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REVERSE LOOP SUBDIVISION FOR GEOMETRY AND TEXTURES

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Abstract. Reverse subdivision aims at constructing a coarser representation of an object given by a fine polygon mesh. In this paper, we first derive a mask for reverse Loop subdivision that can be applied to both regular and extraordinary vertices. The mask is parameterized, and thus can also be used in reversing variants of Loop subdivision, such as those proposed by Warren and Litke.

We apply this mask not only to mesh geometry, but also to texture coordinates. This reverses the texture-mapping process described by DeRose, Kass and Truong, in which a texture originally defined for a coarse mesh was carried to the finer meshes obtained by subdivision. Combined with the forward subdivision, the proposed technique constitutes a multiresolution representation of textured subdivision surfaces. We illustrate its use with a set of examples.

Keywords: Subdivision, Texture mapping, Multiresolution, Meshes

2000 Mathematics subject classification: 68U05, 68U07

1. Introduction

A hierarchy of level-of-detail approximations of a surface can be applied in both *Modeling* and *Rendering*. Subdivision methods can provide the hierarchy of refining approximations; through the use of subdivision, a sequence of meshes in several resolutions is constructed.

In order to have a texture mapping for subdivision surfaces, the rule of transforming vertices of a coarse mesh to a fine one can be used for carrying

texture coordinates from the coarse mesh to the fine mesh. Figure 1 illustrates this concept which was introduced in [4]. By using this technique, we only need a texture map for the given coarse mesh. The texture coordinates of the subsequent finer meshes are carried from the coarse mesh during subdivision. This technique is more advantageous when we encounter time-consuming texture mapping methods such as those that are procedural [4] and interactive [15]. Texture mapping for subdivision surfaces has been also considered in [21] and [16].

Here we consider the inverse problem: How can texture coordinates be carried from a fine mesh over to a coarse one? We describe the possibility of recovering coarse meshes, together with their texture coordinates, from the fine mesh using reverse subdivision. Therefore, with this technique, it is no longer necessary to have coarse meshes and their texture coordinates. This new technique, together with the technique of DeRose, Kass and Truong [4], can generate a hierarchy of meshes with their texture coordinates in different levels-of-detail, so that all meshes and their texture coordinates can be produced from a specific mesh by the use of subdivision and its reverse rules (Figure 10).

This hierarchy is beneficial for various applications, such as view dependent rendering and progressive transmission. Consider looking at an object from close and distant view points. We can use a fine mesh for a close view and a coarse mesh for a distant view; and can easily change the quality or resolution of the mesh both ways by using this hierarchy. Texture coordinates can also be transformed in this process and repeating the texture mapping at each resolution can be avoided. In other words, only one approximation mesh of a surface together with its texture coordinate is needed, and all other approximations in different levels-of-detail, coarse or fine, can be obtained.

Therefore, our framework is a multiresolution hierarchy (MR) that is an extension of subdivision. This is consistent with the framework of [3, 13, 17, 22]. The progressive mesh (PM) [8, 20, 14], is another framework for constructing a hierarchy of several level-of-details of an object. It can be obtained by using vertex-split and collapse operations. PM can be applied to a mesh with an arbitrary topology, while MR can be applied directly only on those meshes that satisfy whatever connectivity conditions are required by the subdivision rule being used. Although, this is a restriction for the MR approach, nevertheless, subdivision methods are widely used in various graphics software and applications. Thus, subdivision connectivity conditions are achievable and not too restrictive in practice. In addition, several techniques have been proposed for *remeshing*, [7, 9, 10], which replace meshes unsuitable for a given subdivision into approximating meshes that are suitable, which broadens the applicability of MR. Furthermore, each step of any subdivision scheme converts a low frequency approximation of a surface to a high one, and analogously each step of reverse subdivision converts a high frequency approximation to a low one. Hence, the hierarchy obtained has a suitable configuration consistent with some specific applications such as (MR) editing and progressive transmission.

A mask for decreasing the resolution of a mesh is an important tool for (MR) surfaces. There is an analogous mask for refining, or increasing, the resolution of a mesh, in the case of subdivision surfaces. While the mask of a subdivision surface usually is a local formula, finding a mask (local formula) for the reverse subdivision is problematic.

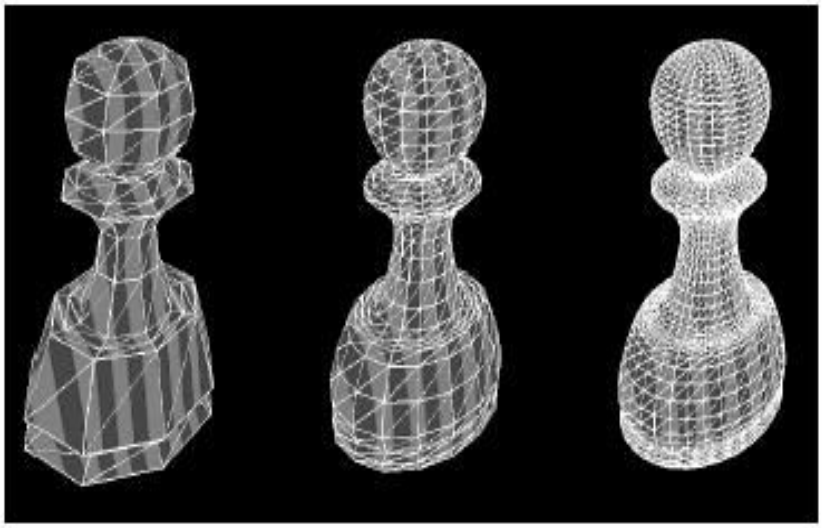

FIGURE 1. Carrying texture from the coarse mesh to fine mesh through Loop subdivision scheme.

