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On the Graphs Related to Green Relations of Finite Semigroups

A. Gharibkhajeh* and H. Doostie

Department of Mathematicse, Science and Research Branch,
Islamic Azad University, Tehran P.O. Box 14515-775, Iran

E-mail: a_gharib@iau-tnb.ac.ir
E-mail: doostih@gmail.com

Abstract. In this paper we develop an analog of the notion of the conjugacy graph of finite groups for the finite semigroups by considering the Green relations of a finite semigroup. More precisely, by defining the new graphs $\Gamma_L(S)$, $\Gamma_R(S)$, $\Gamma_H(S)$, $\Gamma_J(S)$ and $\Gamma_D(S)$ (we name them the Green graphs) related to the Green relations L, R, J, H and D of a finite semigroup S, we first attempt to prove that the graphs $\Gamma_L(S)$ and $\Gamma_H(S)$ have exactly one connected component, and this graphs for regular semigroups are complete. Next, we give a necessary condition for a finite semigroup to be regular. This study shows an intrinsic difference between the conjugacy graphs (of groups) and the Green graphs (of semigroups) as well. Finally, our calculations include two kinds of semigroups, mostly involving the well known Lucas numbers, and examining the proved assertions.

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1. Introduction

Graphs related to various algebraic structures have been actively investigated in the literature. Several classes of graphs associated to groups and

*Corresponding Author

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semigroups have turned out useful and have also been applied in several practical areas. Let us refer the readers to the survey article [11] and articles [8, 15, 17], containing an extensive bibliography devoted to numerous applications of such graphs. In particular, the Cayley graphs of semigroups are closely related to automata theory, as explained, for example, in the monograph [12] and articles [13, 14], where the readers can find many more references on graphs associated to semigroups.

Also, for a finite group G, E.A. Bertram [4] has attached a conjugacy graph $\Gamma(G)$, which is an undirected graph with t vertices $c_1, c_2, ..., c_t$ where c_i's are the conjugacy classes of G and two vertices c_i and c_j are adjacent in $\Gamma(G)$ if and only if $\gcd(|c_i|, |c_j|) > 1$, (gcd is used for the greatest common divisor). The study of this graph is essential for the non-abelian groups and it is proved in [4] that $n(\Gamma(G)) \leq 2$ where $n(\Gamma(G))$ is the number of connected components of $\Gamma(G)$. Also M. Fang [7] studied this graph in the classification of groups for the graphs containing no triangles, in 2003. The recently obtained result is due to Adan-Bante [1] in 2005 which is to study the conjugacy classes of finite p-groups which could be quite related to the study of conjugacy graphs of groups.

In this paper we generalize the notion of conjugacy graph of groups for finitely presented finite semigroups. This paper concentrates on a very specific study, the new Green graphs. They have not been considered in the literature before.

Our notations are fairly standard. In the group and semigroup presentations one may consult [6,10,16] and in the theoretical semigroup ideas one may see [9]. Also our computational examples will be in continuations of the articles [2, 3, 5]. The notation K_n is used for the complete graph with n vertices.

For a finite semigroup S, the left Green relation in S is defined by $aLb \Leftrightarrow S^1a = S^1b$, for every elements a and b of S, where $S^1 = S$ if S posses an identity element, otherwise $S^1 = S \cup \{1\}$, such that, $1s = s1 = s$, for every $s \in S$. The other Green relations R, J, H and D may be defined in a similar way:

$aRb \Leftrightarrow aS^1 = bS^1$,
$aJb \Leftrightarrow S^1aS^1 = S^1bS^1$,
$aHb \Leftrightarrow (aLb \quad \text{and} \quad aRb)$,
$aDb \Leftrightarrow (\exists c \in S, aLc \quad \text{and} \quad cRb)$, for every elements a and b of S.

Definition 1.1. For a finite semigroup S the left Green graph of S, denoted by $\Gamma_L(S)$, is an undirected graph with vertices $L_1, L_2, ..., L_k$ of left Green classes of S (L-classes), and two vertices L_i and L_j are adjacent if and only if $\gcd(|L_i|, |L_j|) > 1$. The other Green graphs $\Gamma_R(S), \Gamma_J(S)$, $\Gamma_H(S)$ and $\Gamma_D(S)$

are defined in a similar way.

