$ contains an odd cycle in $G_1$. This shows that $G_1$ is not bipartite. But this contradicts the hypothesis that $G_1$ is bipartite. Since each $H_i$ is bipartite, it follows that $G_1 \oplus G_2 = H_1 \cup H_2$ is also bipartite.

**Case 2.** Suppose one of $G_1$ and $G_2$ is bipartite, and the other is non-bipartite. Assume that $G_1$ is bipartite. Since $G_2$ is non-bipartite, $G_2$ contains an odd cycle. From Theorem 3.1, $G_1 \oplus G_2$ is connected. Next, we claim that $G_1 \oplus G_2$ is bipartite. If this is not so, then $G_1 \oplus G_2$ is non-bipartite, and hence it contains an odd cycle $Z$. By repeating the same argument as in Case 1, we obtain $G_1 \oplus G_2$ is bipartite.

In either case, we see that $G_1 \oplus G_2$ is bipartite. \hfill \Box

Next, we obtain the finite extension of the above theorem, and its proof follows by the mathematical induction on the number of factors.

**Corollary 4.2.** Let $G_k$ ($1 \leq k \leq n : n \geq 2$) be a connected graph, and let $G = \bigoplus_{k=1}^{n} G_k$. Then $G$ is bipartite if and only if at least one of $G_k$’s is bipartite.

5. **Characterization of Eulerian Tensor Product of Graphs**

An *Euler tour* of a graph $G$ is a closed walk in $G$ that traverses each edge of $G$ exactly once. A graph is eulerian if it contains an Euler tour. It is well-known that a connected graph $G$ is eulerian if and only if every vertex in $G$ has an even degree. For any vertex $(u, v)$ in a tensor product $(G \oplus H)$ of two graphs
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G and H, \( \text{deg}(u, v) = \text{deg}(u) \cdot \text{deg}(v) \). Now, we present a characterization of eulerian tensor product of two graphs.

**Theorem 5.1.** Let G and H be connected graphs such that at most one of them is bipartite. Then \( G \oplus H \) is eulerian if and only if at least one of G and H is eulerian.

**Proof.** Suppose G is eulerian, and it contains an odd cycle. Then by Theorem 4.1, \( G \oplus H \) is connected. Since G is eulerian, \( \text{deg}(u) \) is even for all vertices u in G. Consequently, for any vertex v of H, the pair \((u, v)\) is a vertex in \( G \oplus H \), and \( \text{deg}(u, v) = \text{deg}(u) \cdot \text{deg}(v) \), which is an even degree, because \( \text{deg}(u) \) is even, and \( \text{deg}(v) \geq 1 \). This implies that \( G \oplus H \) is eulerian.

Conversely, assume that \( G \oplus H \) is eulerian. By definition, \( G \oplus H \) is certainly connected. Again by Theorem 4.1, one of G and H contains an odd cycle. To complete the proof, we claim that at least one of G and H is eulerian. On contrary, if possible assume that both G and H are not eulerian graphs. Immediately, there exist at least two odd degree vertices x and y in G and H, respectively. Thus, \((x, y)\) is a vertex in \( G \oplus H \), and also \( \text{deg}(x, y) = \text{deg}(x) \cdot \text{deg}(y) \), which is odd, because both \( \text{deg}(x) \) and \( \text{deg}(y) \) are odd. This shows that \( G \oplus H \) is not eulerian, and it contradicts the hypothesis that \( G \oplus H \) is eulerian. \( \square \)

The finite extension of Theorem 5.1 is the following result, and its proof directly follows by the induction on the number of factors.

**Corollary 5.2.** Let \( G_k \) \((1 \leq k \leq n; \ n \geq 2)\) be a connected graph such that at most one of \( G_k \)'s is bipartite, and let \( G = \bigoplus_{k=1}^{n} G_k \). Then G is eulerian if and only if at least one of \( G_k \)'s is eulerian.

6. **Characterization of Unicyclic Duplicate Graph**

A unicyclic graph is a connected graph which contains exactly one cycle. Next, we obtain a characterization of unicyclic duplicate graph \( G \oplus K_2 \).

**Theorem 6.1.** A non-bipartite graph G is unicyclic if and only if the duplicate graph \( G \oplus K_2 \) is unicyclic.

