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# On Tensor Product of Graphs, Girth and Triangles

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Abstract. The purpose of this paper is to obtain a necessary and sufficient condition for the tensor product of two or more graphs to be connected, bipartite or eulerian. Also, we present a characterization of the duplicate graph $G \oplus K_2$ to be unicyclic. Finally, the girth and the formula for computing the number of triangles in the tensor product of graphs are worked out.

**Keywords:** Tensor product, Bipartite graph, Connected graph, Eulerian graph, Girth, Cycle, Path.

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## 1. Introduction

We shall consider only finite, undirected graphs without loops or multiple edges. We follow the terminology of [1]. For a graph $G$, $V(G)$ and $E(G)$ denote the vertex set and edge set of $G$, respectively. For a connected graph $G$, $nG$ is the graph with $n$ components, each being isomorphic to $G$. It is well-known that a graph is *bipartite* if and only if it contains no odd cycle. We now define the tensor product of two graphs [8] as follows: The *tensor product* of two graphs $G_1$ and $G_2$ is the graph, denoted by $G_1 \oplus G_2$, with vertex set $V(G_1 \oplus G_2) = V(G_1) \times V(G_2)$, and any two of its vertices $(u_1, v_1)$ and $(u_2, v_2)$ are adjacent, whenever $u_1$ is adjacent to $u_2$ in $G_1$ and $v_1$ is adjacent to $v_2$ in $G_2$.

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The graphs $G_1$ and $G_2$ are called *factors* of the product $G_1 \oplus G_2$. Other popular names for tensor product that have appeared in the literature are *Kronecker product, cross product, direct product, conjenction product.* Sampathkumar [6] defines the tensor product of a graph $G$ by $K_2$ as the *duplicate graph* of $G$, and studied its properties and a characterization in great detail. This product is also studied in [5]. Now, we define the two special type of tensor products: $G \oplus nK_2$ and $G \oplus [\oplus_{i=1}^{n} K_2]$ as the *generalized duplicate graphs of a graph* $G$, for any integer $n \geq 2$, and study their structural properties for our later use.

## 2. Structural Properties of the Generalized Duplicate Graphs

The following theorem of Weichsel [8] will be useful in the proof of our results.

**Theorem 2.1.** *If the connected graphs $G$ and $H$ are bipartite, then $G \oplus H$ has exactly two components.*

Next, we present some elementary results of the generalized duplicate graphs for our immediate use.

**Theorem 2.2.** *For any connected, bipartite graph $G$, $G \oplus nK_2 = 2nG$ for $n \geq 1$.*

*Proof.* For $n = 1$, Theorem 2.1 implies that $G \oplus K_2$ has exactly two components. Furthermore, by using the definition of tensor product, we see that each component of $G \oplus K_2$ is isomorphic to $G$. Therefore, $G \oplus K_2 = 2G$. Moreover, corresponding to $n \geq 1$ copies of $K_2$, $G \oplus nK_2$ certainly contains exactly $2n$ copies of $G$. Thus, $G \oplus nK_2 = 2nG$. □

**Theorem 2.3.** *For any connected graph $G$, $G \oplus nK_2$ for $n \geq 1$, is bipartite.*

*Proof.* We discuss two cases depending on $G$.
**Case 1.** Suppose $G$ is bipartite. By Theorem 2.2, $G \oplus nK_2 = 2nG$. Since $G$ is bipartite, it follows immediately that $G \oplus nK_2$ is bipartite.
**Case 2.** Suppose $G$ is non-bipartite. Certainly, $G$ contains a cycle $C_m$ for odd $m \geq 3$. Corresponding to each copy of $K_2$ in $G \oplus nK_2$, there are exactly $n$ distinct subgraphs in $G \oplus nK_2$, each is isomorphic to $C_m \oplus K_2$. It is shown in [2] that $C_m \oplus K_2$ is isomorphic to $C_{2m}$. For even $m \geq 4$, it is also shown in [2] that $C_m \oplus K_2 = C_m \cup C_m$. This proves that $G \oplus nK_2$ has no odd cycles. Hence, $G \oplus nK_2$ is bipartite. □

