On Tensor Product of Graphs, Girth and Triangles

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Abstract. The purpose of this paper is to obtain a necessary and sufficient condition for the tensor product of two or more graphs to be connected, bipartite or eulerian. Also, we present a characterization of the duplicate graph \( G \oplus K_2 \) to be unicyclic. Finally, the girth and the formula for computing the number of triangles in the tensor product of graphs are worked out.

Keywords: Tensor product, Bipartite graph, Connected graph, Eulerian graph, Girth, Cycle, Path.


1. Introduction

We shall consider only finite, undirected graphs without loops or multiple edges. We follow the terminology of [1]. For a graph \( G \), \( V(G) \) and \( E(G) \) denote the vertex set and edge set of \( G \), respectively. For a connected graph \( G \), \( nG \) is the graph with \( n \) components, each being isomorphic to \( G \). It is well-known that a graph is bipartite if and only if it contains no odd cycle. We now define the tensor product of two graphs [8] as follows: The tensor product of two graphs \( G_1 \) and \( G_2 \) is the graph, denoted by \( G_1 \oplus G_2 \), with vertex set \( V(G_1 \oplus G_2) = V(G_1) \times V(G_2) \), and any two of its vertices \((u_1, v_1)\) and \((u_2, v_2)\) are adjacent, whenever \( u_1 \) is adjacent to \( u_2 \) in \( G_1 \) and \( v_1 \) is adjacent to \( v_2 \) in \( G_2 \).

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The graphs $G_1$ and $G_2$ are called factors of the product $G_1 \oplus G_2$. Other popular names for tensor product that have appeared in the literature are Kronecker product, cross product, direct product, conjunction product. Sampathkumar [6] defines the tensor product of a graph $G$ by $K_2$ as the duplicate graph of $G$, and studied its properties and a characterization in great detail. This product is also studied in [5]. Now, we define the two special type of tensor products: $G \oplus nK_2$ and $G \oplus \left[ \bigoplus_{i=1}^{n} K_2 \right]$ as the generalized duplicate graphs of a graph $G$, for any integer $n \geq 2$, and study their structural properties for our later use.

2. STRUCTURAL PROPERTIES OF THE GENERALIZED DUPLICATE GRAPHS

The following theorem of Weichsel [8] will be useful in the proof of our results.

**Theorem 2.1.** If the connected graphs $G$ and $H$ are bipartite, then $G \oplus H$ has exactly two components.

Next, we present some elementary results of the generalized duplicate graphs for our immediate use.

**Theorem 2.2.** For any connected, bipartite graph $G$, $G \oplus nK_2 = 2nG$ for $n \geq 1$.

**Proof.** For $n = 1$, Theorem 2.1 implies that $G \oplus K_2$ has exactly two components. Furthermore, by using the definition of tensor product, we see that each component of $G \oplus K_2$ is isomorphic to $G$. Therefore, $G \oplus K_2 = 2G$. Moreover, corresponding to $n \geq 1$ copies of $K_2$, $G \oplus nK_2$ certainly contains exactly $2n$ copies of $G$. Thus, $G \oplus nK_2 = 2nG$. □

**Theorem 2.3.** For any connected graph $G$, $G \oplus nK_2$ for $n \geq 1$, is bipartite.

**Proof.** We discuss two cases depending on $G$.

**Case 1.** Suppose $G$ is bipartite. By Theorem 2.2, $G \oplus nK_2 = 2nG$. Since $G$ is bipartite, it follows immediately that $G \oplus nK_2$ is bipartite.

**Case 2.** Suppose $G$ is non-bipartite. Certainly, $G$ contains a cycle $C_m$ for odd $m \geq 3$. Corresponding to each copy of $K_2$ in $G \oplus nK_2$, there are exactly $n$ distinct subgraphs in $G \oplus nK_2$, each isomorphic to $C_m \oplus K_2$. It is shown in [2] that $C_m \oplus K_2$ is isomorphic to $C_{2m}$. For even $m \geq 4$, it is also shown in [2] that $C_m \oplus K_2 = C_m \cup C_m$. This proves that $G \oplus nK_2$ has no odd cycles. Hence, $G \oplus nK_2$ is bipartite. □

**Theorem 2.4.** Let $G$ be a connected, bipartite graph and let $H = \bigoplus_{i=1}^{n} K_2$. Then $G \oplus H = 2^nG$ for $n \geq 1$.