The multiresolution masks of Butterfly and Loop subdivision for regular vertices (all vertices have the valence 6), are determined by Samavati and Bartels in [19] and for Doo-Sabin subdivision in [18]. Here we construct a reversal for Loop subdivision that works for extra-ordinary vertices as well. Loop subdivision is a very good selection because its limit surface is smooth, it has a convenient local formula for extra-ordinary vertices, and it is based on triangular meshes. Loop subdivision has been widely used in computer graphics [24]; therefore, constructing a reverse mask of Loop subdivision is advantageous. There are other variations of Loop subdivision such as Warren and Litke that we take them to consideration as well. Therefore, we construct a parametric reverse mask that works for all Loop style subdivisions.

Although, meshes are mostly covered by regular vertices, there are a small number of extra-ordinary vertices on practical meshes. Therefore, determining a local formula for a reverse process is important. This local formula or parametric mask can be applied to both ordinary and extra-ordinary vertices. The main contributions of this work are the construction of this mask, together with the concept of carrying texture coordinates from a fine mesh to a coarse mesh.

The work in [19] provides both local reversal and local *detail* (*error*) masks for ordinary vertices. The detail masks relate to the wavelet coefficients of a

MR surface constructed as a biorthogonal system. The detail coefficients are important only if the fine mesh being reversed is not created by a subdivision, which is not the case we are considering here. The errors or details of the reverse method that relate to the wavelet coefficients are left for future works.

Section 2 gives the necessary background of this work. The construction of the reverse mask is described in section 3. Section 4 demonstrates the concept of carrying texture by using local masks.

2. Background

Subdivision is a repetitive refinement process that gradually converts a given coarse mesh to finer meshes to generate a smooth surface at the limit. An arbitrary mesh M can be denoted by the pair (F, V), where F shows the faces of M, and V denotes the vertices of M. Each element $\nu \in V$ has the spatial coordinates, (x, y, z), and each element $f \in F$ is assigned a list that includes all indices of its adjacent vertices in V.

Catmull-Clarck, Doo-Sabin, Butterfly and Loop subdivisions are some important cases [2, 5, 6, 12]. The input for subdivision methods is $M^0 = (F^0, V^0)$, a control mesh. In each step of a subdivision method, the mesh $M^k = (F^k, V^k)$ is converted to a new and finer mesh $M^{k+1} = (F^{k+1}, V^{k+1})$. This conversion is done through some local affine operations on V^k, together with a mapping process from the faces of F^k to those of F^{k+1}. The affine operations are usually described by masks, or matrices, that are smoothing filters. Consequently, by successively applying a subdivision method, a hierarchy $M^0, M^1, M^2, \ldots, M^k, \ldots$ is obtained that usually converges to a smooth surface.

2.1. Loop Subdivision.

Loop subdivision is an extension of triangular B-spline subdivision to general surfaces. In each step of this subdivision, each triangular face of F^k is replaced by four new triangles that form the faces of F^{k+1} (figure 2).

The set of new vertices V^{k+1} includes two types of vertices; some vertices in V^{k+1} have an analogous vertex in V^k, and are called vertex-vertex or even vertex (vertex ν^{k+1} in figure 2). Some vertices in V^{k+1} have an analogous edge in V^k, which is called edge-vertex or odd vertex (vertex ν_1^{k+1} in figure 2). Assume that vertices $\nu_1^k, \nu_2^k, \ldots, \nu_n^k$ are all neighbors of ν^k in M^k. In addition, ν^{k+1} is the corresponding vertex-vertex of ν^k and $\nu_1^{k+1}, \nu_2^{k+1}, \ldots, \nu_n^{k+1}$ are the corresponding edge-vertices. Then the position of ν^{k+1} is obtained by the vertex-vertex mask

$$\nu^{k+1} = \beta \nu^k + \alpha \sum_{j=1}^{n} \nu_i^k \,, \tag{2.1}$$

where

$$\alpha = \frac{1}{n}\left(\frac{5}{8} - \left(\frac{3}{8} + \frac{1}{4}\cos\left(\frac{2\pi}{n}\right)\right)^2\right), \quad \beta = 1 - n\alpha. \tag{2.2}$$

