

**The Wave Equation in Non-classic Cases:  
Non-self Adjoint with Non-local and Non-periodic Boundary  
Conditions<sup>†</sup>**

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**ABSTRACT.** In this paper the wave equation in some non-classic cases has been studied. In the first case boundary conditions are non-local and non-periodic. At that case the associated spectral problem is a self-adjoint problem and consequently the eigenvalues are real. But in the second case the associated spectral problem is non-self-adjoint and consequently the eigenvalues are complex numbers, in which two cases, the solutions of the problem are constructed by the Fourier method. By compatibility conditions and asymptotic expansions of the Fourier coefficients, the convergence of series solutions are proved. Finally, series solutions are established and the uniqueness of the solution is proved by a special way which has not been used in classic texts.

**Keywords:** Wave equation, Non-local & non-periodic boundary conditions, Asymptotic expansion.

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## 1. INTRODUCTION

Initial and boundary value problems are one of the most important problems in mathematical physics and engineering. In classic texts these problems are given with classic boundary conditions such as Dirichlet and Neumann boundary conditions.[1, 3, 4, 6, 7].

Also some initial-boundary value problems have been considered by non-periodic and non-local boundary conditions [8, 2, 5]. In this paper those problems with non-classic boundary conditions has been investigated such as non-local and non-periodic boundary conditions.

In the first section the associated spectral problem is a self-adjoint problem, in which the adjoint problem of the given problem eigenvalues and eigenfunctions are determined. Next by compatibility conditions and asymptotic explosions of Fourier Coefficients, the convergence of the solution is proved in a special way different from many classical texts. In the second section the associated spectral problem is a non-self-adjoint problem, consequently the eigenvalues are complex numbers and the associated eigenfunctions don't form a complete basis system. In this case the eigenfunctions of problem and adjoint spectral problem must be used. Eigenfunctions of main problem and adjoint spectral problem form a bi-orthogonal complete basis system. These conditions determine the Fourier coefficients in formal series solution in non-self-adjoint case.

## 2. SELF-ADJOINT CASE WITH NON-PERIODIC AND NON-LOCAL BOUNDARY CONDITIONS

**2.1. Mathematical Statement of problem.** The non-homogeneous wave equation is considered in following form

$$\frac{\partial^2 u(x, t)}{\partial t^2} = \frac{\partial^2 u(x, t)}{\partial x^2} + f(x, t) \quad x \in (0, 1), \quad t > 0, \quad (2.1)$$

with non-local and non-periodic boundary conditions

$$u(0, t) + u(1, t) = 0, \quad (2.2)$$

$$\frac{\partial u(0, t)}{\partial x} + \frac{\partial u(1, t)}{\partial x} = 0, \quad t \geq 0.$$

And with initial conditions

$$\frac{\partial^k u(x, 0)}{\partial t^k} = \varphi_k(x), \quad k = 0, 1, \quad x \in [0, 1], \quad (2.3)$$

where the functions  $f(x, t)$  and  $\varphi_k(x)$ ,  $k = 0, 1$  are real valued continuous functions.

**2.2. Eigenvalues and Eigenfunctions.** In this section, the eigenvalues of adjoint equation of (2.1) are determined. In this way, by making use of Fourier method that has been assumed the unknown function is written  $u(x, t) = X(x)T(t)$  then the associated spectral problem will be in the following form

$$X''(x) + \lambda^2 X(x) = 0 \quad (2.4)$$

$$\begin{cases} X(0) + X(1) = 0 \\ X'(0) + X'(1) = 0. \end{cases} \quad (2.5)$$

The general solution of equation (2.4) is

$$X(x) = C_1 \cos \lambda x + C_2 \sin \lambda x, \quad (2.6)$$

where  $C_1, C_2$  are real constants and  $\lambda$  is a complex parameter. Applying the boundary conditions (2.5) to the general solution (2.6) implies

$$\begin{cases} C_1 + C_1 \cos \lambda + C_2 \sin \lambda = 0 \\ \lambda C_2 - \lambda C_1 \sin \lambda + \lambda C_2 \cos \lambda = 0 \end{cases} \quad (2.7)$$