**Theorem 2.4.** *Let $G$ be a connected, bipartite graph and let $H = \oplus_{i=1}^{n} K_2$. Then $G \oplus H = 2^n G$ for $n \geq 1$.*

*Proof.* We proceed by induction on $n$. If $n = 1$, then by Theorem 2.2, $G \oplus H = 2G$. Assume the result holds with at most $n-1$. Consider $G \oplus H = G \oplus [\oplus_{i=1}^{n} K_2]$

$= G\oplus\ [\oplus_{i=1}^{n-1} K_2 \oplus\ K_2] \ = \ [G\oplus\ (\oplus_{i=1}^{n-1} K_2)] \oplus\ K_2$. By induction hypothesis, we have $G\oplus[\oplus_{i=1}^{n-1} K_2] = 2^{n-1}G$. Hence,

$$G\oplus[\oplus_{i=1}^{n} K_2] \ = \ 2^{n-1}G\oplus K_2.\cdots\cdots(2.1)$$

In view of Theorem 2.2 (with $n = 1$), $G\oplus K_2 = 2G$. Using this in (2.1), we get $G\oplus H = 2^n G$. □

## 3. Characterization of Connected Tensor Product of Graphs

Now, we obtain a characterization of connected tensor product of arbitrarily many graphs. We see that Weichsel [6] studied the connectedness of the tensor product of two graphs as follows:

**Theorem 3.1.** *Let $G$ and $H$ be connected graphs. Then $G\oplus H$ is connected if and only if either $G$ or $H$ contains an odd cycle.*

Now, we present the natural finite extension of Weichsel's Theorem as follows:

**Theorem 3.2.** *Let $G_k$ $(1 \le k \le n\ ;\ n \ge 2)$ be connected graph, and let $G = \oplus_{k=1}^{n} G_k$. Then $G$ is connected if and only if at most one of $G_k$'s is bipartite.*

*Proof.* Assume that $G$ is connected. We prove by contradiction. If possible, assume that there are at least two distinct graphs $G_i$ and $G_j$ $(1 \le i, j \le n)$, which are bipartite. By Theorem 2.1, $G_i\oplus G_j$ contains exactly two components say, $F$ and $H$. Now, we have $G = \oplus_{k=1}^{n} G_k = (F\oplus M)\ \cup\ (H\oplus M)$ , where $M = \oplus_{k=1}^{n} G_k$ $(k \ne i, j)$. This shows that $G$ is certainly disconnected, and hence we immediately arrive at a contradiction. Thus, it proves that at most one of $G_k$'s is bipartite.

Conversely, assume that at most one of $G_k$'s is bipartite.

We discuss two cases.

**Case 1.** None of $G_k$'s is bipartite. Immediately, it follows that each $G_k$ contains an odd cycle.

**Case 2.** Exactly one of $G_k$'s is bipartite. Without loss of generality, we assume that $G_1$ is bipartite. The remaining $G_i$ $(2 \le i \le n)$ is non-bipartite, and hence each such $G_i$ contains an odd cycle.

In either case, by applying Theorem 3.1 and the mathematical induction on the number of factors, the result follows. □

## 4. Characterization of Bipartite Tensor Product of Graphs

Now, we shall obtain a necessary and sufficient condition for the tensor product of two or more graphs to be bipartite, (which is proposed in [3]).

**Theorem 4.1.** *Let $G_1$ and $G_2$ be two connected graphs. Then $G_1\oplus G_2$ is bipartite if and only if at least one of $G_1$ and $G_2$ is bipartite.*

