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ON QUASI UNIVERSAL COVERS FOR GROUPS ACTING ON TREES WITH INVERSIONS

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Abstract. In this paper we show that if G is a group acting on a tree X with inversions and if $(T;Y)$ is a fundamental domain for the action of G on X, then there exist a group $\tilde{G}$ and a tree $\tilde{X}$ induced by $(T;Y)$ such that $\tilde{G}$ acts on $\tilde{X}$ with inversions, G is isomorphic to $\tilde{G}$, and X is isomorphic to $\tilde{X}$. The pair $(\tilde{G};\tilde{X})$ is called the quasi universal cover of $(G;X)$ induced by the $(T;Y)$.

Keywords: Groups acting on trees with inversions, Fundamental domains,Quasi universal cover, Isomorphic trees.

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1. Introduction

The structure of groups acting on trees without inversions known as Bass-Serre theory obtained in [6], and the action with inversions obtained by Mahmud in [5]. Let G is a group acting on a tree X without inversions, T be a maximal tree of the quotient graph Y for the action of G on X, and $\tilde{G} = \pi(G, Y, T)$ be the fundamental group of the graph of groups associated with Y relative T as defined in [6, p 42]. Various trees $\tilde{X}$ were constructed on which $\tilde{G}$ acts on $\tilde{X}$ without inversions, G is isomorphic to $\tilde{G}$, and X is isomorphic to $\tilde{X}$. For more details we refer the readers to [1, p 419], or [2, p 205], or [6, p 55]. In this case $(\tilde{G};\tilde{X})$ is called the universal cover of $(G;X)$. In this paper we generalize such result to groups acting on trees with inversions as follows. Let G is a group acting on a tree X with inversions, $(T;Y)$ be a fundamental domain for the action of G on X, and $\tilde{G} = \pi(T;Y)$ be the fundamental group of $(T;Y)$ defined later. Then there exists a tree denoted $\tilde{X} = \widetilde{(T, Y)}$ such that $\tilde{G}$ acts on $\tilde{X}$ with inversions, G is isomorphic to $\tilde{G}$, and X is isomorphic to $\tilde{X}$. The pair $(\tilde{G}; \tilde{X})$ is called the quasi universal cover of $(G; X)$ relative to $(T; Y)$.

We begin by giving preliminary definitions. By a *graphX* we understand a pair of disjoint sets $V(X)$ and $E(X)$ with $V(X)$ non-empty, together with three functions $\partial_0 : E(X) \to V(X)$, $\partial_1 : E(X) \to V(X)$, and $\eta : E(X) \to E(X)$ satisfying the conditions that $\eta\,\partial_0 = \partial_1$, $\eta\,\partial_1 = \partial_0$, and η is an involution fixing some elements of $E(X)$. For simplicity, if $e \in E(X)$, we write $\partial_0(e) = o(e)$, $\partial_1(e) = t(e)$, and $\eta(e) = \bar{e}$. This implies that $o(\bar{e}) = t(e)$, $t(\bar{e}) = o(e)$, and $\bar{\bar{e}} = e$ on which the case $\bar{e} = e$ is allowed. We call the elements of $V(X)$ *vertices* and those of $E(X)$ *edges*. For $e \in E(X)$, we call $o(e)$ the *initial* of e, $t(e)$ the *terminal* of e, and $\bar{e}$ the *inverse* of e. If A is *a* set of edges of X, define $\bar{A}$ to be the set of inverses of the edges of A. That is, $\bar{A} = \{\,\bar{y} : y \in A\}$.

There are obvious definitions of subgraphs, trees, morphisms of graphs and $Aut(X)$, the set of all automorphisms of the graph X which is a group under the composition of morphisms of graphs. For more details we refer the readers to Serre [6], or to Mahmud [5]. We say that a group G acts on a graph X, if there is a group homomorphism $\phi : G \to Aut(X)$. If $x \in X$ (vertex or edge) and $g \in G$, we write $g(x)$ for $(\phi(g))(x)$. If $y \in E(X)$ and $g \in G$, then $g(o(y)) = o(g(y))$, $g(t(y)) = t(g(y))$, and $g(\bar{y}) = \overline{g(y)}$. The case $g(y) = \bar{y}$ for some $g \in G$ and some $y \in E(X)$ may occur. That is, G acts on X with inversions.