Determinant of this system can be written as follows

$$\begin{aligned} \begin{vmatrix} 1 + \cos \lambda & \sin \lambda \\ -\lambda \sin \lambda & \lambda + \lambda \cos \lambda \end{vmatrix} &= \lambda(1 + \cos \lambda)^2 + \lambda \sin^2 \lambda \\ &= \lambda(1 + 2 \cos \lambda + \cos^2 \lambda + \sin^2 \lambda) \\ &= 2\lambda(1 + \cos \lambda) = 4\lambda \cos^2 \frac{\lambda}{2} \end{aligned}$$

Then its roots are:

$$\lambda_0 = 0, \quad \lambda_k = (2k + 1)\pi, \quad k \in \mathbb{Z} - \{0\}$$

It is easy to see that the  $\lambda = 0$  cannot be an eigenvalue. Thus the eigenvalues and related eigenfunctions are

$$\begin{cases} \lambda_k = (2k + 1)\pi \\ X_{k1}(x) = C_1 \cos \lambda_k x, \quad X_{k2}(x) = C_2 \sin \lambda_k x \end{cases}$$

The linearly independent eigenfunctions are only for eigenvalues  $\lambda_k = (2k + 1)\pi$ ,  $k \in \mathbb{N} \cup \{0\}$ .

**Remark 1.** *The spectral problem (2.4)-(2.5) of the given problem (2.1)-(2.3) is a self-adjoint problem.*

**2.3. Calculating formal solution.** The formal solution of equation (2.1) will be given in the following form

$$u(x, t) = \sum_{k=0}^{\infty} [A_k(t)X_{k1}(x) + B_k(t)X_{k2}(x)], \quad (2.8)$$

where the coefficients  $A_k$ ,  $B_k$  are arbitrary functions with respect to time and  $X_{k1}$ ,  $X_{k2}$  are eigenfunctions.

Substituting this series in equation (2.1) yields

$$\begin{aligned} & \sum_{k=0}^{\infty} [A_k''(t)X_{k1}(x) + B_k''(t)X_{k2}(x)] \\ &= - \sum_{k=0}^{\infty} ((2k+1)\pi)^2 [A_k(t)X_{k1}(x) + B_k(t)X_{k2}(x)] + f(x, t) \\ &= -\pi^2 \sum_{k=0}^{\infty} (2k+1)^2 [A_k(t)X_{k1}(x) + B_k(t)X_{k2}(x)] + f(x, t). \end{aligned}$$

The norms of eigenfunctions will be brought in below form:

$$\begin{aligned} \|X_{n1}\|^2 (A_n''(t) + \pi^2(2n+1)^2 A_n(t)) &= \int_0^1 f(x, t)X_{n1}(x)dx, \\ \|X_{n2}\|^2 (B_n''(t) + \pi^2(2n+1)^2 B_n(t)) &= \int_0^1 f(x, t)X_{n2}(x)dx. \end{aligned}$$

The result from above relations is expressed ordinary differential equations for  $A_n(t)$  and  $B_n(t)$ :

$$\begin{cases} A_n''(t) + \pi^2(2n+1)^2 A_n(t) = f_{n1}(t) \equiv \int_0^1 f(x, t)X_{n1}(x)dx \\ B_n''(t) + \pi^2(2n+1)^2 B_n(t) = f_{n2}(t) \equiv \int_0^1 f(x, t)X_{n2}(x)dx \end{cases} \quad (2.9)$$

The solutions response are expressed at below

$$\begin{cases} A_n(t) = A_{n1} \cos(2n+1)\pi t + A_{n2} \sin(2n+1)\pi t \\ \quad + \int_0^t \frac{\sin(2n+1)\pi(t-\tau)}{(2n+1)\pi} f_{n1}(\tau) d\tau \\ B_n(t) = B_{n1} \cos(2n+1)\pi t + B_{n2} \sin(2n+1)\pi t \\ \quad + \int_0^t \frac{\sin(2n+1)\pi(t-\tau)}{(2n+1)\pi} f_{n2}(\tau) d\tau \end{cases} \quad (2.10)$$

where  $A_{n1}$ ,  $A_{n2}$ ,  $B_{n1}$  and  $B_{n2}$  are arbitrary real constants. Replacing these solutions in the series (2.8) gives the formal solution of equation (2.1), that is brought in the below

$$\begin{aligned}
u(x, t) = & \sum_{k=0}^{\infty} [A_{k1} \cos(2k+1)\pi t + A_{k2} \sin(2k+1)\pi t] X_{k1}(x) \\
& + \sum_{k=0}^{\infty} [B_{k1} \cos(2k+1)\pi t + B_{k2} \sin(2k+1)\pi t] X_{k2}(x) \\
& + \sum_{k=0}^{\infty} \left\{ X_{k1}(x) \int_0^t \frac{\sin(2k+1)\pi(t-\tau)}{(2k+1)\pi} f_{k1}(\tau) d\tau \right\} \\
& + \left\{ X_{k2}(x) \int_0^t \frac{\sin(2k+1)\pi(t-\tau)}{(2k+1)\pi} f_{k2}(\tau) d\tau \right\}
\end{aligned} \tag{2.11}$$