*Proof.* Suppose $G_1 \oplus G_2$ is bipartite. We claim that at least one of $G_1$ and $G_2$ is bipartite. If this is not so, then both $G_1$ and $G_2$ are non-bipartite. Consequently, there exist two odd cycles $C_m$ (for $m \geq 3$) and $C_n$ (for $n \geq 3$) in $G_1$ and $G_2$, respectively. Without loss of generality, we consider $m \leq n$. Let $C_m \ : \ u_1, u_2, \ldots, u_m, u_1$ and let $C_n \ : \ v_1, v_2, \ldots, v_m, v_{m+1}, \ldots, v_n, v_1$. Then $C_m \oplus C_n$ contains the cycle $Z$ of length $n$ as follows:
$Z \ : \ (u_1, v_1), (u_2, v_2), \ldots, (u_m, v_m), (u_{m-1}, v_{m+1}), (u_m, v_{m+2}), (u_{m-1}, v_{m+3}),$
$(u_m, v_{m+4}), \ldots, (u_{m-1}, v_{n-1}), (u_m, v_n), (u_1, v_1)$.
So, $G_1 \oplus G_2$ contains the odd cycle $Z$. Hence, $G_1 \oplus G_2$ is non-bipartite. This is a contradiction.
Conversely, assume that at least one of $G_1$ and $G_2$ is bipartite.
We discuss two cases.
**Case 1.** Suppose both $G_1$ and $G_2$ are bipartite. Then by Theorem 2.1, $G_1 \oplus G_2$ contains exactly two components, say $H_1$ and $H_2$. Now, we claim that both $H_1$ and $H_2$ are bipartite. On contrary, if one of $H_i's$ is non-bipartite. Without loss of generality, we assume that $H_1$ is non-bipartite. Then $H_1$ contains an odd cycle, say
$C : (u_1, v_1), (u_2, v_2), \ldots, (u_n, v_n), (u_1, v_1)$ in $G_1 \oplus G_2$, where $u_i \in V(G_1)$ $(1 \leq i \leq n)$, $v_j \in V(G_2)$ $(1 \leq j \leq n)$ . Certainly, the first co-ordinate vertices $u_1, u_2, \ldots, u_n, u_1$ of the cycle $C$ forms a closed odd walk $W$ in $G_1$. Since every closed odd walk in a graph contains an odd cycle, it follows that the walk $W$ contains an odd cycle in $G_1$. This shows that $G_1$ is not bipartite. But this contradicts the hypothesis that $G_1$ is bipartite. Since each $H_i$ is bipartite, it follows that $G_1 \oplus G_2 = H_1 \cup H_2$ is also bipartite.
**Case 2.** Suppose one of $G_1$ and $G_2$ is bipartite, and the other is non-bipartite. Assume that $G_1$ is bipartite. Since $G_2$ is non-bipartite, $G_2$ contains an odd cycle. From Theorem 3.1, $G_1 \oplus G_2$ is connected. Next, we claim that $G_1 \oplus G_2$ is bipartite. If this is not so, then $G_1 \oplus G_2$ is non-bipartite, and hence it contains an odd cycle $Z$. By repeating the same argument as in Case 1, we obtain $G_1 \oplus G_2$ is bipartite.
In either case, we see that $G_1 \oplus G_2$ is bipartite. □

Next, we obtain the finite extension of the above theorem, and its proof follows by the mathematical induction on the number of factors.

**Corollary 4.2.** *Let $G_k$ $(1 \leq k \leq n \ ; \ n \geq 2)$ be a connected graph, and let $G = \oplus_{k=1}^{n} G_k$. Then $G$ is bipartite if and only if at least one of $G_k$'s is bipartite.*

## 5. Characterization of Eulerian Tensor Product of Graphs

An *Euler tour* of a graph $G$ is a closed walk in $G$ that traverses each edge of $G$ exactly once. A graph is eulerian if it contains an Euler tour. It is well-known that a connected graph $G$ is eulerian if and only if every vertex in $G$ has an even degree. For any vertex $(u, v)$ in a tensor product $(G \oplus H)$ of two graphs

$G$ and $H$, $deg(u,v) = deg(u) \bullet deg(v)$. Now, we present a characterization of eulerian tensor product of two graphs.