**Proof.** Suppose a non-bipartite graph G is unicyclic. Then G contains exactly one odd cycle C. Hence by Theorem 4.1, \( G \oplus K_2 \) is connected. Let \( C : u_1, u_2, \ldots, u_{2k+1}, u_1 \) for \( k \geq 1 \). Next, we show that \( G \oplus K_2 \) is unicyclic. For this, let us consider \( V(K_2) = \{v_1, v_2\} \). It is easy to see that the subgraph induced by \( C \oplus K_2 \) in \( G \oplus K_2 \) is certainly isomorphic to an even cycle \( C_{2(2k+1)} \), where \( \langle u_1, v_1 \rangle, \langle u_2, v_2 \rangle, \langle u_3, v_1 \rangle, \ldots, \langle u_{2k-1}, v_1 \rangle, \langle u_{2k}, v_2 \rangle, \langle u_{2k+1}, v_1 \rangle, \langle u_{2k}, v_2 \rangle, \langle u_{2k+1}, v_1 \rangle, \langle u_1, v_2 \rangle, \langle u_2, v_1 \rangle, \langle u_3, v_2 \rangle, \ldots, \langle u_{2k-1}, v_2 \rangle, \langle u_{2k}, v_1 \rangle, \langle u_{2k+1}, v_2 \rangle, \langle u_1, v_1 \rangle \). Since G is unicyclic, it follows that \( G \oplus K_2 \) has no cycles other than \( C_{2(2k+1)} \). If this is not so, then there exists another cycle J in \( G \oplus K_2 \), which is different from \( C_{2(2k+1)} \). Consequently, the first co-ordinates
of the vertices of the cycle $J$, which are in pairs, will form another cycle $C'$ in $G$. Since $J \neq C^2(2k+1)$ in $G \oplus K_2$, it follows $C \neq C'$ in $G$. This is a contradiction to the fact that $G$ is unicyclic. Therefore, $G \oplus K_2$ is unicyclic.

Conversely, suppose that $G \oplus K_2$ is unicyclic. Let $Z$ be the only cycle in $G \oplus K_2$. By Theorem 2.3 (with $n = 1$), $G \oplus K_2$ is bipartite. Hence, $Z$ is a unique even cycle. Clearly, we notice that the first co-ordinate vertices of $G$ in $Z$ forms an odd cycle $C$ in $G$. Since $G \oplus K_2$ is unicyclic, it follows that $C$ is the unique cycle in $G$. Moreover, since $G \oplus K_2$ is connected, it implies that $G$ is connected. Therefore, $G$ is unicyclic. □

7. The Girth and Triangles in Tensor Product Graphs

The girth of a graph $G$, denoted by $g(G)$, is the length of a shortest cycle in $G$, if any. Otherwise, it is undefined if $G$ is a forest. It is clear that the girth of a graph $G$ is the minimum of the girths of its components. Firstly, we determine the girth of the generalized duplicate graphs.

**Theorem 7.1.** Let $G$ be a connected graph with $g(G) = k$. For any positive integer $n \geq 1$, we have

$$g(G \oplus n \, K_2) = g(G \oplus [ \oplus_{i=1}^{n} \, K_2]) = \begin{cases} 
  k & \text{if } G \text{ is bipartite}, \\
  \min\{2p, q\} & \text{otherwise},
\end{cases}$$

where $C_p$ and $C_q$ are the minimal odd and even cycles in a non-bipartite graph $G$, respectively.

**Proof.** First, we discuss the result when $n = 1$.

**Case 1.** Assume $G$ is bipartite. From Theorem 2.2 (with $n = 1$), we have for the duplicate graph $G \oplus K_2 = 2G$. Consequently, $g(G \oplus K_2) = k$.

**Case 2.** Suppose $G$ is not bipartite. Then $G$ contains an odd cycle. Let $C_p$ for $p \geq 3$, be a minimal odd cycle in $G$.

Now, there are two possibilities to discuss:

**2.1.** If $G$ is free-from even cycles, then $C_p \oplus K_2$ contains an even cycle $C_{2p}$ in $G \oplus K_2$.

**2.2.** If $G$ contains a minimal even cycle $C_q$, $q \geq 4$, then $C_q \oplus K_2 = 2C_q$ appears in $G \oplus K_2$.

From the above possibilities, it follows that $g(G \oplus K_2)$ is the minimum of $2p$ and $q$. Thus, $g(G \oplus K_2) = \min\{2p, q\}$.

Finally, consider the result when $n \geq 2$. The result follows immediately if we proceed as above by applying Theorem 2.2 or 2.4 repeatedly. □

Next, we derive a formula (which is proposed in [4]) for computing the number of triangles in the tensor product of two graphs. For this, firstly we establish the following lemma.
Lemma 7.2. Let $G_k$ ($1 \leq k \leq n ; n \geq 2$) be a connected graph. Then the product $\otimes_{k=1}^{n}G_k$ contains a triangle if and only if each $G_k$ contains a triangle.

Proof. Now, we discuss the case when $n = 2$. Suppose $G_1 \oplus G_2$ contains a triangle $T$, and let $(a_1,b_1), (a_2,b_2)$ and $(a_3,b_3)$ be any three vertices of $T$. By definition, $(a_1,b_1)(a_2,b_2), (a_2,b_2)(a_3,b_3)$ and $(a_3,b_3)(a_1,b_1)$ are the edges of $T$ in $G_1 \oplus G_2$ if and only if the edges: $a_1a_2, a_2a_3$ and $a_3a_1$ constitute a triangle $T_1$ in $G_1$ and also the edges : $b_1b_2, b_2b_3$ and $b_3b_1$ constitute a triangle $T_2$ in $G_2$. But this is so if and only if both $G_1$ and $G_2$ have triangles $T_1$ and $T_2$, respectively. Finally, we discuss the case when $n \geq 3$. The result follows immediately if we proceed by applying induction on the number of factors. □

Theorem 7.3. Let $G_i$ ($1 \leq i \leq 2$) be a connected graph having the number of triangles $n_i$. Then the product $G_1 \oplus G_2$ contains $6n_1n_2$ triangles.