**Remark 2.** *The series solution (2.11) is satisfied equation (2.1) and each term of this series is satisfied boundary conditions (2.2).*

The coefficients of  $A_{n1}$ ,  $A_{n2}$ ,  $B_{n1}$  and  $B_{n2}$  are determined by imposing the given initial conditions (2.3) for the  $u(x, t)$  and  $\frac{\partial u(x, t)}{\partial t}$ .

$$\begin{cases} \sum_{k=0}^{\infty} [A_{k1} X_{k1}(x) + B_{k1} X_{k2}(x)] = \varphi_0(x) \\ \sum_{k=0}^{\infty} (2k+1)\pi [A_{k2} X_{k1}(x) + B_{k2} X_{k2}(x)] = \varphi_1(x) \end{cases}$$

Thus these coefficients are calculated in the following form

$$\begin{cases} A_{n1} = \int_0^1 \varphi_0(\xi) X_{n1}(\xi) d\xi, & B_{n1} = \int_0^1 \varphi_0(\xi) X_{n2}(\xi) d\xi \\ A_{n2} = \frac{1}{(2n+1)\pi} \int_0^1 \varphi_1(\xi) X_{n1}(\xi) d\xi & B_{n2} = \frac{1}{(2n+1)\pi} \int_0^1 \varphi_1(\xi) X_{n2}(\xi) d\xi \end{cases} \tag{2.12}$$

**2.4. Compatibility and Sufficient Conditions for convergence of the Formal Series solution.** In this section, some compatibility and sufficient conditions will be obtained for the formal series (2.8) to have a classic solution. To this end, we need to prove following theorem.

**Theorem 1.** *Suppose the functions  $\varphi_0$  and  $\varphi_1$  in the initial conditions (2.3) of problem (2.1)-(2.3) satisfy the following conditions:*

$$\varphi_0^{(k)}(1) = \varphi_0^{(k)}(0), \quad k = 0, 1, 2, 3 \quad \text{and} \quad \varphi_0(x) \in C^{(4)}(0, 1) \tag{2.13}$$

$$\varphi_1^{(k)}(1) = \varphi_1^{(k)}(0), \quad k = 0, 1, \quad \text{and} \quad \varphi_1(x) \in C^{(3)}(0, 1), \tag{2.14}$$

and suppose the function  $f(x, t)$  satisfies:

$$f(1, t) = f(0, t), \quad t > 0 \tag{2.15}$$

and also its derivatives  $\frac{\partial^2 f(x, t)}{\partial x \partial t}$  is differentiable at  $x = 0, x = 1, t > 0$  and there is  $M \in \mathbb{N}$  such that

$$\left| \frac{\partial^3 f(x, t)}{\partial x^2 \partial t} \right| \leq M \quad (2.16)$$

Then the series solution (2.8) is a uniformly convergent series with respect to  $x$  and  $t$ .

In the other words the classic solution of main problem (2.1)-(2.3) is given by formal solution (2.11).

*Proof.* For establishing uniformly convergence of series (2.11), consider the asymptotic expansion of the coefficients of  $A_{n1}$ ,  $A_{n2}$ ,  $B_{n1}$  and  $B_{n2}$ . we give some sufficient conditions. For this by applying repeatedly integration by parts, we get