We have the following notations related to the action of the group G on the graph X.

(1) If $x \in X$ (vertex or edge), we define $G(x) = \{g(x) : g \in G\}$, and this set is called the orbit of x. (2) If $x, y \in X$, we define $G(x \to y) = \{g \in G : g(x) = y\}$ and $G(x \to x) = G_x$, the stabilizer of x. Thus $G(x \to y) \neq \emptyset$ if and only if x and y are in the same orbit. It is clear that if $v \in V(X)$, $y \in E(X)$, and $u \in \{o(y), t(y)\}$, then $G(v, y) = \emptyset$, $G_{\bar{y}} = G_y$, and $G_y \leq G_u$.

2. Structure of groups acting on trees with inversions

The aim of this section is to establish various notational conventions and results that we shall use throughout the paper.

Let G be a group acting on a tree X with inversions. Let T and Y be two subtrees of $X, T \subseteq Y$ satisfying the conditions that T contains exactly one vertex from each vertex orbit, and each edge of Y has at least one end in T and Y contains exactly one edge y from each edge orbit such that $G(y \to \bar{y}) = \emptyset$, and exactly one pair x and $\bar{x}$ from each edge orbit such that $G(x \to \bar{x}) \neq \emptyset$. The pair $(T; Y)$ is called a *fundamental domain* for the action of G on X. It is clear that the structure of Y implies that if e_1 and e_2 are two edges of Y such that e_1 and e_2 are in tha same $G-$edge orbit, then $e_1 = e_2$, or $e_1 = \bar{e}_2$.
For the existence of T and Y we refer the readers to [3].

For the rest of this section G, X, T and Y will be as above. We have the following notations.

(i) Let $+Y$ and $-Y$ be the sets defined as follows. $+Y = \{y \in E(Y) : o(y) \in V(T), t(y) \notin V(T), G(y \to \bar{y}) = \emptyset\}$, and $-Y = \{x \in E(Y) : o(x) \in V(T), t(x) \notin V(T), G(x \to \bar{x}) \neq \emptyset\}$. It is clear that $Y = E(T) \cup +Y \cup \overline{+Y} \cup -Y \cup \overline{-Y}$.
(ii) For each vertex v of X, let v^* be the unique vertex of T such that $G(v^* \to v) \neq \emptyset$. That is, v and v^* are in the same vertex orbit.
(iii) For each edge e of $E(T) \cup +Y \cup -Y$ define $[e]$ be an element be an arbitrary element of $G(t(e) \to (t(e)^*)$. That is, $[e]((t(e))^*) = t(e)$ to be chosen as follows. $[e] = 1$ if $e \in E(T)$, and $[e](e) = \bar{e}$ if $e \in -Y$.
It is clear that $[e]^{-1}G_e[e]$ is a subgroup of $G_{(t(e))^*}$, and if $e \in -Y$, then $[e]^2 \in G_e$.

Proposition 2.1. *G is generated by the elements $[e]$ and by the generators of G_v, where e runs over the edges of Y and v runs over the vertices of T.*

Proof. By Theorem 5.1 of [5].

3. Quasi universal covers for groups acting on trees with inversions

Throughout this section G will be a group acting on a tree X with inversions, and $(T; Y)$ be a fundamental domain for the action G on X. In [4], Mahmood introduced the concept of a subfundamental domain $(T_1; Y_1)$ for the action of G on X, and defined it is fundamental group $\pi(T_1;\ Y_1)$, and then showed that there exists a tree denoted $\widetilde{(T_1;\ Y_1)}$ on which $\pi(T_1;\ Y_1)$ acts with inversions. In this section we take T_1 and Y_1 of Definition 4.1 of [4] to be $T_1 = T$ and $Y_1 = Y$, and $\tilde{G}_v = G_v$, $\tilde{G}_y = [y]^{-1}G_y[y]$ such that $\phi_y : [y]^{-1}G_y[y] \to G_y$ is given by $\phi_y(g) = [y]g[y]^{-1}$ and $\phi_y(g) = [y]g[y]^{-1}$ if $G(y, \bar{y}) \neq \emptyset$ for any vertex v of T and any edge y of Y. Then by Proposition 5.2 of [4] implies that the group $\pi(T, Y)$ has the presentation