**Proof.** We proceed by induction on $n$. If $n = 1$, then by Theorem 2.2, $G \oplus H = 2G$. Assume the result holds with at most $n-1$. Consider $G \oplus H = G \oplus \left[ \bigoplus_{i=1}^{n} K_2 \right]$
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\[ G \oplus \left[ \bigoplus_{i=1}^{n-1} K_2 \oplus K_2 \right] = \left[ G \oplus \left( \bigoplus_{i=1}^{n-1} K_2 \right) \right] \oplus K_2. \]

By induction hypothesis, we have \( G \oplus \left[ \bigoplus_{i=1}^{n-1} K_2 \right] = 2^{n-1}G. \) Hence,

\[ G \oplus \left[ \bigoplus_{i=1}^{n} K_2 \right] = 2^{n-1}G \oplus K_2, \cdots \quad (2.1) \]

In view of Theorem 2.2 (with \( n = 1 \)), \( G \oplus K_2 = 2G. \) Using this in (2.1), we get \( G \oplus H = 2^nG. \) □

3. Characterization of Connected Tensor Product of Graphs

Now, we obtain a characterization of connected tensor product of arbitrarily many graphs. We see that Weichsel [6] studied the connectedness of the tensor product of two graphs as follows:

**Theorem 3.1.** Let \( G \) and \( H \) be connected graphs. Then \( G \oplus H \) is connected if and only if either \( G \) or \( H \) contains an odd cycle.

Now, we present the natural finite extension of Weichsel’s Theorem as follows:

**Theorem 3.2.** Let \( G_k \) (\( 1 \leq k \leq n \); \( n \geq 2 \)) be connected graph, and let \( G = \bigoplus_{k=1}^{n} G_k. \) Then \( G \) is connected if and only if at most one of \( G_k \)’s is bipartite.

**Proof.** Assume that \( G \) is connected. We prove by contradiction. If possible, assume that there are at least two distinct graphs \( G_i \) and \( G_j \) (\( 1 \leq i, j \leq n \)), which are bipartite. By Theorem 2.1, \( G_i \oplus G_j \) contains exactly two components say, \( F \) and \( H. \) Now, we have \( G = \bigoplus_{k=1}^{n} G_k = (F \oplus M) \cup (H \oplus M), \) where \( M = \bigoplus_{k=1}^{n} G_k \) (\( k \neq i, j \)). This shows that \( G \) is certainly disconnected, and hence we immediately arrive at a contradiction. Thus, it proves that at most one of \( G_k \)’s is bipartite.

Conversely, assume that at most one of \( G_k \)’s is bipartite. We discuss two cases.

**Case 1.** None of \( G_k \)’s is bipartite. Immediately, it follows that each \( G_k \) contains an odd cycle.

**Case 2.** Exactly one of \( G_k \)’s is bipartite. Without loss of generality, we assume that \( G_1 \) is bipartite. The remaining \( G_i \) (\( 2 \leq i \leq n \)) is non-bipartite, and hence each such \( G_i \) contains an odd cycle.

In either case, by applying Theorem 3.1 and the mathematical induction on the number of factors, the result follows. □

4. Characterization of Bipartite Tensor Product of Graphs

Now, we shall obtain a necessary and sufficient condition for the tensor product of two or more graphs to be bipartite, (which is proposed in [3]).

**Theorem 4.1.** Let \( G_1 \) and \( G_2 \) be two connected graphs. Then \( G_1 \oplus G_2 \) is bipartite if and only if at least one of \( G_1 \) and \( G_2 \) is bipartite.
Proof. Suppose \( G_1 \oplus G_2 \) is bipartite. We claim that at least one of \( G_1 \) and \( G_2 \) is bipartite. If this is not so, then both \( G_1 \) and \( G_2 \) are non-bipartite. Consequently, there exist two odd cycles \( C_m \) (for \( m \geq 3 \)) and \( C_n \) (for \( n \geq 3 \)) in \( G_1 \) and \( G_2 \), respectively. Without loss of generality, we consider \( m \leq n \). Let \( C_m : u_1, u_2, \ldots, u_m, u_1 \) and let \( C_n : v_1, v_2, \ldots, v_m, v_{m+1}, \ldots, v_n, v_1 \). Then \( C_m \oplus C_n \) contains the cycle \( Z \) of length \( n \) as follows:

\[
Z : (u_1, v_1), (u_2, v_2), \ldots, (u_m, v_m), (u_{m-1}, v_{m+1}), (u_m, v_{m+2}), (u_{m-1}, v_{m+3}), (u_m, v_{m+4}), \ldots, (u_{m-1}, v_{n-1}), (u_m, v_n), (u_1, v_1).
\]