$$\begin{aligned} A_{n1} &= \sqrt{2} \int_0^1 \varphi_0(\xi) \cos \lambda_n \xi \, d\xi = \sqrt{2} \varphi_0(\xi) \frac{\sin(2n+1)\pi\xi}{(2n+1)\pi} \Big|_0^1 \\ &\quad - \sqrt{2} \int_0^1 \varphi_0'(\xi) \frac{\sin(2n+1)\pi\xi}{(2n+1)\pi} \, d\xi \\ &= \frac{\sqrt{2} \varphi_0'(\xi)}{(2n+1)^2 \pi^2} \cos(2n+1)\pi\xi \Big|_0^1 \\ &\quad - \frac{\sqrt{2}}{(2n+1)^2 \pi^2} \int_0^1 \varphi_0''(\xi) \cos(2n+1)\pi\xi \, d\xi \\ &= -\sqrt{2} \frac{\varphi_0'(1) - \varphi_0'(0)}{(2n+1)^2 \pi^2} + \frac{\sqrt{2}}{(2n+1)^4 \pi^4} (\varphi_0'''(1) - \varphi_0'''(0)) \\ &\quad + \frac{\sqrt{2}}{(2n+1)^4 \pi^4} \int_0^1 \varphi_0^{(4)}(\xi) \cos(2n+1)\pi\xi \, d\xi \end{aligned} \quad (2.17)$$

By the same way for  $B_{n1}$ ,  $A_{n2}$  and  $B_{n2}$  we will get:

$$\begin{aligned} B_{n1} &= \sqrt{2} \frac{\varphi_0(1) - \varphi_0(0)}{(2n+1)\pi} - \sqrt{2} \frac{\varphi_0''(1) - \varphi_0''(0)}{(2n+1)^3 \pi^3} \\ &\quad + \frac{\sqrt{2}}{(2n+1)^4 \pi^4} \int_0^1 \varphi_0^{(4)}(\xi) \sin(2n+1)\pi\xi \, d\xi, \end{aligned} \quad (2.18)$$

$$A_{n2} = -\sqrt{2} \frac{\varphi_1'(1) - \varphi_1'(0)}{(2n+1)^3 \pi^3} - \frac{\sqrt{2}}{(2n+1)^3 \pi^3} \int_0^1 \varphi_1''(\xi) \cos(2n+1)\pi\xi \, d\xi \quad (2.19)$$

and

$$B_{n2} = \sqrt{2} \frac{\varphi_1(1) - \varphi_1(0)}{(2n+1)^2 \pi^2} - \frac{\sqrt{2}}{(2n+1)^3 \pi^3} \int_0^1 \varphi_1''(\xi) \sin(2n+1)\pi\xi \, d\xi \quad (2.20)$$

Relations (2.13)-(2.14) and the factors  $\frac{\sqrt{2}}{(2n+1)^3\pi^3}$  and  $\frac{\sqrt{2}}{(2n+1)^4\pi^4}$  of integrals in (2.17), (2.18), (2.19), and (2.20) guaranty the uniformly convergence of series solution (2.11). We need only to consider the asymptotic expansion of coefficients  $f_{k1}$  and  $f_{k2}$ . For this we can write

$$\begin{aligned} f_{k1}(\tau) &= \sqrt{2} \int_0^1 f(x, \tau) \cos(2k+1)\pi x dx \\ &= \frac{\sqrt{2}}{(2k+1)^2\pi^2} \frac{\partial f(x, \tau)}{\partial x} \cos(2k+1)\pi x \Big|_{x=0}^{x=1} \\ &\quad - \frac{\sqrt{2}}{(2k+1)^2\pi^2} \int_0^1 \frac{\partial^2 f(x, \tau)}{\partial x^2} \cos(2k+1)\pi x dx \\ &= \frac{\sqrt{2}}{(2k+1)^2\pi^2} \left[ \frac{\partial f(x, \tau)}{\partial x} \Big|_{x=1} - \frac{\partial f(x, \tau)}{\partial x} \Big|_{x=0} \right] \\ &\quad - \frac{\sqrt{2}}{(2k+1)^2\pi^2} \int_0^1 \frac{\partial^2 f(x, \tau)}{\partial x^2} \cos(2k+1)\pi x dx \end{aligned}$$

consequently we have:

$$\begin{aligned} &\int_0^t \frac{\sin(2k+1)\pi(t-\tau)}{(2k+1)\pi} f_{k1}(\tau) d\tau \\ &= \frac{\sqrt{2}}{(2k+1)^3\pi^3} \int_0^t \left[ \frac{\partial f(x, \tau)}{\partial x} \Big|_{x=1} - \frac{\partial f(x, \tau)}{\partial x} \Big|_{x=0} \right] \sin(2k+1)\pi(t-\tau) \\ &\quad - \frac{\sqrt{2}}{(2k+1)^3\pi^3} \int_0^t \sin 2k\pi(t-\tau) d\tau \int_0^1 \frac{\partial^2 f(x, \tau)}{\partial x^2} \cos(2k+1)\pi x dx \end{aligned} \quad (2.21)$$

