

On Tensor Product of Graphs, Girth and Triangles

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ABSTRACT. The purpose of this paper is to obtain a necessary and sufficient condition for the tensor product of two or more graphs to be connected, bipartite or eulerian. Also, we present a characterization of the duplicate graph $G \oplus K_2$ to be unicyclic. Finally, the girth and the formula for computing the number of triangles in the tensor product of graphs are worked out.

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1. INTRODUCTION

We shall consider only finite, undirected graphs without loops or multiple edges. We follow the terminology of [1]. For a graph G , $V(G)$ and $E(G)$ denote the vertex set and edge set of G , respectively. For a connected graph G , nG is the graph with n components, each being isomorphic to G . It is well-known that a graph is *bipartite* if and only if it contains no odd cycle. We now define the tensor product of two graphs [8] as follows: The *tensor product* of two graphs G_1 and G_2 is the graph, denoted by $G_1 \oplus G_2$, with vertex set $V(G_1 \oplus G_2) = V(G_1) \times V(G_2)$, and any two of its vertices (u_1, v_1) and (u_2, v_2) are adjacent, whenever u_1 is adjacent to u_2 in G_1 and v_1 is adjacent to v_2 in G_2 .

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The graphs G_1 and G_2 are called *factors* of the product $G_1 \oplus G_2$. Other popular names for tensor product that have appeared in the literature are *Kronecker product*, *cross product*, *direct product*, *conjunction product*. Sampathkumar [6] defines the tensor product of a graph G by K_2 as the *duplicate graph* of G , and studied its properties and a characterization in great detail. This product is also studied in [5]. Now, we define the two special type of tensor products: $G \oplus nK_2$ and $G \oplus [\oplus_{i=1}^n K_2]$ as the *generalized duplicate graphs of a graph G* , for any integer $n \geq 2$, and study their structural properties for our later use.

2. STRUCTURAL PROPERTIES OF THE GENERALIZED DUPLICATE GRAPHS

The following theorem of Weichsel [8] will be useful in the proof of our results.

Theorem 2.1. *If the connected graphs G and H are bipartite, then $G \oplus H$ has exactly two components.*

Next, we present some elementary results of the generalized duplicate graphs for our immediate use.

Theorem 2.2. *For any connected, bipartite graph G , $G \oplus nK_2 = 2nG$ for $n \geq 1$.*

Proof. For $n = 1$, Theorem 2.1 implies that $G \oplus K_2$ has exactly two components. Furthermore, by using the definition of tensor product, we see that each component of $G \oplus K_2$ is isomorphic to G . Therefore, $G \oplus K_2 = 2G$. Moreover, corresponding to $n \geq 1$ copies of K_2 , $G \oplus nK_2$ certainly contains exactly $2n$ copies of G . Thus, $G \oplus nK_2 = 2nG$. \square

Theorem 2.3. *For any connected graph G , $G \oplus nK_2$ for $n \geq 1$, is bipartite.*

Proof. We discuss two cases depending on G .

Case 1. Suppose G is bipartite. By Theorem 2.2, $G \oplus nK_2 = 2nG$. Since G is bipartite, it follows immediately that $G \oplus nK_2$ is bipartite.

Case 2. Suppose G is non-bipartite. Certainly, G contains a cycle C_m for odd $m \geq 3$. Corresponding to each copy of K_2 in $G \oplus nK_2$, there are exactly n distinct subgraphs in $G \oplus nK_2$, each is isomorphic to $C_m \oplus K_2$. It is shown in [2] that $C_m \oplus K_2$ is isomorphic to C_{2m} . For even $m \geq 4$, it is also shown in [2] that $C_m \oplus K_2 = C_m \cup C_m$. This proves that $G \oplus nK_2$ has no odd cycles. Hence, $G \oplus nK_2$ is bipartite. \square

Theorem 2.4. *Let G be a connected, bipartite graph and let $H = \oplus_{i=1}^n K_2$. Then $G \oplus H = 2^n G$ for $n \geq 1$.*

Proof. We proceed by induction on n . If $n = 1$, then by Theorem 2.2, $G \oplus H = 2G$. Assume the result holds with at most $n-1$. Consider $G \oplus H = G \oplus [\oplus_{i=1}^n K_2]$

$= G \oplus [\oplus_{i=1}^{n-1} K_2 \oplus K_2] = [G \oplus (\oplus_{i=1}^{n-1} K_2)] \oplus K_2$. By induction hypothesis, we have $G \oplus [\oplus_{i=1}^{n-1} K_2] = 2^{n-1}G$. Hence,

$$G \oplus [\oplus_{i=1}^n K_2] = 2^{n-1}G \oplus K_2. \dots \dots (2.1)$$

In view of Theorem 2.2 (with $n = 1$), $G \oplus K_2 = 2G$. Using this in (2.1), we get $G \oplus H = 2^n G$. \square