$$\pi(T, Y) = \langle G_v, t_y, t_x |\, relG_v, G_m = G_{\bar{m}},\\ t_y.[y]^{-1}G_y[y].t_y^{-1} = G_y, t_x.G_x.t_x^{-1} = G_x, t_x^2 = [y]^2\rangle$$

where $v \in V(T)$, $m \in E(T)$, $y \in +Y$, and $x \in -Y$.

The notations of the presentation of $\pi(T, Y)$ are defined as follows.

(i) $\langle G_v \mid relG_v\rangle$ is any presentation of G_v.
(ii) $G_m = G_{\bar{m}}$ is the set of relations $w(g) = w'(g)$, where $w(g)$ and $w'(g)$ are words in the generating symbols of $G_{t(m)}$ and $G_{o(m)}$ respectively of value g, where g is an element in the set of the generators of G_m.
(iii) $t_y.[y]^{-1}G_y[y].\, t_y^{-1} = G_y$ is the set of relations $t_y w([y]^{-1}g[y]\ )\ t_y^{-1} = w(g)$, where $w([y]^{-1}g[y])$ and $w(g)$ are words in the sets of generating symbols of $G_{(t(y))^*}$ and $G_{o(y)}$ of values $[y]^{-1}g[y]$ and g respectively , where g is an element in the set of the generators of G_y .
(iv) $t_x\,.\,G_x\,.\,t_x^{-1} = G_x$ is the set of relations $t_x\, w(g\,)\, t_x^{-1} = w'(g)$, where $w(g)$ and $w'(g)$ are words in the set of generating symbols of $G_{o(x)}$ of values g and $[x]\, g\, [x]^{-1}$ respectively, where g is an element in the set of the generators of G_x.
(v) $t_x^2 = [x]^2$ is the relation $x^2 = w([x]^2)$, where $w([x]^2)$ is a word in the set of the generating symbols of $G_{o(x)}$ of value $[x]^2$.

By Theorem 7.14 of [4], we have the tree $\widetilde{(T,Y)}$ defined as follows. $V(\widetilde{(T,Y)}) = \{[g,v] : g \in \pi(T,Y), v \in V(T)\}$, and $E(\widetilde{(T,Y)}) = \{[g,y] : g \in \pi(T,Y), y \in E(T) \cup +Y \cup \overline{+Y} \cup -Y\}$, where $[g,v]$ is the ordered pair (gG_v, v) and $[g,y]$ is the ordered pair $(gG_y,\ y)$. (Note that if $g \in G_v$, or $g \in G_y$, then $[g,v] = [1,v]$, and $[g,y] = [1,y]$). Define the ends and the inverse of the edge $[g,y]$ of $\widetilde{(T,Y)}$ to be as follows. $o([g,y]) = [g,(o(y))^*]$, $t([g,y]) = [gt_y,\ (t(y))^*]$, and

$$\overline{[g,y]} = \begin{cases} [gt_y, \bar{y}] if y \in E(T) \cup +Y \cup \overline{+Y} \\ [gt_y, y] if y \in -Y \end{cases}.$$

Proposition 7.4 of [4], implies that $\pi(T,Y)$ acts on $\widetilde{(T,Y)}$ with inversions as follows. If $f \in \pi(T,Y)$, $[g,v] \in V(\widetilde{(T,Y)})$, and $[g,v] \in E(\widetilde{(T,Y)})$, then $f[g,v] = [fg,v]$, and $f[g,y] = [fg,y]$.
We note that Corollary 7.5 of [4], implies that if $y \in -Y$, then $\pi(T,\ Y)$ inverts all edges $[g,y]$ in $\widetilde{(T,\ Y)}$.
For example, the element t_y takes the edge $[1,y]$ into its inverse $[t_y,y]$, because $\overline{[1,y]} = [t_y,y] = t_y[1,y]$.
Now we show that $\pi(T,Y)$ is isomorphic to G and $\pi(T,Y)$ is isomorphic to X. First we start by the following definitions and propositions.