By the same way we have for  $f_{k2}$

$$\begin{aligned} f_{k2}(\tau) &= \sqrt{2} \int_0^1 f(x, \tau) \sin(2k+1)\pi x dx \\ &= -\sqrt{2} \frac{f(1, \tau) - f(0, \tau)}{(2k+1)\pi} + \frac{\sqrt{2}}{(2k+1)^2\pi^2} \int_0^1 \frac{\partial^2 f(x, \tau)}{\partial x^2} \sin(2k+1)\pi x dx \end{aligned}$$

And this implies

$$\begin{aligned} &\int_0^t \frac{\sin(2k+1)\pi(t-\tau)}{(2k+1)\pi} f_{k2}(\tau) d\tau \\ &= -\frac{\sqrt{2}}{(2k+1)^2\pi^2} \int_0^1 [f(1, \tau) - f(0, \tau)] \sin(2k+1)\pi x dx \\ &\quad - \frac{\sqrt{2}}{(2k+1)^3\pi^3} \int_0^t \sin(2k+1)\pi(t-\tau) d\tau \int_0^1 \frac{\partial^2 f(x, \tau)}{\partial x^2} \sin(2k+1)\pi x dx \end{aligned} \quad (2.22)$$

Finally the relation (2.15), (2.16), (2.17), (2.21) and (2.22) guaranty the uniformly convergence of series solution (2.11).  $\square$

**2.5. Uniqueness of solution.** In this section the uniqueness of solution will be shown. The solution of initial-boundary value problem (2.1)-(2.3) which was presented by (2.11) is unique. For this, we prove that the homogeneous problem corresponding to initial-boundary value problem (2.1)-(2.3) has only a trivial solution.

Multiplying  $\frac{\partial u(x, \tau)}{\partial \tau}$  to the both side homogeneous wave equation and integration on  $[0, 1] \times [0, t]$  implies

$$\int_0^1 dx \int_0^t \frac{\partial^2 u(x, \tau)}{\partial \tau^2} \frac{\partial u(x, \tau)}{\partial \tau} d\tau = \int_0^t d\tau \int_0^1 \frac{\partial^2 u(x, \tau)}{\partial x^2} \frac{\partial u(x, \tau)}{\partial \tau} dx$$

Integrating by parts method yields

$$\begin{aligned} \frac{1}{2} \int_0^1 dx \int_0^t \frac{\partial}{\partial \tau} \left( \frac{\partial u(x, \tau)}{\partial \tau} \right)^2 d\tau &= \int_0^t d\tau \left[ \frac{\partial u(x, \tau)}{\partial x} \frac{\partial u(x, \tau)}{\partial \tau} \Big|_{x=0}^1 \right. \\ &\left. - \int_0^1 \frac{\partial u(x, \tau)}{\partial x} \frac{\partial^2 u(x, \tau)}{\partial x \partial \tau} dx \right] \end{aligned}$$

Applying the boundary and compatibility conditions to the first integral on the right-hand side leads to

$$\begin{aligned} &\int_0^t d\tau \frac{\partial u(x, \tau)}{\partial x} \frac{\partial u(x, \tau)}{\partial \tau} \Big|_{x=0}^{x=1} \\ &= \int_0^t d\tau \left[ \left( -\frac{\partial u(x, \tau)}{\partial x} \Big|_{x=0} \right) \left( -\frac{\partial u(x, \tau)}{\partial \tau} \Big|_{x=0} \right) \right. \\ &\quad \left. - \frac{\partial u(x, \tau)}{\partial x} \Big|_{x=0} \frac{\partial u(x, \tau)}{\partial \tau} \Big|_{x=0} \right] = 0 \end{aligned}$$

Consequently we will have:

$$\frac{1}{2} \int_0^1 \left( \frac{\partial u(x, t)}{\partial t} \right)^2 dx = -\frac{1}{2} \int_0^1 dx \int_0^t \frac{\partial}{\partial \tau} \left( \frac{\partial u(x, \tau)}{\partial x} \right)^2 d\tau$$

or

$$\int_0^1 \left[ \left( \frac{\partial u(x, t)}{\partial t} \right)^2 + \left( \frac{\partial u(x, t)}{\partial x} \right)^2 \right] dx = 0$$

Hence  $u(x, t)$  is a constant. Regarding the initial and boundary value conditions, we get  $u(x, t) \equiv 0$ .