Let $n(\Gamma_L(S))$ be the number of connected components of the graph $\Gamma_L(S)$, (a similar notation is used for $\Gamma_R(S)$, $\Gamma_J(S)$, $\Gamma_H(S)$ and $\Gamma_D(S)$). For a presentation $\pi = \langle X|R\rangle$, let $G = Gp(\pi)$ and $S = Sg(\pi)$ be the group and the semigroup presented by π, respectively. A natural question may be posed here is that: for a semigroup S when the Green graphs may be isomorphic, and also which Green graphs of which semigroups may have just one non-zero connected component. In the next sections we investigate these questions by providing certain infinite classes of finite semigroups.

Our main results in this paper are:

Theorem A. For every finite and non-commutative semigroup S, $n(\Gamma_L(S)) = n(\Gamma_H(S)) = 1$.

Theorem B. Let S be a finite regular semigroup. Then, $\Gamma_D(S) \simeq \Gamma_J(S) \simeq \Gamma_R(S)$. Moreover, $\Gamma_L(S) \simeq \Gamma_H(S)$ which is a complete graph.

Theorem C. If all of five Green graphs are isomorphic, then the semigroup S is regular.

Corollary D. The Green graphs of a commutative semigroup S, coincide and are not the zero graphs (in spite of the conjugacy graph of an abelian finite group being a zero graph).

2. Green graphs with one connected component

For a finite semigroup S let $\gcd(|L_1|, |L_2|) \geq 1$ where, $L_1 = [x]_L$ and $L_2 = [y]_L$ are two arbitrary left Green classes of S. Then there are two cases. If $\gcd(|L_1|, |L_2|) > 1$ then obviously $\Gamma_L(S)$ has exactly one connected component. However, for the case $\gcd(|L_1|, |L_2|) = 1$ we have the following lemma which will be used in the proof of Theorem A.

Lemma 2.1. Let S be a finite semigroup, and let L_1 and L_2 be two left Green classes of S with representatives a_1 and a_2, respectively. If $C_S(a_i)$ is the centralizer of a_i in S, (i=1, 2) and $\gcd(|L_1|, |L_2|) = 1$, then the following conditions hold:

(i) $S = C_S(a_1)C_S(a_2)$,

(ii) $|L_1| = 1$ or $|L_2| = 1$,

(iii) L_1L_2 is a left Green class of S.

Proof. (i). Let p^n be the highest power of a prime p dividing $|S|$ and $n \geq 1$. Clearly it suffices to show that either $p^n||C_S(a_1)|$ or $p^n||C_S(a_2)|$. If p doesn't divide $|L_1|$ then $p^n||C_S(a_1)|$, and if p divides $|L_1|$ then p doesn't divide $|L_2|$. Since $(|L_1|,|L_2|)=1$, we get $p^n||C_S(a_2)|$.

On the other hand, $C_S(a_1)C_S(a_2)\subseteq S$ and $p^n\leq |C_S(a_1)C_S(a_2)|\leq |S|=p^n$. So, $S=C_S(a_1)C_S(a_2)$.

(ii) If $|L_1|,|L_2|>1$, then $|L_1|=(\frac{|S|}{|C_S(a_1)|})>1$ and $|L_2|=(\frac{|S|}{|C_S(a_2)|})>1$, so $(|L_1|,|L_2|)=1$ yields $(\frac{|S|}{|C_S(a_1)|},\frac{|S|}{|C_S(a_2)|})=1$. Consequently, $p|\frac{|S|}{|C_S(a_1)|}$ and $p|\frac{|S|}{|C_S(a_2)|}$ thus $p|1$, which is a contradiction.

(iii) We have to show that for every $x,y\in L_1L_2$, $xS=yS$. By considering (ii), we can suppose that $L_1=[a_1]_L=\{a_1\}$, thus if $x\in L_1L_2$ then, there exists an element $c_1\in L_2$ such that $x=a_1c_1$. Also if $y\in L_1L_2$, there exists an element $c_2\in L_2$ such that $y=a_1c_2$. Then we get:

$xS=a_1c_1S=a_1(c_1S)=a_1(a_2S)=a_1(c_2S)=a_1c_2S=yS.$ □

Proof of Theorem A. Let L_1 and L_2 be two left Green classes of S. If $\gcd(|L_1|,|L_2|)=1$ then $|L_1|=1$ or $|L_2|=1$ (by the Lemma 2.1.) So the non-central left Green classes are the vertices of a unique component of $\Gamma_L(S)$ and then, $\Gamma_L(S)$ has exactly one non-zero component such that every two vertices are adjacent, this means that $n(\Gamma_L(S))=1$. Proving $n(\Gamma_H(S))=1$ is now quite easy by considering the definition of H relation. □

3. Green graphs of regular semigroups

Let S be a finite semigroup, then $\Gamma_D(S)\simeq\Gamma_J(S)$ is a quick result of the identification of Green relations D and J (see [9]).