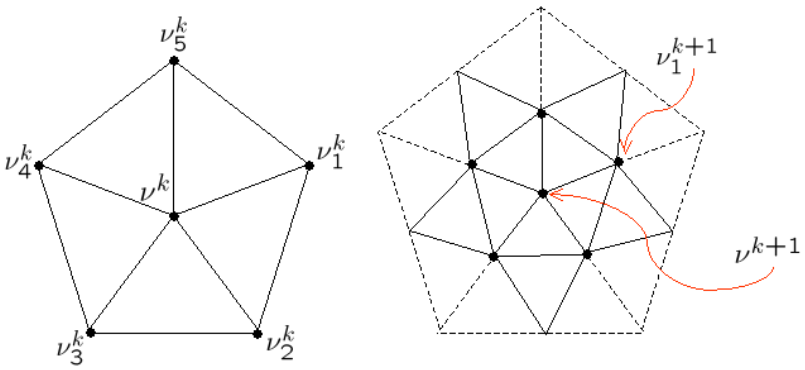

FIGURE 2. Situation around a vertex ν^k before and after subdivision.

The weight α is a function of n, and has been selected such that the limit surface is smooth.

The edge-vertex mask is

$$\nu_j^{k+1} = \frac{3}{8}\nu^k + \frac{3}{8}\nu_j^k + \frac{1}{8}\nu_{j+1}^k + \frac{1}{8}\nu_{j-1}, \quad j = 1, 2, \ldots, n \tag{2.3}$$

where indices are in module n.

For triangular meshes, a *regular vertex* (or ordinary vertex) has the valance of 6, i.e. $n = 6$. Otherwise, the vertex is called *extra-ordinary* ($n \neq 6$). In the regular case, α is $\frac{1}{16}$, and β is $\frac{5}{8}$ and these are also obtainable from the triangular B-spline surface. But masks (2.2) and (2.3) are more general and can be applied for any type of vertex including ordinary and extra-ordinary. The local formula (2.2) is one of the advantages of Loop subdivision. Note that both masks of equations (2.1) and (2.3) are affine operations. Figure (3) shows two consecutive meshes of Loop subdivision and their limit surface.

2.2. **Boundary.** When we encounter boundary vertices, we need to use boundary masks that are usually different from the interior mask. It is important that subdivision at any point on the boundary be independent of any point in the interior of the mesh [24]. This permits two surfaces to be joined along a boundary curve. Therefore, cubic B-spline subdivision masks for curves can be used as the boundary masks of Loop subdivision

$$\begin{aligned} \nu^{k+1} &= \tfrac{1}{8}\nu_1^k + \tfrac{3}{4}\nu^k + \tfrac{1}{8}\nu_2^k, \\ \nu_1^{k+1} &= \tfrac{1}{2}\nu^k + \tfrac{1}{2}\nu_1^k, \\ \nu_2^{k+1} &= \tfrac{1}{2}\nu^k + \tfrac{1}{2}\nu_2^k, \end{aligned} \tag{2.4}$$

where ν_1^k and ν_2^k are two direct neighbors of ν^k on the boundary (see figure 4).

In figure 11 the left one is a mesh with boundary and the middle one shows the obtained mesh by Loop subdivision.

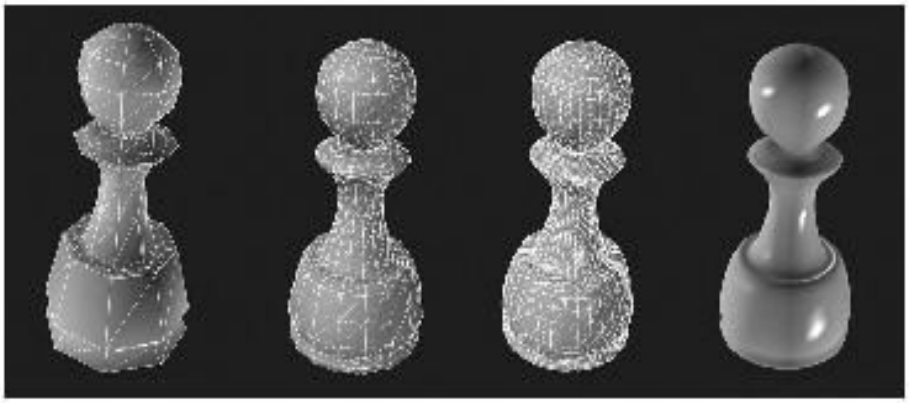

FIGURE 3. Loop subdivision.

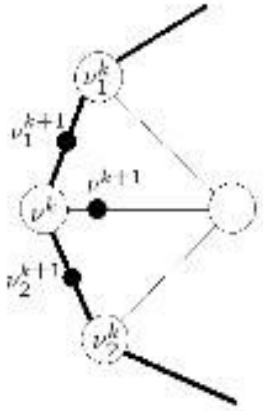

FIGURE 4. An illustration for the boundary mask.