**Theorem 5.1.** *Let $G$ and $H$ be connected graphs such that at most one of them is bipartite. Then $G \oplus H$ is eulerian if and only if at least one of $G$ and $H$ is eulerian.*

*Proof.* Suppose $G$ is eulerian, and it contains an odd cycle. Then by Theorem 4.1, $G \oplus H$ is connected. Since $G$ is eulerian, $deg(u)$ is even for all vertices $u$ in $G$. Consequently, for any vertex $v$ of $H$, the pair $(u,v)$ is a vertex in $G \oplus H$, and $deg(u,v) = deg(u) \bullet deg(v)$, which is an even degree, because $deg(u)$ is even, and $deg(v) \geq 1$ . This implies that $G \oplus H$ is eulerian.
Conversely, assume that $G \oplus H$ is eulerian. By definition, $G \oplus H$ is certainly connected. Again by Theorem 4.1, one of $G$ and $H$ contains an odd cycle. To complete the proof, we claim that at least one of $G$ and $H$ is eulerian. On contrary, if possible assume that both $G$ and $H$ are not eulerian graphs. Immediately, there exist at least two odd degree vertices $x$ and $y$ in $G$ and $H$, respectively. Thus, $(x,y)$ is a vertex in $G \oplus H$, and also $deg(x,y) = deg(x) \bullet deg(y)$, which is odd, because both $deg(x)$ and $deg(y)$ are odd. This shows that $G \oplus H$ is not eulerian, and it contradicts the hypothesis that $G \oplus H$ is eulerian. ☐

The finite extension of Theorem 5.1 is the following result, and its proof directly follows by the induction on the number of factors.

**Corollary 5.2.** *Let $G_k$ $(1 \leq k \leq n \ ; \ n \geq 2)$ be a connected graph such that at most one of $G_k$'s is bipartite, and let $G = \oplus_{k=1}^{n} G_k$. Then $G$ is eulerian if and only if at least one of $G_k$'s is eulerian.*

## 6. Characterization of Unicyclic Duplicate Graph

A *unicyclic graph* is a connected graph which contains exactly one cycle. Next, we obtain a characterization of unicyclic duplicate graph $G \oplus K_2$.

**Theorem 6.1.** *A non-bipartite graph $G$ is unicyclic if and only if the duplicate graph $G \oplus K_2$ is unicyclic.*

*Proof.* Suppose a non-bipartite graph $G$ is unicyclic. Then $G$ contains exactly one odd cycle $C$. Hence by Theorem 4.1, $G \oplus K_2$ is connected. Let $C$ : $u_1, u_2, \ldots, u_{2k+1}, u_1$ for $k \geq 1$. Next, we show that $G \oplus K_2$ is unicyclic. For this, let us consider $V(K_2) = \{v_1, v_2\}$. It is easy to see that the subgraph induced by $C \oplus K_2$ in $G \oplus K_2$ is certainly isomorphic to an even cycle $C_{2(2k+1)}$, where $C_{2(2k+1)} : (u_1, v_1), (u_2, v_2), (u_3, v_1), \ldots, (u_{2k-1}, v_1), (u_{2k}, v_2), (u_{2k+1}, v_1),$ $(u_{2k}, v_2), (u_{2k+1}, v_1), (u_1, v_2), (u_2, v_1), (u_3, v_2), \ldots, (u_{2k-1}, v_2), (u_{2k}, v_1),$ $(u_{2k+1}, v_2), (u_1, v_1)$. Since $G$ is unicyclic, it follows that $G \oplus K_2$ has no cycles other than $C_{2(2k+1)}$. If this is not so, then there exists another cycle $J$ in $G \oplus K_2$, which is different from $C_{2(2k+1)}$. Consequently, the first co-ordinates of the vertices of the cycle $J$, which are in pairs, will form another cycle $C\,'$ in $G$. Since $J \neq C_{2(2k+1)}$ in $G \oplus K_2$, it follows $C \neq C\,'$ in $G$. This is a contradiction to the fact that $G$ is unicyclic. Therefore, $G \oplus K_2$ is unicyclic.