Proof of Theorem B. By the above comment, $\Gamma_J(S)\simeq\Gamma_D(S)$. So it is sufficient to show that $\Gamma_R(S)\simeq\Gamma_J(S)$. For every J-class $[x]_J$, if $y\in[x]_J$, there exist $u_1,v_1,u_2,v_2\in S$ such that $y=u_1xv_1$ and $x=u_2yv_2$. So, $y=u_1xv_1=(u_1x)v_1=x'v_1$ and then $y\in[x']_R$. Also if $y\in[z]_R$, then there exists $k\in S$ such that $y=zk$, and by the regularity of S, there exists $r\in S$, such that $z=zrz$. So $y=zrzk=z(rz)k=zlk\in[l]_J$. Thus, every J-class is equal to an R-class. Now, if $[x_1]_J=[x_1']_R$ and $[x_2]_J=[x_2']_R$ such that $\gcd(|[x_1]_J|,|[x_2]_J|)>1$, then $\gcd(|[x_1']_R|,|[x_2']_R|)>1$. Thus there is a bijection between the vertex sets of $\Gamma_R(S)$ and $\Gamma_J(S)$ such that two adjacent vertices of $\Gamma_R(S)$ map to the adjacent vertices of $\Gamma_J(S)$. This shows that $\Gamma_R(S)\simeq\Gamma_J(S)$. For the other part of theorem it is obvious that each H-class of the semigroup S is a subset of a L-class. Let $x\in[a]_L$ then, there exist $u,v\in S$ such that $x=ua$ and $a=vx$. By regularity of S there exists $r\in S$ such that $a=ara$. So $x=ua=uara=ua(ra)=uak\in[a]_J=[a]_R$, and $x\in[a]_L\cap[a]_R=[a]_H$. This proves that $\Gamma_H(S)\simeq\Gamma_L(S)$ which is a complete graph by considering H relation. □

Note that, Theorem B is valid for finite periodic semigroups and for inverse semigroups.

Proof of Theorem C. Let S be a finite semigroup that all of five Green graphs are isomorphic. Then for a fixed element $x \in S$, there exists a J-class $[a]_J$ such that $x \in [a]_J$. So, there are $u, v, u', v' \in S$ such that $x = uav$ and $a = u'xv'$. The relations $x = uav$ and $a = u'xv'$ imply $u \in [x]_R$ and $v \in [x]_L$. Consequently, there exist $r, s \in S$ such that $u = xr$ and $v = sx$. So $x = uav = (xr)a(sx) = x(ras)x = xkx$. This implies that x is a regular element and consequently S is a regular semigroup. □

Proof of Corollary D. For a commutative semigroup S, it may be easily checked that $\Gamma_L(S) \simeq \Gamma_R(S) \simeq \Gamma_J(S) \simeq \Gamma_H(S) \simeq \Gamma_D(S)$. However $\Gamma_R(S)$ may have at least two connected vertices, in general. This is in contrary by the fact that for an abelian group, the conjugacy graph is always a zero graph. □

4. Conclusion

In this chapter we examine five classes of non-commutative finitely presented semigroups as follows:

$S_1 =< a, b | a^3 = a, b^n a = a, abab^2 = b >$,

$S_2 =< a, b | a^3 = a, b^{n+1} = b, abab^2 = b >$,

$S_3 =< a, b | a^3 = a, b^{2n+1} = b, ab^2ab^{n-1} = ba >$,

$S_4 =< a, b | a^3 = a, b^{2n+1} = b, ab^{n-1}ab^2 = ba >$,

$S_5 =< a, b | ab = ba^{1+p^{\alpha-\gamma}}, a = a^{1+p^{\alpha}}, b = b^{1+p^{\beta}} >$,

where, $p \geq 2$ is a prime, n, α, β and γ are integers such that $n \geq 2$, $\alpha \geq 2\gamma$, $\beta \geq \gamma \geq 1$, and $\alpha + \beta \geq 3$. For more information on the finiteness and the orders of these semigroups one may see [2,3,6]. For the classes S_1, S_2 and S_5 we prove their regularity and construct the related Green graphs which will examine the results of Chapter 3. In the two remained classes S_3 and S_4, by showing the non-regularity we compute the Green graphs which are also an examining of the results of Chapter 3. There are two comments on Theorems A and B.