2.3. **Loop Style Subdivisions.** We derive a reverse mask for Loop subdivision. This mask is parameterized and can also be applied in reversing other variants such as triangle averaging of Warren and Weimer [23] and quasi-interpolation scheme of Litke, Levin and Schröder [11]. The triangle averaging scheme can produce a smooth surface but not necessarily at extra-ordinary vertices. The α , β values of this scheme are

$$\alpha = \frac{3}{8n}, \quad \beta = \frac{5}{8}. \tag{2.5}$$

Therefore α for this scheme is simpler than (2.2). In figure 5, there is a comparison between Warren's scheme and the original Loop subdivision. In the quasi-interpolation scheme different mask values are used to obtain a quasi-interpolation limit surface. The mask values are

$$\alpha = -\tfrac{1}{2n}, \quad \beta = \tfrac{3}{2}. \tag{2.6}$$

The right part of figure 5 shows an example of this scheme.

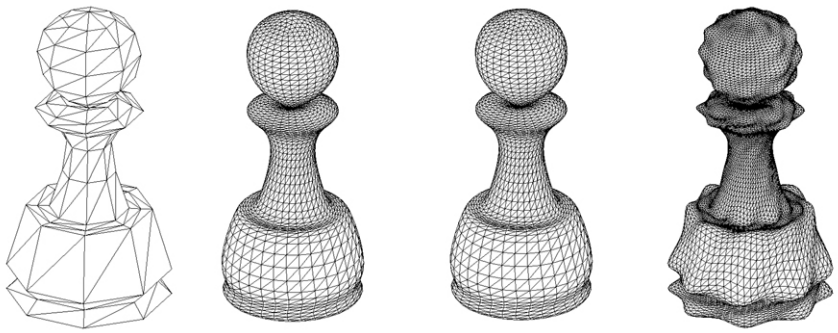

FIGURE 5. From left to right: the coarse mesh, Loop subdivision, Warren's scheme and the quasi-interpolation scheme.

2.4. **Texture Mapping.** The conventional texture mapping of subdivision surfaces is close to the technique of texture mapping of polyhedron meshes. The control polyhedron M is mapped to a fine mesh M^k such that M^k is smooth enough for the graphics pipeline and rendering. Then texture mapping techniques of polyhedron meshes are employed for M^k. In practical applications, sometimes we need a finer mesh, since, it is necessary to change the view point or the object's position. In this situation we have to continue the subdivision process on M^k in order to obtain a smoother mesh M^ℓ where $\ell > k$. Consequently, the texture mapping must be repeated for the new mesh, M^k. DeRose, Kass and Troung [4] provide the concept of carrying texture coordinates from the coarse mesh to the finer one by using the same subdivision rule that is applied to the vertices. More specifically, we assume that texture coordinates of the control mesh M^0 are somehow given. Each texture coordinate is a pair (s_j, t_j) assigned to the vertex $\nu_j \in V$. These coordinates can be obtained by some methods that are potentially expensive such as procedural, manual and interactive textures.

The pair (s_j, t_j) presents a point in the texture space whose value defines the intensity color of ν_j. t_j and s_j are numeric scalars and can be added to the spatial coordinates of ν_j. Therefore, the corresponding attribute of ν_j is $(x_j, y_j, z_j, s_j, t_j)$ where the first three components are the spatial coordinates and the two last components are the texture coordinates. Now, it is sufficient to apply the subdivision mask to elements of the five dimensional Euclidian space $\mathbf{E}^5$. By using this technique, not only it is not necessary to repeat texture mapping for finer surfaces, but also a smooth correlation between the texture of the coarse mesh and the fine mesh is provided (Figure 1). The

efficiency of this technique is substantial when complicated texture maps are encountered. The concept of increasing the space dimension of control points might be extended to include other rasterization features such as multi-textures and bump maps. DeRose at el.[4] have considered texture mapping for Catmull-Clark subdivision. Seng and Zhiyong [21] have considered it for Doo-Sabin subdivision. Piponi and Broshukov [16] provide a seamless texture mapping of subdivision surfaces. We extend this concept to offer the ability of carrying texture coordinates from a fine mesh to a coarse one. It relates to the reversal of subdivision and constructing a MR representation.

2.5. **Reverse Subdivision and Multiresolution Surfaces.** Subdivision methods give a hierarchy

$$M^0, M^1, M^2, \ldots, M^k, \ldots$$

where

$$M^k = (V^k, F^k).$$

Each $\nu \in V$ has a 5 space coordinate (x, y, z, s, t). Suppose M^0 is the control mesh of an object and M^k is a good approximation of the limit surface for rendering. Now, if the object position is changed to a new position that is closer to the view point, it will be necessary to construct a finer mesh M^{k+i}, $(i > 0)$. Thus, it is sufficient to repeat the subdivision rule for i times over M^k. In other words, having M^k is enough for all finer approximations. Contrarily, if the object position is moved to a distant position, it will be better to have a coarser approximation in respect to M^k. Therefore, we encounter this problem: *"How can coarser meshes $M^{k-\ell}$ be obtained from M^k?"* This question is interesting in two ways. Firstly, for $1 \leq \ell \leq k$, procedural reversal relieves us of the necessity of storing any information about $M^0, \ldots, M^{k-1}$, trading time for space. Secondly, for $\ell > k$, reversal provides us coarser meshes than M^0, which might have been given at a fairly high level of detail to begin with, due perhaps to the design process.