So, \( G_1 \oplus G_2 \) contains the odd cycle \( Z \). Hence, \( G_1 \oplus G_2 \) is non-bipartite. This is a contradiction.

Conversely, assume that at least one of \( G_1 \) and \( G_2 \) is bipartite.

We discuss two cases.

**Case 1.** Suppose both \( G_1 \) and \( G_2 \) are bipartite. Then by Theorem 2.1, \( G_1 \oplus G_2 \) contains exactly two components, say \( H_1 \) and \( H_2 \). Now, we claim that both \( H_1 \) and \( H_2 \) are bipartite. On contrary, if one of \( H_i \)'s is non-bipartite. Without loss of generality, we assume that \( H_1 \) is non-bipartite. Then \( H_1 \) contains an odd cycle, say \( C : (u_1, v_1), (u_2, v_2), \ldots, (u_n, v_n), (u_1, v_1) \) in \( G_1 \oplus G_2 \), where \( u_i \in V(G_1) \) (\( 1 \leq i \leq n \)), \( v_j \in V(G_2) \) (\( 1 \leq j \leq n \)). Certainly, the first co-ordinate vertices \( u_1, u_2, \ldots, u_n, u_1 \) of the cycle \( C \) forms a closed odd walk \( W \) in \( G_1 \). Since every closed odd walk in a graph contains an odd cycle, it follows that the walk \( W \) contains an odd cycle in \( G_1 \). This shows that \( G_1 \) is not bipartite. But this contradicts the hypothesis that \( G_1 \) is bipartite. Since each \( H_i \) is bipartite, it follows that \( G_1 \oplus G_2 = H_1 \cup H_2 \) is also bipartite.

**Case 2.** Suppose one of \( G_1 \) and \( G_2 \) is bipartite, and the other is non-bipartite. Assume that \( G_1 \) is bipartite. Since \( G_2 \) is non-bipartite, \( G_2 \) contains an odd cycle. From Theorem 3.1, \( G_1 \oplus G_2 \) is connected. Next, we claim that \( G_1 \oplus G_2 \) is bipartite. If this is not so, then \( G_1 \oplus G_2 \) is non-bipartite, and hence it contains an odd cycle \( Z \). By repeating the same argument as in Case 1, we obtain \( G_1 \oplus G_2 \) is bipartite.

In either case, we see that \( G_1 \oplus G_2 \) is bipartite. \( \square \)

Next, we obtain the finite extension of the above theorem, and its proof follows by the mathematical induction on the number of factors.

**Corollary 4.2.** Let \( G_k \) (\( 1 \leq k \leq n \); \( n \geq 2 \)) be a connected graph, and let \( G = \oplus_{k=1}^{n} G_k \). Then \( G \) is bipartite if and only if at least one of \( G_k \)'s is bipartite.

5. **Characterization of Eulerian Tensor Product of Graphs**

An **Euler tour** of a graph \( G \) is a closed walk in \( G \) that traverses each edge of \( G \) exactly once. A graph is eulerian if it contains an Euler tour. It is well-known that a connected graph \( G \) is eulerian if and only if every vertex in \( G \) has an even degree. For any vertex \((u, v)\) in a tensor product \((G \oplus H)\) of two graphs
G and H, \( \text{deg}(u,v) = \text{deg}(u) \cdot \text{deg}(v) \). Now, we present a characterization of eulerian tensor product of two graphs.

**Theorem 5.1.** Let G and H be connected graphs such that at most one of them is bipartite. Then G \( \oplus \) H is eulerian if and only if at least one of G and H is eulerian.