3. CHARACTERIZATION OF CONNECTED TENSOR PRODUCT OF GRAPHS

Now, we obtain a characterization of connected tensor product of arbitrarily many graphs. We see that Weichsel [6] studied the connectedness of the tensor product of two graphs as follows:

Theorem 3.1. *Let G and H be connected graphs. Then $G \oplus H$ is connected if and only if either G or H contains an odd cycle.*

Now, we present the natural finite extension of Weichsel's Theorem as follows:

Theorem 3.2. *Let G_k ($1 \leq k \leq n$; $n \geq 2$) be connected graph, and let $G = \oplus_{k=1}^n G_k$. Then G is connected if and only if at most one of G_k 's is bipartite.*

Proof. Assume that G is connected. We prove by contradiction. If possible, assume that there are at least two distinct graphs G_i and G_j ($1 \leq i, j \leq n$), which are bipartite. By Theorem 2.1, $G_i \oplus G_j$ contains exactly two components say, F and H . Now, we have $G = \oplus_{k=1}^n G_k = (F \oplus M) \cup (H \oplus M)$, where $M = \oplus_{k=1}^n G_k$ ($k \neq i, j$). This shows that G is certainly disconnected, and hence we immediately arrive at a contradiction. Thus, it proves that at most one of G_k 's is bipartite.

Conversely, assume that at most one of G_k 's is bipartite.

We discuss two cases.

Case 1. None of G_k 's is bipartite. Immediately, it follows that each G_k contains an odd cycle.

Case 2. Exactly one of G_k 's is bipartite. Without loss of generality, we assume that G_1 is bipartite. The remaining G_i ($2 \leq i \leq n$) is non-bipartite, and hence each such G_i contains an odd cycle.

In either case, by applying Theorem 3.1 and the mathematical induction on the number of factors, the result follows. \square

4. CHARACTERIZATION OF BIPARTITE TENSOR PRODUCT OF GRAPHS

Now, we shall obtain a necessary and sufficient condition for the tensor product of two or more graphs to be bipartite, (which is proposed in [3]).

Theorem 4.1. *Let G_1 and G_2 be two connected graphs. Then $G_1 \oplus G_2$ is bipartite if and only if at least one of G_1 and G_2 is bipartite.*

Proof. Suppose $G_1 \oplus G_2$ is bipartite. We claim that at least one of G_1 and G_2 is bipartite. If this is not so, then both G_1 and G_2 are non-bipartite. Consequently, there exist two odd cycles C_m (for $m \geq 3$) and C_n (for $n \geq 3$) in G_1 and G_2 , respectively. Without loss of generality, we consider $m \leq n$. Let $C_m : u_1, u_2, \dots, u_m, u_1$ and let $C_n : v_1, v_2, \dots, v_m, v_{m+1}, \dots, v_n, v_1$. Then $C_m \oplus C_n$ contains the cycle Z of length n as follows:

$$Z : (u_1, v_1), (u_2, v_2), \dots, (u_m, v_m), (u_{m-1}, v_{m+1}), (u_m, v_{m+2}), (u_{m-1}, v_{m+3}), \\ (u_m, v_{m+4}), \dots, (u_{m-1}, v_{n-1}), (u_m, v_n), (u_1, v_1).$$

So, $G_1 \oplus G_2$ contains the odd cycle Z . Hence, $G_1 \oplus G_2$ is non-bipartite. This is a contradiction.

Conversely, assume that at least one of G_1 and G_2 is bipartite.

We discuss two cases.

Case 1. Suppose both G_1 and G_2 are bipartite. Then by Theorem 2.1, $G_1 \oplus G_2$ contains exactly two components, say H_1 and H_2 . Now, we claim that both H_1 and H_2 are bipartite. On contrary, if one of H_i 's is non-bipartite. Without loss of generality, we assume that H_1 is non-bipartite. Then H_1 contains an odd cycle, say

$C : (u_1, v_1), (u_2, v_2), \dots, (u_n, v_n), (u_1, v_1)$ in $G_1 \oplus G_2$, where $u_i \in V(G_1)$ ($1 \leq i \leq n$), $v_j \in V(G_2)$ ($1 \leq j \leq n$). Certainly, the first co-ordinate vertices $u_1, u_2, \dots, u_n, u_1$ of the cycle C forms a closed odd walk W in G_1 . Since every closed odd walk in a graph contains an odd cycle, it follows that the walk W contains an odd cycle in G_1 . This shows that G_1 is not bipartite. But this contradicts the hypothesis that G_1 is bipartite. Since each H_i is bipartite, it follows that $G_1 \oplus G_2 = H_1 \cup H_2$ is also bipartite.