Conversely, suppose that $G \oplus K_2$ is unicyclic. Let $Z$ be the only one cycle in $G \oplus K_2$. By Theorem 2.3 (with $n = 1$), $G \oplus K_2$ is bipartite. Hence, $Z$ is a unique even cycle. Clearly, we notice that the first co-ordinate vertices of $G$ in $Z$ forms an odd cycle $C$ in $G$. Since $G \oplus K_2$ is unicyclic, it follows that $C$ is the unique cycle in $G$. Moreover, since $G \oplus K_2$ is connected, it implies that $G$ is connected. Therefore, $G$ is unicyclic. □

## 7. The Girth and Triangles in Tensor Product Graphs

The *girth* of a graph $G$, denoted by $g(G)$, is the length of a shortest cycle in $G$, if any. Otherwise, it is undefined if $G$ is a forest. It is clear that the girth of a graph $G$ is the minimum of the girths of its components. Firstly, we determine the girth of the generalized duplicate graphs.

**Theorem 7.1.** *Let $G$ be a connected graph with $g(G) = k$. For any positive integer $n \geq 1$, we have*

$$g(G \oplus n\ K_2) \ = g(G \oplus [\ \oplus_{i=1}^{n}\ K_2]) \ = \begin{cases} k & if\ G\ is\ bipartite, \\ min\{2p, q\} & otherwise, \end{cases}$$

*where $C_p$ and $C_q$ are the minimal odd and even cycles in a non-bipartite graph $G$, respectively.*

*Proof.* First, we discuss the result when $n = 1$.

**Case 1.** Assume $G$ is bipartite. From Theorem 2.2 (with $n = 1$), we have for the duplicate graph $G \oplus K_2 = 2G$. Consequently, $g(G \oplus K_2) = k$.

**Case 2.** Suppose $G$ is not bipartite. Then $G$ contains an odd cycle. Let $C_p$ for $p \geq 3$, be a minimal odd cycle in $G$.

Now, there are two possibilities to discuss:

**2.1.** If $G$ is free-from even cycles, then $C_p \oplus K_2$ contains an even cycle $C_{2p}$ in $G \oplus K_2$.

**2.2.** If $G$ contains a minimal even cycle $C_q$, $q \geq 4$, then $C_q \oplus K_2 = 2C_q$ appears in $G \oplus K_2$.

From the above possibilities, it follows that $g(G \oplus K_2)$ is the minimum of $2p$ and $q$. Thus, $g(G \oplus K_2) = min\{2p, q\}$.

Finally, consider the result when $n \geq 2$. The result follows immediately if we proceed as above by applying Theorem 2.2 or 2.4 repeatedly.

□

Next, we derive a formula (which is proposed in [4]) for computing the number of triangles in the tensor product of two graphs. For this, firstly we establish the following lemma.

**Lemma 7.2.** *Let $G_k$ $(1 \leq k \leq n\ ;\ n \geq 2)$ be a connected graph. Then the product $\oplus_{k=1}^{n} G_k$ contains a triangle if and only if each $G_k$ contains a triangle.*

*Proof.* Now, we discuss the case when $n = 2$. Suppose $G_1 \oplus G_2$ contains a triangle $T$, and let $(a_1, b_1), (a_2, b_2)$ and $(a_3, b_3)$ be any three vertices of $T$. By definition, $(a_1, b_1)(a_2, b_2), (a_2, b_2)(a_3, b_3)$ and $(a_3, b_3)(a_1, b_1)$ are the edges of $T$ in $G_1 \oplus G_2$ if and only if the edges: $a_1a_2$, $a_2a_3$ and $a_3a_1$ constitute a triangle $T_1$ in $G_1$ and also the edges : $b_1b_2$, $b_2b_3$ and $b_3b_1$ constitute a triangle $T_2$ in $G_2$. But this is so if and only if both $G_1$ and $G_2$ have triangles $T_1$ and $T_2$, respectively. Finally, we discuss the case when $n \geq 3$. The result follows immediately if we proceed by applying induction on the number of factors. □

**Theorem 7.3.** *Let $G_i$ $(1 \leq i \leq 2)$ be a connected graph having the number of triangles $n_i$. Then the product $G_1 \oplus G_2$ contains $6n_1n_2$ triangles.*

*Proof.* First, let us compute the actual number of triangles in the product $T_1 \oplus T_2$, when $T_i$ is any triangle in $G_i$ (for $i = 1, 2$). It is easy to see that there are exactly 6 distinct triangles in $T_1 \oplus T_2$. But each $G_i$ contains $n_i$ triangles. Consequently, the product $G_1 \oplus G_2$ contains $6n_1n_2$ triangles, and there are no more other triangles because of Lemma 7.2. □

The immediate consequence of the above theorem is the following corollary.