**Proof.** Suppose G is eulerian, and it contains an odd cycle. Then by Theorem 4.1, G \( \oplus \) H is connected. Since G is eulerian, \( \text{deg}(u) \) is even for all vertices u in G. Consequently, for any vertex v of H, the pair \( (u,v) \) is a vertex in G \( \oplus \) H, and \( \text{deg}(u,v) = \text{deg}(u) \cdot \text{deg}(v) \), which is an even degree, because \( \text{deg}(u) \) is even, and \( \text{deg}(v) \geq 1 \). This implies that G \( \oplus \) H is eulerian.

Conversely, assume that G \( \oplus \) H is eulerian. By definition, G \( \oplus \) H is certainly connected. Again by Theorem 4.1, one of G and H contains an odd cycle. To complete the proof, we claim that at least one of G and H is eulerian. On contrary, if possible assume that both G and H are not eulerian graphs. Immediately, there exist at least two odd degree vertices x and y in G and H, respectively. Thus, \( (x,y) \) is a vertex in G \( \oplus \) H, and also \( \text{deg}(x,y) = \text{deg}(x) \cdot \text{deg}(y) \), which is odd, because both \( \text{deg}(x) \) and \( \text{deg}(y) \) are odd. This shows that G \( \oplus \) H is not eulerian, and it contradicts the hypothesis that G \( \oplus \) H is eulerian. \( \square \)

The finite extension of Theorem 5.1 is the following result, and its proof directly follows by the induction on the number of factors.

**Corollary 5.2.** Let \( G_k \) \( (1 \leq k \leq n ; n \geq 2) \) be a connected graph such that at most one of \( G_k \)'s is bipartite, and let \( G = \oplus_{k=1}^{n} G_k \). Then G is eulerian if and only if at least one of \( G_k \)'s is eulerian.

6. CHARACTERIZATION OF UNICYCLIC DUPLICATE GRAPH

A unicyclic graph is a connected graph which contains exactly one cycle. Next, we obtain a characterization of unicyclic duplicate graph G \( \oplus \) \( K_2 \).

**Theorem 6.1.** A non-bipartite graph G is unicyclic if and only if the duplicate graph G \( \oplus \) \( K_2 \) is unicyclic.

**Proof.** Suppose a non-bipartite graph G is unicyclic. Then G contains exactly one odd cycle C. Hence by Theorem 4.1, G \( \oplus \) \( K_2 \) is connected. Let C : \( u_1, u_2, \ldots, u_{2k+1}, u_1 \) for \( k \geq 1 \). Next, we show that G \( \oplus \) \( K_2 \) is unicyclic. For this, let us consider \( V(K_2) = \{v_1, v_2\} \). It is easy to see that the subgraph induced by C \( \oplus \) \( K_2 \) in G \( \oplus \) \( K_2 \) is certainly isomorphic to an even cycle C\( _{2(2k+1)} \), where C\( _{2(2k+1)} \) : \( (u_1, v_1), (u_2, v_2), (u_3, v_1), \ldots, (u_{2k-1}, v_1), (u_{2k}, v_2), (u_{2k+1}, v_1), (u_{2k}, v_2), (u_{2k+1}, v_1), (u_1, v_2), (u_2, v_1), (u_3, v_2), \ldots, (u_{2k-1}, v_2), (u_{2k}, v_1), (u_{2k+1}, v_2), (u_1, v_1) \). Since G is unicyclic, it follows that G \( \oplus \) \( K_2 \) has no cycles other than C\( _{2(2k+1)} \). If this is not so, then there exists another cycle J in G \( \oplus \) \( K_2 \), which is different from C\( _{2(2k+1)} \). Consequently, the first co-ordinates
of the vertices of the cycle \( J \), which are in pairs, will form another cycle \( C' \) in \( G \). Since \( J \neq C_{2(2k+1)} \) in \( G \oplus K_2 \), it follows \( C \neq C' \) in \( G \). This is a contradiction to the fact that \( G \) is unicyclic. Therefore, \( G \oplus K_2 \) is unicyclic.