Proof. First, let us compute the actual number of triangles in the product $T_1 \oplus T_2$, when $T_i$ is any triangle in $G_i$ (for $i = 1, 2$). It is easy to see that there are exactly 6 distinct triangles in $T_1 \oplus T_2$. But each $G_i$ contains $n_i$ triangles. Consequently, the product $G_1 \oplus G_2$ contains $6n_1n_2$ triangles, and there are no more triangles because of Lemma 7.2. □

The immediate consequence of the above theorem is the following corollary.

Corollary 7.4. Let $G_k$ ($1 \leq k \leq n ; n \geq 2$) be a connected graph having the number of triangles $n_k$. Then the product $\otimes_{k=1}^{n}G_k$ contains $6^{n-1}(\Pi_{k=1}^{n}n_k)$ triangles.

Corollary 7.5. The number of triangles in $K_m \oplus K_n$ is $\frac{1}{6}|mn(m-1)(n-1)(m-2)(n-2)|$.

Proof. We know that the number of triangles in $K_p = pC_3$. Therefore from Theorem 7.3, the number of triangles in $K_m \oplus K_n$ is $6(mC_3)(nC_3) = \frac{1}{6}|mn(m-1)(n-1)(m-2)(n-2)|$. □

Finally to determine the girth of the tensor product of graphs, we need to establish the following lemma.

Lemma 7.6. Let $G_k$ ($1 \leq k \leq n ; n \geq 2$) be a connected, triangle-free graph such that each $G_k$ contains an induced subgraph isomorphic to $P_3$. Then $g(\otimes_{k=1}^{n}G_k) = 4$.

Proof. We discuss the case when $n = 2$. Let $a_i$ ($1 \leq i \leq 3$) and $b_i$ ($1 \leq i \leq 3$) be the vertices of a subgraph isomorphic to $P_3$ in $G_1$ and $G_2$, respectively. Then the subgraph $\{a_1,a_2,a_3\} \oplus \{b_1,b_2,b_3\}$ is isomorphic to $P_3 \oplus P_3$ in $G_1 \oplus G_2$. It is easy to see that $P_3 \oplus P_3 = K_{1,4} \cup C_4$. Immediately, a 4-cycle $C_4$ appears as a subgraph in $G_1 \oplus G_2$. However from Lemma 7.2, there is no triangle in $G_1 \oplus G_2$. Consequently, $C_4$ is the smallest cycle in $G_1 \oplus G_2$. 

Therefore, \( g(G_1 \oplus G_2) = 4 \).
When \( n \geq 3 \), the result follows easily if we proceed by induction on the number of factors. \( \square \)

The following result gives the girth of tensor product of arbitrarily many graphs.

**Theorem 7.7.** Let \( G_k \) (1 ≤ \( k \) ≤ \( n \); \( n \geq 2 \)) be a connected graph of order \( \geq 3 \). Then \( g(\oplus_{k=1}^{n} G_k) \) is either 3 or 4.

**Proof.** We discuss three cases when \( n = 2 \).

**Case 1.** Suppose both \( G_1 \) and \( G_2 \) have triangles. By Lemma 7.2, \( G_1 \oplus G_2 \) contains a triangle. Hence, \( g(G_1 \oplus G_2) = 3 \).

**Case 2.** Suppose one of \( G_1 \) and \( G_2 \) is triangle-free. Without loss of generality, we assume that \( G_1 \) contains a triangle, and \( G_2 \) has an induced subgraph isomorphic to \( P_3 \). It is easy to check that \( K_3 \oplus P_3 \) contains a 4-cycle \( C_4 \). Consequently, this \( C_4 \) appears in \( (G_1 \oplus G_2) \). However again by Lemma 7.2, \( G_1 \oplus G_2 \) is triangle-free. This implies that \( C_4 \) is the smallest cycle in \( G_1 \oplus G_2 \). Therefore, \( g(G_1 \oplus G_2) = 4 \).

**Case 3.** Suppose \( G_1 \) and \( G_2 \) are triangle-free. Then each \( G_1 \) and \( G_2 \) contains an induced subgraph isomorphic to \( P_3 \). From Lemma 7.6, \( g(G_1 \oplus G_2) = 4 \).

From the above cases, it follows that \( g(G_1 \oplus G_2) = 3 \) or 4.

When \( n \geq 3 \), it is not difficult to prove the result if we proceed by induction on the number of factors. \( \square \)

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