The reverse mask together with the subdivision mask provide a (MR) of the given object that is suitable for applications such as; view dependent rendering, flexible editing and progressive transmission. In section 3, we construct the reverse mask from the Loop subdivision mask. The method is general enough to be extended to other subdivision schemes, but in this work we just focus one Loop subdivision and its variants.

3. Reverse Mask of Loop Subdivision

For the reverse process, it is necessary to construct a mask to map V^{k+1} to V^k. Assume a general subdivision situation for an extra-ordinary vertex in Figure 6. In this Figure, we know $\nu^{k+1}, \nu_1^{k+1}, \nu_2^{k+1}, \ldots, \nu_n^{k+1}$ and we want to find ν^k by a new mask such that the following conditions are met:
(i) The operation of the new mask must be affine(to obtain a mask that maps points to points).
(ii) Weights of neighbors of ν^{k+1} in the mask must be equal (similar to Loop mask of equation (2.1).

(iii) The new mask must be a reverse of the subdivision mask *i.e.* the action of subdivision mask of Equation (2.1) and (2.2) on ν^k and its neighbors must exactly reconstruct ν^{k+1}.

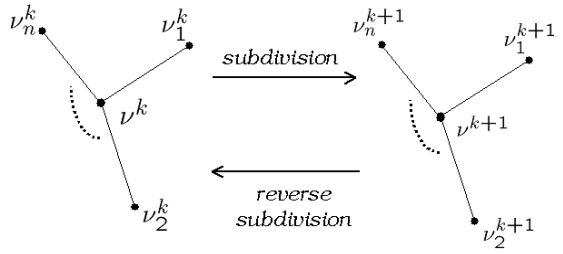

FIGURE 6. General situation for an extra-ordinary vertex.

Condition (ii) provides the diagram of figure 7 for the reverse mask. In this diagram μ is the weight of ν^{k+1} and η is the weight of the neighbors in the reverse mask. Note that the weight of all neighbors are equal to η. These weights are determined such that conditions (i) and (iii), also become true.

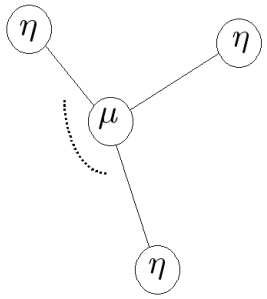

FIGURE 7. Reverse mask.

For condition (iii) (reversal), it is necessary to have

$$\mu\,\nu^{k+1} + \eta \sum_{j=1}^{n} \nu_j^{k+1} = \nu^k.$$

By use of Equation 2.1, we get

$$\mu(\beta\nu^k) + \mu\left(\alpha \sum_{j=1}^{n} \nu_j^k\right) + \eta\left(\frac{3}{8} n\nu^k\right) + \eta\left(\frac{5}{8} \sum_{j=1}^{n} \nu_j^k\right) = \nu^k,$$

or equivalently

$$\left(\mu\beta + \frac{3}{8} n\eta\right)\nu^k + \left(\mu\alpha + \frac{5}{8}\eta\right) \sum_{j=1}^{n} \nu_j^k = \nu^k.$$

To enforce the equality, we must have

$$\begin{cases} \mu\beta + \frac{3}{8}n\eta = 1 \\ \mu\alpha + \frac{5}{8}\eta = 0. \end{cases} \tag{3.1}$$

In this system, α, β are parameters of Loop subdivision mask in equation (2.2), and they satisfy the relation

$$\beta = 1 - n\alpha.$$

If we solve equation (3.1) with respect to β, then

$$\mu = \frac{5}{8\beta - 3}, \quad \eta = \frac{\beta - 1}{n\left(\beta - \frac{3}{8}\right)}. \tag{3.2}$$

The equation (3.2) is a parametric formula for the reverse mask and can be applied to both regular and extra-ordinary vertices. For example, in the case of regular vertex, i.e. $n = 6, \alpha = \frac{1}{16}, \beta = \frac{5}{8}$, equation (3.2) gives $\mu = \frac{5}{2}$ and $\eta = -\frac{1}{4}$. figure 8 diagrammatically presents this result, which exactly matches with the **A** mask of width 1 in [19]. However, the diagram in figure 7 together with the formula (3.2) can also be used for extra-ordinary cases.

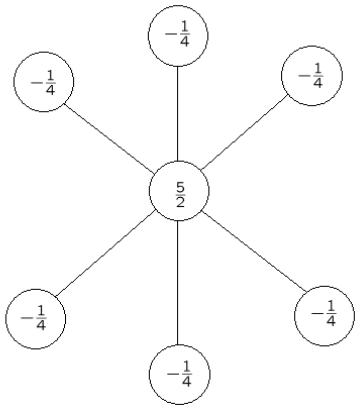

FIGURE 8. Reverse mask for regular vertex.