Conversely, suppose that \( G \oplus K_2 \) is unicyclic. Let \( Z \) be the only one cycle in \( G \oplus K_2 \). By Theorem 2.3 (with \( n = 1 \)), \( G \oplus K_2 \) is bipartite. Hence, \( Z \) is a unique even cycle. Clearly, we notice that the first co-ordinate vertices of \( G \) in \( Z \) forms an odd cycle \( C \) in \( G \). Since \( G \oplus K_2 \) is unicyclic, it follows that \( C \) is the unique cycle in \( G \). Moreover, since \( G \oplus K_2 \) is connected, it implies that \( G \) is connected. Therefore, \( G \) is unicyclic. □

7. The Girth and Triangles in Tensor Product Graphs

The girth of a graph \( G \), denoted by \( g(G) \), is the length of a shortest cycle in \( G \), if any. Otherwise, it is undefined if \( G \) is a forest. It is clear that the girth of a graph \( G \) is the minimum of the girths of its components. Firstly, we determine the girth of the generalized duplicate graphs.

**Theorem 7.1.** Let \( G \) be a connected graph with \( g(G) = k \). For any positive integer \( n \geq 1 \), we have

\[
g(G \oplus n K_2) = \min\{g(G), g(G \oplus K_2)\} = \begin{cases} k & \text{if } G \text{ is bipartite,} \\ \min\{2p, q\} & \text{otherwise,} \end{cases}
\]

where \( C_p \) and \( C_q \) are the minimal odd and even cycles in a non-bipartite graph \( G \), respectively.

**Proof.** First, we discuss the result when \( n = 1 \).

**Case 1.** Assume \( G \) is bipartite. From Theorem 2.2 (with \( n = 1 \)), we have for the duplicate graph \( G \oplus K_2 = 2G \). Consequently, \( g(G \oplus K_2) = k \).

**Case 2.** Suppose \( G \) is not bipartite. Then \( G \) contains an odd cycle. Let \( C_p \) for \( p \geq 3 \), be a minimal odd cycle in \( G \).

Now, there are two possibilities to discuss:

**2.1.** If \( G \) is free-from even cycles, then \( C_p \oplus K_2 \) contains an even cycle \( C_{2p} \) in \( G \oplus K_2 \).

**2.2.** If \( G \) contains a minimal even cycle \( C_q \), \( q \geq 4 \), then \( C_q \oplus K_2 = 2C_q \) appears in \( G \oplus K_2 \).

From the above possibilities, it follows that \( g(G \oplus K_2) \) is the minimum of \( 2p \) and \( q \). Thus, \( g(G \oplus K_2) = \min\{2p, q\} \).

Finally, consider the result when \( n \geq 2 \). The result follows immediately if we proceed as above by applying Theorem 2.2 or 2.4 repeatedly. □

Next, we derive a formula (which is proposed in [4]) for computing the number of triangles in the tensor product of two graphs. For this, firstly we establish the following lemma.
Lemma 7.2. Let \( G_k \) (\( 1 \leq k \leq n ; n \geq 2 \)) be a connected graph. Then the product \( \oplus^k_{i=1} G_k \) contains a triangle if and only if each \( G_k \) contains a triangle.

Proof. Now, we discuss the case when \( n = 2 \). Suppose \( G_1 \oplus G_2 \) contains a triangle \( T \), and let \((a_1, b_1), (a_2, b_2)\) and \((a_3, b_3)\) be any three vertices of \( T \). By definition, \((a_1, b_1)(a_2, b_2), (a_2, b_2)(a_3, b_3), (a_3, b_3)(a_1, b_1)\) are the edges of \( T \) in \( G_1 \oplus G_2 \) if and only if the edges: \( a_1a_2, a_2a_3 \) and \( a_3a_1 \) constitute a triangle \( T_1 \) in \( G_1 \) and also the edges: \( b_1b_2, b_2b_3 \) and \( b_3b_1 \) constitute a triangle \( T_2 \) in \( G_2 \). But this is so if and only if both \( G_1 \) and \( G_2 \) have triangles \( T_1 \) and \( T_2 \), respectively. Finally, we discuss the case when \( n \geq 3 \). The result follows immediately if we proceed by applying induction on the number of factors. □

Theorem 7.3. Let \( G_i \) (\( 1 \leq i \leq 2 \)) be a connected graph having the number of triangles \( n_i \). Then the product \( G_1 \oplus G_2 \) contains \( 6n_1n_2 \) triangles.