Case 2. Suppose one of G_1 and G_2 is bipartite, and the other is non-bipartite. Assume that G_1 is bipartite. Since G_2 is non-bipartite, G_2 contains an odd cycle. From Theorem 3.1, $G_1 \oplus G_2$ is connected. Next, we claim that $G_1 \oplus G_2$ is bipartite. If this is not so, then $G_1 \oplus G_2$ is non-bipartite, and hence it contains an odd cycle Z . By repeating the same argument as in Case 1, we obtain $G_1 \oplus G_2$ is bipartite.

In either case, we see that $G_1 \oplus G_2$ is bipartite. \square

Next, we obtain the finite extension of the above theorem, and its proof follows by the mathematical induction on the number of factors.

Corollary 4.2. *Let G_k ($1 \leq k \leq n$; $n \geq 2$) be a connected graph, and let $G = \oplus_{k=1}^n G_k$. Then G is bipartite if and only if at least one of G_k 's is bipartite.*

5. CHARACTERIZATION OF EULERIAN TENSOR PRODUCT OF GRAPHS

An *Euler tour* of a graph G is a closed walk in G that traverses each edge of G exactly once. A graph is eulerian if it contains an Euler tour. It is well-known that a connected graph G is eulerian if and only if every vertex in G has an even degree. For any vertex (u, v) in a tensor product $(G \oplus H)$ of two graphs

G and H , $\deg(u, v) = \deg(u) \bullet \deg(v)$. Now, we present a characterization of eulerian tensor product of two graphs.

Theorem 5.1. *Let G and H be connected graphs such that at most one of them is bipartite. Then $G \oplus H$ is eulerian if and only if at least one of G and H is eulerian.*

Proof. Suppose G is eulerian, and it contains an odd cycle. Then by Theorem 4.1, $G \oplus H$ is connected. Since G is eulerian, $\deg(u)$ is even for all vertices u in G . Consequently, for any vertex v of H , the pair (u, v) is a vertex in $G \oplus H$, and $\deg(u, v) = \deg(u) \bullet \deg(v)$, which is an even degree, because $\deg(u)$ is even, and $\deg(v) \geq 1$. This implies that $G \oplus H$ is eulerian.

Conversely, assume that $G \oplus H$ is eulerian. By definition, $G \oplus H$ is certainly connected. Again by Theorem 4.1, one of G and H contains an odd cycle. To complete the proof, we claim that at least one of G and H is eulerian. On contrary, if possible assume that both G and H are not eulerian graphs. Immediately, there exist at least two odd degree vertices x and y in G and H , respectively. Thus, (x, y) is a vertex in $G \oplus H$, and also $\deg(x, y) = \deg(x) \bullet \deg(y)$, which is odd, because both $\deg(x)$ and $\deg(y)$ are odd. This shows that $G \oplus H$ is not eulerian, and it contradicts the hypothesis that $G \oplus H$ is eulerian. \square

The finite extension of Theorem 5.1 is the following result, and its proof directly follows by the induction on the number of factors.

Corollary 5.2. *Let G_k ($1 \leq k \leq n$; $n \geq 2$) be a connected graph such that at most one of G_k 's is bipartite, and let $G = \bigoplus_{k=1}^n G_k$. Then G is eulerian if and only if at least one of G_k 's is eulerian.*

6. CHARACTERIZATION OF UNICYCLIC DUPLICATE GRAPH

A *unicyclic graph* is a connected graph which contains exactly one cycle. Next, we obtain a characterization of unicyclic duplicate graph $G \oplus K_2$.