For an extra-ordinary example, let $n = 3$; thus, equation (2.2) gives

$$\alpha = \frac{3}{16}\,,\ \beta = \frac{7}{16}.$$

If we substitute α, β into equation (3.2), we obtain

$$\mu = 10\,,\ \eta = -3,$$

or diagrammatically as in figure 9.

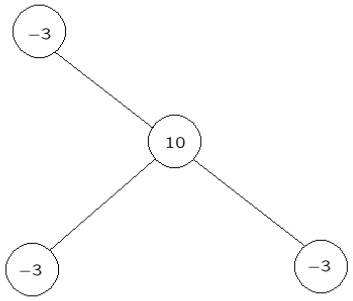

FIGURE 9. Reverse mask for an extra-ordinary vertex (n=3).

3.1. **Affine Invariance Property.** Both examples figure 8 and figure 9 form affine operations i.e. the sum of the weights is one. This property is generally correct, since

$$\mu + n\eta = 1,$$

where μ and η are determined from equation (3.2). In fact, the property (condition (i)) automatically becomes true for all reverse masks (condition (iii)). There is a short argument for this fact in Appendix.

3.2. **Reverse of Loop Style Subdivisions.** The formula (3.2) is a parameterized, and thus can be applied directly to Warren scheme (2.5) and quasi-interpolation scheme (2.6). This implies the following μ and η values for the reverse mask of the Warren scheme

$$\mu = \tfrac{5}{2}, \ \ \eta = -\tfrac{3}{2n}. \tag{3.3}$$

And the following values for the reverse mask of the quasi-interpolation scheme

$$\mu = \tfrac{5}{9}, \quad \eta = \tfrac{4}{9n}. \tag{3.4}$$

3.3. **Reverse Mask of the Boundary Vertices.** We have used cubic B-spline mask for boundary vertices as a pure curve scheme. Therefore, we need to find a reverse mask for the cubic B-spline subdivision. In Bartels and Samavati [1], several masks for cubic B-spline subdivision are provided. The simplest one is

$$\nu^k = -\frac{1}{2}\nu_1^{k+1} + 2\nu^{k+1} - \frac{1}{2}\nu_2^{k+1}. \tag{3.5}$$

Here the same notation of the section 2.2 is used. In figure 11, the boundary of the right surface has been obtained by application of the mask (3.5) on the boundary of the middle surface.

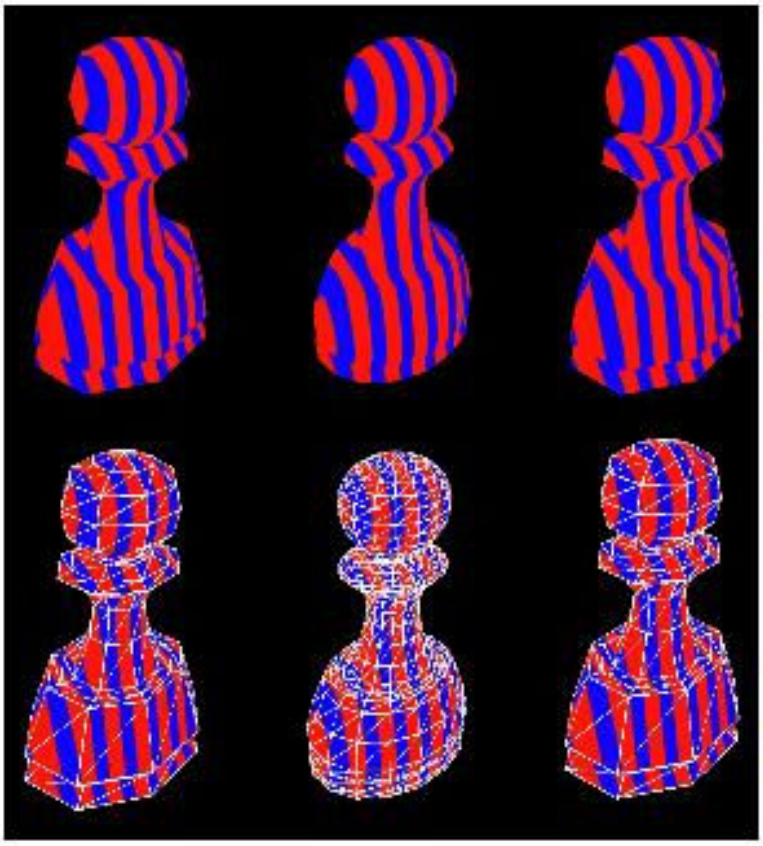

FIGURE 10. The top-left shows the textured control mesh of a pawn, and the top-middle is the fine mesh obtained by Loop subdivision, and top-right shows the obtained mesh from the reverse process. The second row shows the same meshes together with grid lines.

4. RESULTS

To demonstrate the quality of the proposing technique we show several examples. In these examples, Loop subdivision has been used for both increasing the resolution of a given mesh and transforming its texture coordinates. Our reverse mask (equation (3.2) has been used for decreasing the resolution together with texture coordinates.