### 3. NON SELF-ADJOINT CASE WITH COMPLEX EIGENVALUES

**3.1. Mathematical Statement of problem.** In this section the wave equation with following boundary and initial conditions is considered:

$$\frac{\partial^2 u(x, t)}{\partial t^2} = \frac{\partial^2 u(x, t)}{\partial x^2}, \quad x \in (0, 1), \quad t > 0, \quad (3.1)$$

$$2u(0, t) = u(1, t), \quad (3.2)$$

$$\frac{\partial u(x, t)}{\partial x} \Big|_{x=0} = 0, \quad t \geq 0.$$

$$\frac{\partial^k u(x, 0)}{\partial t^k} = \varphi_k(x), \quad k = 0, 1, \quad x \in [0, 1] \quad (3.3)$$

**3.2. Non-Self-adjoint spectral problem.** At first the associated spectral problem are obtained. For this, by using the similar manner which was applied in section 1, the associated spectral problem

$$\begin{cases} X''(x) - \lambda^2 X(x) = 0 \\ X'(0) = 0, \quad X(1) - 2X(0) = 0 \end{cases} \quad (3.4)$$

is not a self-adjoint problem. Note that the resulted boundary conditions for adjoint problem will be different from boundary conditions of main problem (3.4). The related eigenvalues and eigenfunctions for main spectral problem (3.4) are

$$\begin{cases} \lambda_{k1} = \ln(2 + \sqrt{3}) + 2k\pi i, & \lambda_{k2} = \ln(2 - \sqrt{3}) + 2k\pi i \\ X_{k1} = e^{\lambda_{k1}x} + e^{-\lambda_{k1}x}, & X_{k2} = e^{\lambda_{k2}x} + e^{-\lambda_{k2}x} \end{cases} \quad (3.5)$$

and its related adjoint problem is

$$\begin{cases} Z''(x) - \rho^2 Z(x) = 0 \\ Z(1) = 0, \quad Z'(0) - 2Z'(1) = 0, \end{cases} \quad (3.6)$$

also the related eigenvalues and eigenfunctions are

$$\begin{cases} \rho_{k1} = \ln(2 + \sqrt{3}) + 2k\pi i, & \rho_{k2} = \ln(2 - \sqrt{3}) + 2k\pi i \\ Z_{k1} = e^{\rho_{k1}x} + e^{-\rho_{k1}x}, & Z_{k2} = e^{\rho_{k2}x} + e^{-\rho_{k2}x} \end{cases} \quad (3.7)$$

One can verify that the eigenfunctions  $X_{k1}$  and  $X_{k2}$  of problem (3.4) are not linear independent. Also the eigenfunctions  $Z_{k1}$  and  $Z_{k2}$  of related adjoint problem (3.6) are not linear independent. This subject causes the eigenfunctions of the main problem (3.4) and adjoint problem (3.6) can not form a complete basis system separately. For this, we should join eigenfunctions of these problems together for constructing a complete basis system.

At first these functions are orthogonal, that is

$$(X_k, Z_n) = \int_0^1 (e^{\lambda_k x} + e^{-\lambda_k x})(e^{\bar{\rho}_n x} - e^{2\bar{\rho}_n - \bar{\rho}_n x}) dx = 0, \quad k \neq n \quad (3.8)$$

$$(X_n, Z_n) = -6 - 4\sqrt{3} \neq 0$$

Generally

$$(X_k, Z_n) = \delta_{kn}[-2(3 + 2\sqrt{3})], \quad k, n \in \mathbb{Z} \quad (3.9)$$

The relations (3.8) and (3.9) show that the eigenfunctions (3.5) and (3.7) form a bi-orthogonal basis complete system.