Theorem 6.1. *A non-bipartite graph G is unicyclic if and only if the duplicate graph $G \oplus K_2$ is unicyclic.*

Proof. Suppose a non-bipartite graph G is unicyclic. Then G contains exactly one odd cycle C . Hence by Theorem 4.1, $G \oplus K_2$ is connected. Let $C : u_1, u_2, \dots, u_{2k+1}, u_1$ for $k \geq 1$. Next, we show that $G \oplus K_2$ is unicyclic. For this, let us consider $V(K_2) = \{v_1, v_2\}$. It is easy to see that the subgraph induced by $C \oplus K_2$ in $G \oplus K_2$ is certainly isomorphic to an even cycle $C_{2(2k+1)}$, where $C_{2(2k+1)} : (u_1, v_1), (u_2, v_2), (u_3, v_1), \dots, (u_{2k-1}, v_1), (u_{2k}, v_2), (u_{2k+1}, v_1), (u_{2k}, v_2), (u_{2k+1}, v_1), (u_1, v_2), (u_2, v_1), (u_3, v_2), \dots, (u_{2k-1}, v_2), (u_{2k}, v_1), (u_{2k+1}, v_2), (u_1, v_1)$. Since G is unicyclic, it follows that $G \oplus K_2$ has no cycles other than $C_{2(2k+1)}$. If this is not so, then there exists another cycle J in $G \oplus K_2$, which is different from $C_{2(2k+1)}$. Consequently, the first co-ordinates

of the vertices of the cycle J , which are in pairs, will form another cycle C' in G . Since $J \neq C_{2(2k+1)}$ in $G \oplus K_2$, it follows $C \neq C'$ in G . This is a contradiction to the fact that G is unicyclic. Therefore, $G \oplus K_2$ is unicyclic.

Conversely, suppose that $G \oplus K_2$ is unicyclic. Let Z be the only one cycle in $G \oplus K_2$. By Theorem 2.3 (with $n = 1$), $G \oplus K_2$ is bipartite. Hence, Z is a unique even cycle. Clearly, we notice that the first co-ordinate vertices of G in Z forms an odd cycle C in G . Since $G \oplus K_2$ is unicyclic, it follows that C is the unique cycle in G . Moreover, since $G \oplus K_2$ is connected, it implies that G is connected. Therefore, G is unicyclic. \square

7. THE GIRTH AND TRIANGLES IN TENSOR PRODUCT GRAPHS

The *girth* of a graph G , denoted by $g(G)$, is the length of a shortest cycle in G , if any. Otherwise, it is undefined if G is a forest. It is clear that the girth of a graph G is the minimum of the girths of its components. Firstly, we determine the girth of the generalized duplicate graphs.

Theorem 7.1. *Let G be a connected graph with $g(G) = k$. For any positive integer $n \geq 1$, we have*

$$g(G \oplus n K_2) = g(G \oplus [\oplus_{i=1}^n K_2]) = \begin{cases} k & \text{if } G \text{ is bipartite,} \\ \min\{2p, q\} & \text{otherwise,} \end{cases}$$

where C_p and C_q are the minimal odd and even cycles in a non-bipartite graph G , respectively.

Proof. First, we discuss the result when $n = 1$.

Case 1. Assume G is bipartite. From Theorem 2.2 (with $n = 1$), we have for the duplicate graph $G \oplus K_2 = 2G$. Consequently, $g(G \oplus K_2) = k$.

Case 2. Suppose G is not bipartite. Then G contains an odd cycle. Let C_p for $p \geq 3$, be a minimal odd cycle in G .

Now, there are two possibilities to discuss:

2.1. If G is free-from even cycles, then $C_p \oplus K_2$ contains an even cycle C_{2p} in $G \oplus K_2$.

2.2. If G contains a minimal even cycle C_q , $q \geq 4$, then $C_q \oplus K_2 = 2C_q$ appears in $G \oplus K_2$.

From the above possibilities, it follows that $g(G \oplus K_2)$ is the minimum of $2p$ and q . Thus, $g(G \oplus K_2) = \min\{2p, q\}$.

Finally, consider the result when $n \geq 2$. The result follows immediately if we proceed as above by applying Theorem 2.2 or 2.4 repeatedly. \square

Next, we derive a formula (which is proposed in [4]) for computing the number of triangles in the tensor product of two graphs. For this, firstly we establish the following lemma.

Lemma 7.2. *Let G_k ($1 \leq k \leq n$; $n \geq 2$) be a connected graph. Then the product $\oplus_{k=1}^n G_k$ contains a triangle if and only if each G_k contains a triangle.*

Proof. Now, we discuss the case when $n = 2$. Suppose $G_1 \oplus G_2$ contains a triangle T , and let $(a_1, b_1), (a_2, b_2)$ and (a_3, b_3) be any three vertices of T . By definition, $(a_1, b_1)(a_2, b_2), (a_2, b_2)(a_3, b_3)$ and $(a_3, b_3)(a_1, b_1)$ are the edges of T in $G_1 \oplus G_2$ if and only if the edges: a_1a_2, a_2a_3 and a_3a_1 constitute a triangle T_1 in G_1 and also the edges : b_1b_2, b_2b_3 and b_3b_1 constitute a triangle T_2 in G_2 . But this is so if and only if both G_1 and G_2 have triangles T_1 and T_2 , respectively. Finally, we discuss the case when $n \geq 3$. The result follows immediately if we proceed by applying induction on the number of factors. \square