**Corollary 7.4.** *Let $G_k$ $(1 \leq k \leq n\ ;\ n \geq 2)$ be a connected graph having the number of triangles $n_k$. Then the product $\oplus_{k=1}^{n} G_k$ contains $6^{n-1}(\Pi_{k=1}^{n} n_k)$ triangles.*

**Corollary 7.5.** *The number of triangles in $K_m \oplus K_n$ is $\frac{1}{6}[\ mn(m-1)(n-1)(m-2)(n-2)]$.*

*Proof.* We know that the number of triangles in $K_p = pC_3$. Therefore from Theorem 7.3, the number of triangles in $K_m \oplus K_n$ is $6(mC_3)(nC_3) = \frac{1}{6}[\ mn(m-1)(n-1)(m-2)(n-2)]$. □

Finally to determine the girth of the tensor product of graphs, we need to establish the following lemma.

**Lemma 7.6.** *Let $G_k$ $(1 \leq k \leq n\ ;\ n \geq 2)$ be a connected, triangle-free graph such that each $G_k$ contains an induced subgraph isomorphic to $P_3$. Then $g(\oplus_{k=1}^{n}\ G_k) = 4$.*

*Proof.* We discuss the case when $n = 2$. Let $a_i$ $(1 \leq i \leq 3)$ and $b_i$ $(1 \leq i \leq 3)$ be the vertices of a subgraph isomorphic to $P_3$ in $G_1$ and $G_2$, respectively. Then the subgraph $< \{a_1, a_2, a_3\} > \oplus < \{b_1, b_2, b_3\} >$ is isomorphic to $P_3 \oplus P_3$ in $G_1 \oplus G_2$. It is easy to see that $P_3 \oplus P_3 = K_{1,4} \cup C_4$. Immediately, a 4-cycle $C_4$ appears as a subgraph in $G_1 \oplus G_2$. However from Lemma 7.2, there is no triangle in $G_1 \oplus G_2$. Consequently, $C_4$ is the smallest cycle in $G_1 \oplus G_2$.

Therefore, $g(G_1 \oplus G_2) = 4$.
When $n \geq 3$, the result follows easily if we proceed by induction on the number of factors. □

The following result gives the girth of tensor product of arbitrarily many graphs.

**Theorem 7.7.** *Let $G_k$ $(1 \leq k \leq n\ ;\ n \geq 2)$ be a connected graph of order $\geq 3$. Then $g(\oplus_{k=1}^{n} G_k)$ is either 3 or 4.*

*Proof.* We discuss three cases when $n = 2$.
**Case 1.** Suppose both $G_1$ and $G_2$ have triangles. By Lemma 7.2, $G_1 \oplus G_2$ contains a triangle. Hence, $g(G_1 \oplus G_2) = 3$.
**Case 2.** Suppose one of $G_1$ and $G_2$ is triangle-free. Without loss of generality, we assume that $G_1$ contains a triangle, and $G_2$ has an induced subgraph isomorphic to $P_3$. It is easy to check that $K_3 \oplus P_3$ contains a 4-cycle $C_4$. Consequently, this $C_4$ appears in $(G_1 \oplus G_2)$. However again by Lemma 7.2, $G_1 \oplus G_2$ is triangle-free. This implies that $C_4$ is the smallest cycle in $G_1 \oplus G_2$. Therefore, $g(G_1 \oplus G_2) = 4$.
**Case 3.** Suppose $G_1$ and $G_2$ are triangle-free. Then each $G_1$ and $G_2$ contains an induced subgraph isomorphic to $P_3$. From Lemma 7.6, $g(G_1 \oplus G_2) = 4$.
From the above cases, it follows that $g(G_1 \oplus G_2) = 3$ or 4.
When $n \geq 3$, It is not difficult to prove the result if we proceed by induction on the number of factors. □

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