First of all, we mention that the results of this chapter have been influenced by computational evidence running on computer by GAP software [18].

S_1 and S_2 are examples of finite regular semigroups. Proving the regularity is easy by using the relators of the semigroups, and using GAP [18], we get the numerical results as follows:

$\Gamma_R(S_1) = \Gamma_J(S_1) = \Gamma_D(S_1) = K_1$,

$\Gamma_L(S_1) = \Gamma_H(S_1) = K_2$.

$\Gamma_R(S_2) = \Gamma_J(S_2) = \Gamma_D(S_2) = K_2$,

$\Gamma_L(S_2) = \Gamma_H(S_2) = K_4$.

This computation shows that $n(\Gamma_R(S_1)) = 0$ and $n(\Gamma_R(S_2)) = 1$. So, the assertion of Theorem A doesn't hold for Green R relation in general.

The semigroup S_5 is an example of regular semigroup as well which distinguishing Theorems B and C. Indeed we can easily show that $\Gamma_L(S_5) \simeq \Gamma_R(S_5) \simeq \Gamma_J(S_5) \simeq \Gamma_H(S_5) \simeq \Gamma_D(S_5) \simeq K_3$. So the converse of Theorem B doesn't hold.

Since the semigroups S_3 and S_4 are not regular, for, we may use the general forms of their elements to show that the regularity condition doesn't hold for at least one element in each of the semigroups S_3 and S_4.

Our main results of computations in this chapter are the following propositions:

Proposition 4.1. For every integer $n \geq 2$,

(i) $\Gamma_R(S_3) = K_5$,

(ii) $\Gamma_J(S_3) = \Gamma_D(S_3) = K_4$,

(iii) $\Gamma_L(S_3) = \begin{cases} K_{2n+4}, & \text{if n is odd,} \\ K_{2n+3}, & \text{if n is even,} \end{cases}$

(iv) $\Gamma_H(S_3) = \begin{cases} 4nK_1 \cup K_4, & \text{if n is odd,} \\ 4nK_1 \cup K_3, & \text{if n is even.} \end{cases}$

Proof. First of all we note that for every integers i, j and n where ($n \geq 2$, $1 \leq i, j \leq 2n$) the following relations hold in S_3:

$ba = a^2ba, \quad b^ia = (a^2b)^ia, \quad b^ia^2 = (a^2b)^ia^2,$

$b^iab^j = (a^2b)^i(ab^j), \quad b^ia^2b^j = (a^2b)^{i+1}b^{j-1},$

(the proof of these relations are easy by considering the relators of S_3). Also we may note that every element of S_3 is in the form $b^{\alpha_1}a^{\epsilon_1}b^{\alpha_2}a^{\epsilon_2}...b^{\alpha_k}a^{\epsilon_k}.b^{\delta}$, where, $\epsilon_i \in \{1, 2\}$ and $\alpha_i, \delta \in \{1, 2, ..., 2n\}$, and this element will be decomposed as a product of the elements of R-classes of S_3.

(i) For every $n \geq 2$ the order of S_3 is $(4ng_n + 2 + 6n)$, or $(2ng_n + 2 + 10n)$, if n is odd or even, respectively ($\{g_n\}$ is the sequence of Lucas numbers defined as $g_0 = 2$, $g_1 = 1$, $g_n = g_{n-1} + g_{n-2}$, $(n \geq 2)$). Then, by using the relators $a^3 = a$, $b^{2n+1} = b$ and $ab^2ab^{n-1} = ba$ of S_3 and GAP [11] we get the following R-classes for S_3:

$R_1 = [a]_R = \{a, a^2\}$, $\quad R_2 = [b]_R = \{b^i | i = 1, ..., 2n\}$,

$R_3 = [ab]_R = \{ab^i | i = 1, ..., 2n\}$, $\quad R_4 = [a^2b]_R = \{a^2b^i | i = 1, ..., 2n\}$,

$R_5 = [ba]_R = S_3 - (R_1 \cup R_2 \cup R_3 \cup R_4)$,

where, $|R_1| = 2$, $|R_2| = |R_3| = |R_4| = 2n$ and

$|R_5| = \begin{cases} 4ng_n, & \text{if n is odd,} \\ 2ng_n + 4n, & \text{if n is even.} \end{cases}$

Hence, $\gcd(|R_i|, |R_j|) \geq 2$, for every $1 \leq i, j \leq 5$. This proves that $\Gamma_R(S_3) = K_5$.