Theorem 7.3. *Let G_i ($1 \leq i \leq 2$) be a connected graph having the number of triangles n_i . Then the product $G_1 \oplus G_2$ contains $6n_1n_2$ triangles.*

Proof. First, let us compute the actual number of triangles in the product $T_1 \oplus T_2$, when T_i is any triangle in G_i (for $i = 1, 2$). It is easy to see that there are exactly 6 distinct triangles in $T_1 \oplus T_2$. But each G_i contains n_i triangles. Consequently, the product $G_1 \oplus G_2$ contains $6n_1n_2$ triangles, and there are no more other triangles because of Lemma 7.2. \square

The immediate consequence of the above theorem is the following corollary.

Corollary 7.4. *Let G_k ($1 \leq k \leq n$; $n \geq 2$) be a connected graph having the number of triangles n_k . Then the product $\oplus_{k=1}^n G_k$ contains $6^{n-1}(\prod_{k=1}^n n_k)$ triangles.*

Corollary 7.5. *The number of triangles in $K_m \oplus K_n$ is $\frac{1}{6}[mn(m-1)(n-1)(m-2)(n-2)]$.*

Proof. We know that the number of triangles in $K_p = pC_3$. Therefore from Theorem 7.3, the number of triangles in $K_m \oplus K_n$ is $6(mC_3)(nC_3) = \frac{1}{6}[mn(m-1)(n-1)(m-2)(n-2)]$. \square

Finally to determine the girth of the tensor product of graphs, we need to establish the following lemma.

Lemma 7.6. *Let G_k ($1 \leq k \leq n$; $n \geq 2$) be a connected, triangle-free graph such that each G_k contains an induced subgraph isomorphic to P_3 . Then $g(\oplus_{k=1}^n G_k) = 4$.*

Proof. We discuss the case when $n = 2$. Let a_i ($1 \leq i \leq 3$) and b_i ($1 \leq i \leq 3$) be the vertices of a subgraph isomorphic to P_3 in G_1 and G_2 , respectively. Then the subgraph $\langle \{a_1, a_2, a_3\} \rangle \oplus \langle \{b_1, b_2, b_3\} \rangle$ is isomorphic to $P_3 \oplus P_3$ in $G_1 \oplus G_2$. It is easy to see that $P_3 \oplus P_3 = K_{1,4} \cup C_4$. Immediately, a 4-cycle C_4 appears as a subgraph in $G_1 \oplus G_2$. However from Lemma 7.2, there is no triangle in $G_1 \oplus G_2$. Consequently, C_4 is the smallest cycle in $G_1 \oplus G_2$.

Therefore, $g(G_1 \oplus G_2) = 4$.

When $n \geq 3$, the result follows easily if we proceed by induction on the number of factors. \square

The following result gives the girth of tensor product of arbitrarily many graphs.

Theorem 7.7. *Let G_k ($1 \leq k \leq n$; $n \geq 2$) be a connected graph of order ≥ 3 . Then $g(\bigoplus_{k=1}^n G_k)$ is either 3 or 4.*

Proof. We discuss three cases when $n = 2$.

Case 1. Suppose both G_1 and G_2 have triangles. By Lemma 7.2, $G_1 \oplus G_2$ contains a triangle. Hence, $g(G_1 \oplus G_2) = 3$.

Case 2. Suppose one of G_1 and G_2 is triangle-free. Without loss of generality, we assume that G_1 contains a triangle, and G_2 has an induced subgraph isomorphic to P_3 . It is easy to check that $K_3 \oplus P_3$ contains a 4-cycle C_4 . Consequently, this C_4 appears in $(G_1 \oplus G_2)$. However again by Lemma 7.2, $G_1 \oplus G_2$ is triangle-free. This implies that C_4 is the smallest cycle in $G_1 \oplus G_2$. Therefore, $g(G_1 \oplus G_2) = 4$.

Case 3. Suppose G_1 and G_2 are triangle-free. Then each G_1 and G_2 contains an induced subgraph isomorphic to P_3 . From Lemma 7.6, $g(G_1 \oplus G_2) = 4$.

From the above cases, it follows that $g(G_1 \oplus G_2) = 3$ or 4.

When $n \geq 3$, It is not difficult to prove the result if we proceed by induction on the number of factors. \square

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