For the pawn example in figure 10, a simple texture has been used for a better illustration of the concept. This figure shows one step of Loop subdivision and

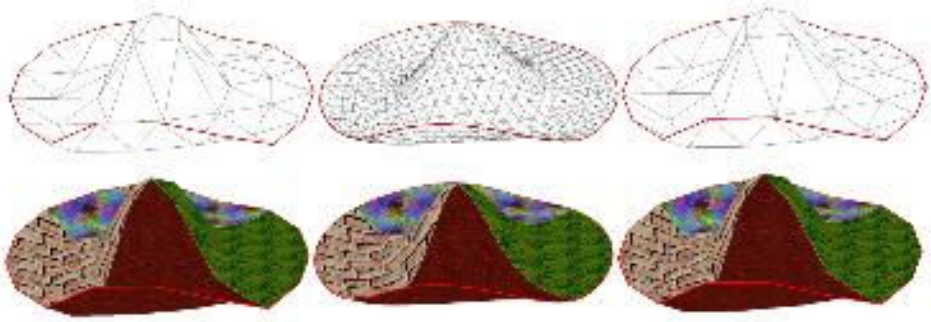

FIGURE 11. The top-left shows a simple control mesh with boundary, and the top-middle is the fine mesh obtained by Loop subdivision, and top-right shows the obtained mesh from the reverse process. The second row shows the same meshes together with a simple texture.

one level of the reverse process. In order to enhance clarity, a grid based outline versions of those meshes are also provided. The coarse mesh includes 154 vertices while the fine mesh includes 610 vertices.

The fish example in figure 14 shows the results after two steps of Loop subdivision, as well as two levels of the reverse process. The control mesh has 46 and the finest mesh has 390 vertices.

Figure 11 demonstrates the impact of the reverse subdivision on different kinds of textures. In this example, the boundary is highlighted to show the effect of the boundary masks.

The example in figure 12 shows the effect of Loop subdivision and its reverse scheme on a complicated mesh.

FIGURE 12. Left is a given control mesh, middle is the resulting mesh after one step of Loop subdivision and right is the resulting mesh after reversal.

Figure 13 shows the impact of repositioning the vertices in the fine mesh in the reverse process. The reverse mask of the quasi-interpolation scheme (3.4) produces a better result. This is due to the fact that Loop subdivision is a contracting scheme, therefore its inverse must expand the objects. In the other hand, the quasi-interpolating subdivision is an expansion scheme(see the equation (2.6), consequently its inverse must be a contraction scheme and reduces the impact of the perturbation. In fact, the values of μ and η in this case are positive, and consequently the resulting coarse mesh is contained in the convex hull of the fine mesh.

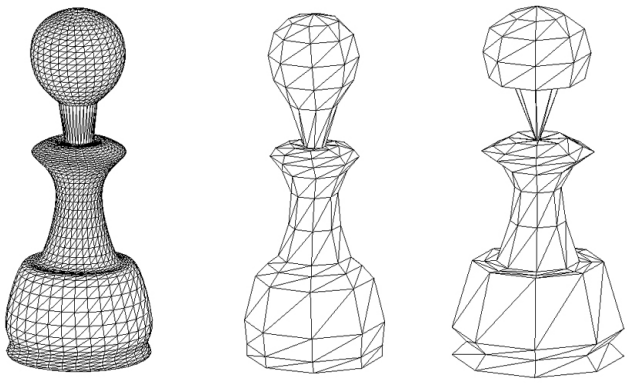

FIGURE 13. The left shows a modified fine mesh of the pawn, and the middle is the resulting coarse mesh via the mask 3.4, and the right shows the obtained mesh from the reverse Loop.

5. Conclusion

A reverse mask for Loop subdivision has been constructed. This mask is a parameterized formula and can be applied to other Loop style subdivisions. We have also described a technique for texture mapping of multiresolution surfaces using the reverse mask. We have demonstrated the effectiveness of the resulting method mostly when there is no large modification in the fine meshes.

In the case of a significant change to the fine mesh the reverse mask of Loop and Warren schemes show an unstable behavior, however the reverse mask of quasi-interpolation produce better results.

Bartels and Samavati [1] have constructed more stable reverse masks for curve scheme by increasing the width of curve masks. It is possible to employ and extend that concept to obtain wider but more stable reverse Loop masks. However, the resulting masks will be too complicated to compute and implement especially in extra-ordinary cases.