Proof. First, let us compute the actual number of triangles in the product \( T_1 \oplus T_2 \), when \( T_i \) is any triangle in \( G_i \) (for \( i = 1, 2 \)). It is easy to see that there are exactly 6 distinct triangles in \( T_1 \oplus T_2 \). But each \( G_i \) contains \( n_i \) triangles. Consequently, the product \( G_1 \oplus G_2 \) contains \( 6n_1n_2 \) triangles, and there are no more other triangles because of Lemma 7.2. □

The immediate consequence of the above theorem is the following corollary.

Corollary 7.4. Let \( G_k \) (\( 1 \leq k \leq n ; n \geq 2 \)) be a connected graph having the number of triangles \( n_k \). Then the product \( \oplus^k_{i=1} G_k \) contains \( 6^{n-1}(\Pi_k=1 n_k) \) triangles.

Corollary 7.5. The number of triangles in \( K_m \oplus K_n \) is \( \frac{1}{6}mn(m-1)(n-1)(m-2)(n-2) \).

Proof. We know that the number of triangles in \( K_p = pC_3 \). Therefore from Theorem 7.3, the number of triangles in \( K_m \oplus K_n \) is \( 6(mC_3)(nC_3) = \frac{1}{6}mn(m-1)(n-1)(m-2)(n-2) \). □

Finally to determine the girth of the tensor product of graphs, we need to establish the following lemma.

Lemma 7.6. Let \( G_k \) (\( 1 \leq k \leq n ; n \geq 2 \)) be a connected, triangle-free graph such that each \( G_k \) contains an induced subgraph isomorphic to \( P_3 \). Then \( g(\oplus^k_{i=1} G_k) = 4 \).

Proof. We discuss the case when \( n = 2 \). Let \( a_i \) (\( 1 \leq i \leq 3 \)) and \( b_i \) (\( 1 \leq i \leq 3 \)) be the vertices of a subgraph isomorphic to \( P_3 \) in \( G_1 \) and \( G_2 \), respectively. Then the subgraph \(<a_1a_2a_3> \oplus <b_1b_2b_3>\) is isomorphic to \( P_3 \oplus P_3 \) in \( G_1 \oplus G_2 \). It is easy to see that \( P_3 \oplus P_3 = K_{1,4} \cup C_4 \). Immediately, a 4-cycle \( C_4 \) appears as a subgraph in \( G_1 \oplus G_2 \). However from Lemma 7.2, there is no triangle in \( G_1 \oplus G_2 \). Consequently, \( C_4 \) is the smallest cycle in \( G_1 \oplus G_2 \).
Therefore, \( g(G_1 \oplus G_2) = 4 \).
When \( n \geq 3 \), the result follows easily if we proceed by induction on the number of factors.

The following result gives the girth of tensor product of arbitrarily many graphs.

**Theorem 7.7.** Let \( G_k \ (1 \leq k \leq n ; \ n \geq 2) \) be a connected graph of order \( \geq 3 \). Then \( g(\oplus_{k=1}^{n} G_k) \) is either 3 or 4.

**Proof.** We discuss three cases when \( n = 2 \).

**Case 1.** Suppose both \( G_1 \) and \( G_2 \) have triangles. By Lemma 7.2, \( G_1 \oplus G_2 \) contains a triangle. Hence, \( g(G_1 \oplus G_2) = 3 \).

**Case 2.** Suppose one of \( G_1 \) and \( G_2 \) is triangle-free. Without loss of generality, we assume that \( G_1 \) contains a triangle, and \( G_2 \) has an induced subgraph isomorphic to \( P_3 \). It is easy to check that \( K_3 \oplus P_3 \) contains a 4-cycle \( C_4 \). Consequently, this \( C_4 \) appears in \( (G_1 \oplus G_2) \). However again by Lemma 7.2, \( G_1 \oplus G_2 \) is triangle-free. This implies that \( C_4 \) is the smallest cycle in \( G_1 \oplus G_2 \). Therefore, \( g(G_1 \oplus G_2) = 4 \).

**Case 3.** Suppose \( G_1 \) and \( G_2 \) are triangle-free. Then each \( G_1 \) and \( G_2 \) contains an induced subgraph isomorphic to \( P_3 \). From Lemma 7.6, \( g(G_1 \oplus G_2) = 4 \).

From the above cases, it follows that \( g(G_1 \oplus G_2) = 3 \) or 4.

When \( n \geq 3 \), it is not difficult to prove the result if we proceed by induction on the number of factors.

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**References**