(ii) By using the same method as above we get the D-classes as follows:

$$D_1 = [a]_D = R_1, \qquad D_2 = [b]_D = R_2,$$
$$D_3 = [a^2b]_D = \{a^ib^j | 1 \leq i \leq 2, 1 \leq j \leq 2n\}, \qquad D_4 = [ba]_D = R_5.$$

Then, $|D_1| = 2$, $|D_2| = 2n$, $|D_3| = 4n$, $|D_4| = |R_5|$.
Thus, $\gcd(|D_i|, |D_j|) \geq 2$, for every $1 \leq i, j \leq 4$, which proves that $\Gamma_J(S_3) \simeq \Gamma_D(S_3) = K_4$.

(iii) For every even values of n, there are exactly $(2n+3)$ L-classes for S_3 as follows:

$L_1 = [a]_L = R_1, \qquad L_2 = [b]_L = R_2,$
$L_i = [ab^{i-2}]_L = \{ab^{i-2}, a^2b^{i-2}\}$, $(i = 3, 4, ..., (2n+2))$,
$L_{2n+3} = [ba^2]_L = S_3 - (L_1 \cup L_2 \cup ... \cup L_{2n+2})$.

So, $|L_1| = |L_3| = |L_4| = ... = |L_{2n+2}| = 2$, $|L_2| = 2n$. Since n is even, then by considering the order of S_3 we get $|L_{2n+3}| = 2ng_n + 4n$. Consequently, $\gcd(|L_i|, |L_j|) \geq 2$, for every $1 \leq i, j \leq (2n+3)$ and then $\Gamma_L(S_3) = K_{2n+3}$, for all even values of n.

For the odd values of n we have exactly $(2n+4)$ L-classes as follows:

$L_1 = [a]_L = R_1, \qquad L_2 = [b]_L = R_2,$
$L_i = [ab^{i-2}]_L = \{ab^{i-2}, a^2b^{i-2}\}$, $(i = 3, 4, ..., (2n+2))$,
$L_{2n+3} = [ba^2]_L, \qquad L_{2n+4} = [ba^2b]_L,$

where, $L_{2n+3} \cup L_{2n+4} = S_3 - (L_1 \cup L_2 \cup ... \cup L_{2n+2})$. So,

$$\begin{cases} |L_1| = |L_3| = |L_4| = ... = |L_{2n+2}| = 2, \\ |L_2| = 2n, \\ |L_{2n+3}| = |L_{2n+4}| = 2ng_n. \end{cases}$$

Thus, $\gcd(|L_i|, |L_j|) \geq 2$, for every $1 \leq i, j \leq (2n+4)$ which shows that $\Gamma_L(S_3) = K_{2n+4}$, for all odd values of n.

(iv) There are exactly $(4n+3)$ H-classes of S_3 for even values of n, and they are:

$H_1 = [a]_H = \{a, a^2\}, \qquad H_{1+i} = [ab^i]_H = \{ab^i\}, (i = 1, 2, ..., 2n),$
$H_{2n+1+j} = [a^2b^j]_H = \{a^2b^j\}, (j = 1, 2, ..., 2n),$
$H_{4n+2} = [b]_H = \{b, b^2, b^3, ..., b^{2n}\},$
$H_{4n+3} = [ba^2]_H = S_3 - (H_1 \cup H_2 \cup ... \cup H_{4n+2})$. So,

$$\begin{cases} |H_2| = |H_3| = |H_4| = ... = |H_{4n+1}| = 1, \\ |H_1| = 2, \\ |H_{4n+2}| = 2n, \\ |H_{4n+3}| = 2ng_n + 4n. \end{cases}$$

Consequently, $\gcd(|H_i|, |H_j|) \geq 2$, just for the values $i, j = 1, 4n+2, 4n+3$. Hence $\Gamma_H(S_3) = 4nK_1 \cup K_3$, for all even values of n.