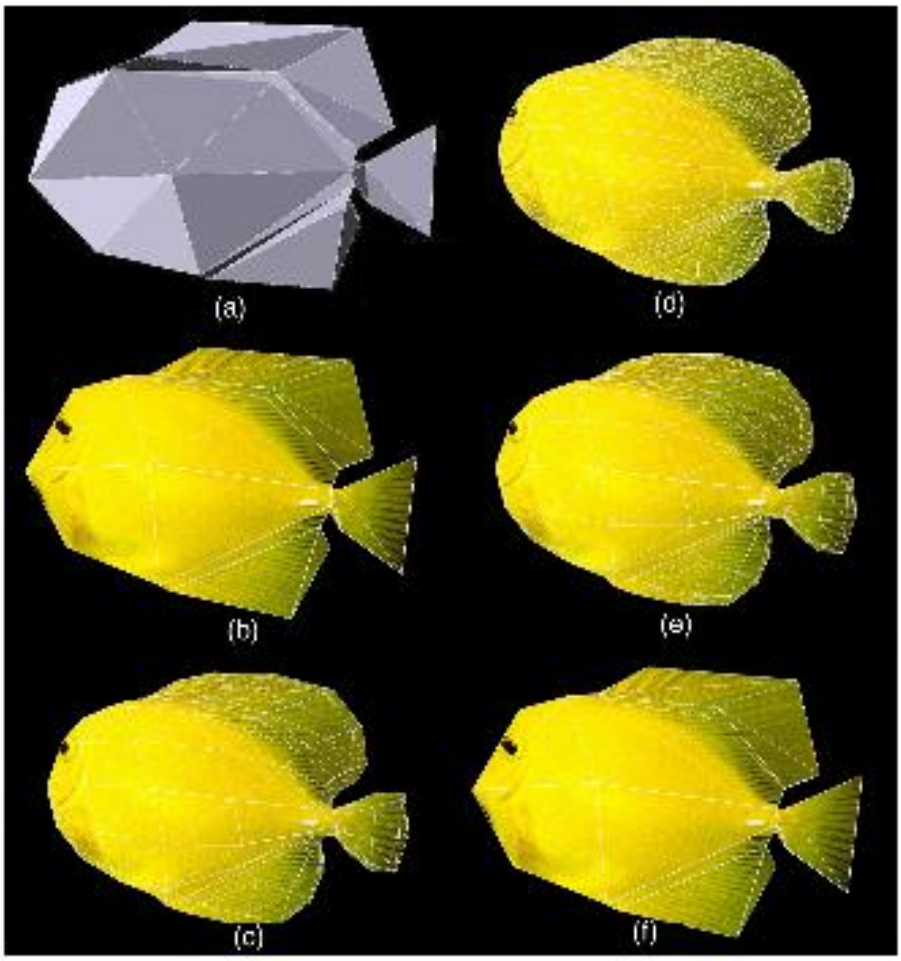

FIGURE 14. (a) The control mesh. (b) The textured control mesh. (c) The resulting mesh after one step of Loop subdivision. (d) After two steps of Loop subdivision. (e) The resulting mesh after applying the reverse scheme on mesh (d). (f) The resulting mesh after applying the reverse scheme on mesh (e).

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Appendix We want to show that each reverse mask of width one for Loop subdivision forms an affine operation. Consider, the following matrix relation:

$$\begin{bmatrix} \nu^{k+1} \\ \nu_1^{k+1} \\ \nu_2^{k+1} \\ \vdots \\ \nu_n^{k+1} \end{bmatrix} = \begin{bmatrix} \beta & \alpha & \alpha & \alpha & \dots & \alpha \\ \frac{3}{8} & \frac{3}{8} & \frac{1}{8} & 0 & \dots & \frac{1}{8} \\ \frac{3}{8} & \frac{1}{8} & \frac{3}{8} & \frac{1}{8} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{3}{8} & \frac{1}{8} & 0 & 0 & \dots & \frac{3}{8} \end{bmatrix} \begin{bmatrix} \nu^{k} \\ \nu_1^{k} \\ \nu_2^{k} \\ \vdots \\ \nu_n^{k} \end{bmatrix}, \tag{5.1}$$

or

$$V^{k+1} = \mathbf{S}.\, V^k,$$

where $\mathbf{S}$ is a local subdivision matrix [23]. Furthermore, assume that the row matrix, $\mathbf{R} = [a_0, a_1, \dots, a_n]$, represents the reverse mask, then we have:

$$\mathbf{R}.\, V^{k+1} = \nu^k,$$

therefore,

$$\mathbf{R}.\mathbf{S}.\, V^k = \nu^k,$$

or equivalently

$$\mathbf{R}.\mathbf{S} = [1, 0, 0, \dots, 0]^T.$$

If we set $\mathbf{S} = [S_1, S_2, \dots, S_n]$ where S_i is i-th column of $\mathbf{S}$, then we obtain, $\mathbf{R}S_1 = 1$ and $\mathbf{R}S_i = 0\,,\ i = 2 \dots, n$. Therefore

$$\mathbf{R}(S_1 + S_2 + \dots + S_n) = 1, \tag{5.2}$$

Since rows of $\mathbf{S}$ have the unit summation property [23], we must have:

$$S_1 + S_2 + \dots + S_n = [1, 1, \dots, 1]^T$$

by using recent equation (5.2), we obtain

$$\mathbf{R}.\begin{bmatrix} 1 \\ 1 \\ \dots \\ 1 \end{bmatrix} = 1,$$

and this equation shows affine property of $\mathbf{R}$ mask.