**3.3. Constructing formal solution.** Firstly from time equation

$$T''(t) - \lambda^2 T(t) = 0 \quad (3.10)$$

we have

$$T_n(t) = A_n e^{-\lambda_n t} + B_n e^{\lambda_n t}. \quad (3.11)$$

Then we can write the formal solution for the main boundary-initial value problem (3.1)-(3.3) as follows:

$$u(x, t) = \sum_{n \in \mathbb{Z}} (A_n e^{-\lambda_n t} + B_n e^{\lambda_n t}) X_n(x) \quad (3.12)$$

Now the unknown coefficients  $A_n$  and  $B_n$  in (3.12) are determined. Imposing the initial condition (3.3) to the  $u(x, t)$  and  $\frac{\partial u(x, t)}{\partial t}$  yields

$$\begin{aligned} \sum_{n=0}^{\infty} [A_n + B_n] X_n(x) &= \varphi_0(x) \\ \sum_{n=0}^{\infty} \lambda_n [-A_n + B_n] X_n(x) &= \varphi_1(x) \end{aligned}$$

Multiplying the eigenfunctions  $Z_k(x)$  to the both side of above relations gives:

$$\begin{aligned} \sum_{n=0}^{\infty} [A_n + B_n] (X_n, Z_k) &= (\varphi_0, Z_k) = -2(A_k + B_k)(3 + 2\sqrt{3}) \\ \sum_{n=0}^{\infty} \lambda_n [-A_n + B_n] (X_n, Z_k) &= (\varphi_1, Z_k) = 2\lambda_k (A_k - B_k)(3 + 2\sqrt{3}) \end{aligned}$$

Then

$$\begin{aligned} A_k &= \frac{(\varphi_1, Z_k) - \lambda_k (\varphi_0, Z_k)}{2\lambda_k (3 + 2\sqrt{3})} \\ B_k &= \frac{(\varphi_1, Z_k) + \lambda_k (\varphi_0, Z_k)}{2\lambda_k (3 + 2\sqrt{3})} \end{aligned} \quad (3.13)$$

Finally the formal solution of problem (3.1)-(3.3) is given by

$$u(x, t) = \sum_{n=-\infty}^{\infty} \frac{(e^{-\lambda_n t} - e^{\lambda_n t})(\varphi_1, Z_n) - \lambda_n(e^{-\lambda_n t} + e^{\lambda_n t})(\varphi_0, Z_n)}{2\lambda_n(3 + 2\sqrt{3})} X_n(x) \quad (3.14)$$

At the end of the following theorem, some sufficient conditions are given to have a classic solution for the problem (3.1)-(3.3).

**Theorem 2.** *Suppose the functions  $\varphi_0(x)$  and  $\varphi_1(x)$  in the problem (3.1)-(3.3) satisfy the following conditions*

$$\varphi_0(0) = \varphi_0(1) = \varphi_0'(0) = \varphi_0'(1) = \varphi_0''(0) = \varphi_0''(1), \quad \varphi_0 \in C^{(4)}(0, 1) \quad (3.15)$$

$$\varphi_1(0) = \varphi_1(1) = \varphi_1'(0) = \varphi_1'(1), \quad \varphi_1 \in C^{(3)}(0, 1) \quad (3.16)$$

Then the series solution (3.14) and its second derivatives with respect to  $x$  and  $t$  are uniformly convergence.

*Proof.* Consider the asymptotic expansion of  $(\varphi_0, Z_n)$

$$\begin{aligned} (\varphi_0, Z_n) &= \int_0^1 \varphi_0(x) \overline{Z_n(x)} dx = \int_0^1 \varphi_0(x) [e^{\overline{\rho_n}x} - e^{2\overline{\rho_n} - \overline{\rho_n}x}] dx \\ &= \varphi_0(x) \frac{e^{\overline{\rho_n}x} + e^{2\overline{\rho_n} - \overline{\rho_n}x}}{\overline{\rho_n}} \Big|_0^1 - \varphi_0'(x) \frac{e^{\overline{\rho_n}x}}{\overline{\rho_n}^2} + \int_0^1 \varphi_0''(x) \frac{e^{\overline{\rho_n}x}}{\overline{\rho_n}^2} dx \\ &\quad - e^{2\overline{\rho_n}} \varphi_0'(x) \frac{e^{-\overline{\rho_n}x}}{-\overline{\rho_n}^2} \Big|_0^1 - \frac{e^{2\overline{\rho_n}}}{\overline{\rho_n}^2} \int_0^1 \varphi_0''(x) e^{\overline{\rho_n}x} dx \end{aligned}$$

Continuing this process and considering the relations (3.15) and the factors  $\frac{e^{2\overline{\rho_n}}}{-\overline{\rho_n}^2}$  of integral and by writing same asymptotic expansion for  $(\varphi_1, Z_n)$  and relation (3.16) is resulted uniformly convergence of series solution (3.14).  $\square$

The uniqueness of the solution (3.14) can be proved by similar manner which was used in Subsection 1.5.

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