There are $(4n+4)$ numbers of H-classes for S_3 as follows:

$$\begin{aligned}
&H_1 = [a]_H = \{a, a^2\}, \qquad H_{1+i} = [ab^i]_H = \{ab^i\}, (i = 1, 2, ..., 2n),\\
&H_{2n+1+j} = [a^2b^j]_H = \{a^2b^j\}, (j = 1, 2, ..., 2n),\\
&H_{4n+2} = [b]_H = \{b, b^2, b^3, ..., b^{2n}\},\\
&H_{4n+3} = [ba^2]_H, \quad H_{4n+4} = [ba^2b]_H,
\end{aligned}$$

for odd values of n, where $H_{4n+3} \cup H_{4n+4} = S_3 - (H_1 \cup H_2 \cup ... \cup H_{4n+2})$. So,

$$\begin{cases}
|H_2| = |H_3| = |H_4| = ... = |H_{4n+1}| = 1,\\
|H_1| = 2,\\
|H_{4n+2}| = 2n,\\
|H_{4n+3}| = |H_{4n+4}| = 2ng_n.
\end{cases}$$

Consequently, $\gcd(|H_i|, |H_j|) \geq 2$, just for the values $i, j = 1, 4n+2, 4n+3, 4n+4$. Hence $\Gamma_H(S_3) = 4nK_1 \cup K_4$. This complete the proof. □

Proposition 4.2. For every integer $n \geq 2$,
(i) $\Gamma_R(S_4) = K_5$,
(ii) $\Gamma_J(S_4) = \Gamma_D(S_4) = K_4$,
(iii) $\Gamma_L(S_4) = K_{2n+4}$,
(iv) $\Gamma_H(S_4) = 4nK_1 \cup K_4$.

Proof. A similar method to that of Proposition 4.1 may be used for S_4. □

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References

1. E. Adan-Bante, Conjugacy classes and finite p-groups, *Arch. Math.*, **85**, (2005), 297-303.
2. K. Ahmadidelir, H. Doostie, Two classes of finite semigroups and monoids involving Lucas numbers, *Semigroup Forum*, **78**, (2009), 200 − 209.
3. A. Arjomandfar, C. M. Campbell, H. Doostie, Semigroups related to certain group presentations, *Semigroup Forum.*, (Online first).
4. E. A. Bertram, M. Herzog, A. Mann, On a graph related to conjugacy classes of groups, *Bull. London Math. Soc.*, **22**, (1990), 569 − 575.
5. C. M. Campbell, E. F. Robertson, R. M. Thomas, On a class of semigroups with symmetric presentations, it Semigroup Forum, **46**, (1993), 286 − 306.
6. C. M. Campbell, E. F. Robertson, R. M. Thomas, *Semigroup presentation and number sequences. In*: Bergum, G.E. etal. (eds.) Applications of Fibonacci numbers, Vol. 5, 77 − 83, Springer Berlin (1993).
7. M. Fang, P. Zhang, Finite group containing no Triangles, *J. Algebra*, **204**, (2003), 613 − 619.

8. M. Fouroudi Ghasemabadi, N. Ahanjideh, Characterizations of the simple group $D_n(3)$ by prime graph and spectrum, *Iranian Journal of Mathematical science and Informatics*, **7**(1), (2012), $91-106$.
9. J.M. Howie, *Fundamentals of Semigroup Theory*, Oxford University Press Inc. New York (1995).
10. D. L. Johnson, *Presentations of Groups*, Cambridge University Press, Cambridge (1997).
11. A. Kelarev, J. Ryan, J. Yearwood, Cayley graphs as classifiers for data mining: The influence of asymmetries, *Discrete Mathematics*, **309** (17), (2009), $5360-5369$.
12. A. V. Kelarev, *Graph Algebras and Automata*, Marcel Dekker, New York, 2003.
13. A. V. Kelarev, Labelled Cayley graphs and minimal automata, *Australasian J. Combinatorics*, **30**, (2004), $95-101$.
14. A. V. Kelarev, On Cayley graphs of inverse semigroups, *Semigroup Forum*, 72 (3), (2006), $411-418$.
15. H. R. Maimani, Median and Center of Zero-Divisor graph of commutative semigroups, *Iranian Journal of Mathematical science and Informatics*, **3**(2), 2008, $69-76$.
16. E. F. Robertson, Y. Unlu, On Semigroup presentations, *Proc. Edinburgh Math. Soc.*, **36**, (1993), $55-68$.
17. A. Tehranian, H. R. Maimani, A study of the Total graph, *Iranian Journal of Mathematical science and Informatics*, **6**(2), 2011, $75-80$.
18. The GAP Group. GAP-Groups, Algorithms and progeramming, Version 4.4 Available from http://www.gap-system.org, 2005.