Definition 3.1. Define the mapping $\theta : \pi(T,Y) \to G$ by the identity mapping on G_v and by the mapping $t_y \to [y]$, $t_x \to [x]$, where $v \in V(T)$, $y \in +Y$, and $x \in -Y$.

Proposition 3.2. *θ is an onto homomorphism.*

Proof. It is clear that the images $[y]$ and $[x]$ of y and x respectively under the given mapping $t_y \to [y]$, $t_x \to [x]$, where $y \in +Y$, and $x \in -Y$ satisfy the defining relations $t_y.[y]^{-1}G_y[y].\,t_y^{-1} = G_y$, $t_x\,.\,G_x\,.\,t_x^{-1} = G_x$, and $t_x^2 = [x]^2$ of $\pi(T,Y)$. So, by Dyck's Theorem [2, Th.14. p.19] the given mapping defines the given homomorphism $\theta : \pi(T,Y) \to G$. Since by Proposition 2.1, G is generated by G_v and by $[y]$ and $[x]$, where $v \in V(T)$, $y \in +Y$, and $x \in -Y$, therefore θ is an onto homomorphism. This completes the proof.

Definition 3.3. Define $\sigma : \widetilde{(T,Y)} \to X$ by $\sigma([g,v]) = (\theta(g))(v)$, and

$$\sigma([g,\ y] = \begin{cases} (\theta(g))(y)\ \ if\, o(y) \in V(T) \\ (\theta(g))[y](y)\ \ if\, o(y) \notin V(T) \end{cases}$$

where $v \in V(T)$, and $y \in E(T) \cup +Y \cup \overline{+Y} \cup -Y\}$.

Proposition 3.4. *σ is an onto morphism.*

Proof. It is clear that σ maps vertices to vertices and edges to edges. If $[f,u]$ and $[g,v]$ are two vertices of $\widetilde{(T,Y)}$ such that $[f,u] = [g,v]$, then $u = v$ and $fG_v = gG_v$.

Then $f = gh$, $h \in G_v$, and

$$\begin{aligned}
\sigma([f,u]) &= \sigma([gh,v]) \\
&= (\theta(gh))(v) \\
&= (\theta(g)\theta(h))(v) \text{ because } \theta \text{ is a homomorphism} \\
&= \theta(g)(\theta(h))(v) \\
&= \theta(g)(\sigma[h,v]) \\
&= \theta(g)(\sigma[1,v]) \text{ because} h \in G_v \\
&= \theta(g)(\theta(1))(v) \\
&= \theta(g)(1)(v) \\
&= \theta(g)(v) \\
&= \sigma([g,v]).
\end{aligned}$$

Similarly, if $[f,x]$ and $[g,y]$ are two edges of $\widetilde{(T,Y)}$ such that $[f,x] = [g,y]$, then $\sigma[f,x] = \sigma[g,y]$.

This implies that σ is a well-defined mapping.

Now let $[g,y]$ be an edge of $\widetilde{(T,Y)}$. We need to prove the following.
(i) $\sigma(o[g,y]) = o(\sigma[g,y])$,
(ii) $\sigma(t[g,y]) = t(\sigma[g,y])$, and
(iii) $\sigma(\overline{[g,\, y]}) = \overline{\sigma[g,\, y]}$.

Now

$$\begin{aligned}
\sigma(o([g,y])) &= \sigma([g,(o(y))^*]) \\
&= (\theta(g))(o(y))^*) \\
&= (\theta(g))(o(y)^*) \\
&= \begin{cases} (\theta(g))(o(y)) & if\, o(y) \in V(T) \\ (\theta(g))[y](o(y)) & if\, o(y) \notin V(T) \end{cases} \\
&= \begin{cases} o((\theta(g))(y)) & if\, o(y) \in V(T) \\ o((\theta(g))[y](y)) & if\, o(y) \notin V(T) \end{cases} \\
&= o(\sigma[g,y]).
\end{aligned}$$

$$\begin{aligned}
\sigma(t([g,y])) &= \sigma([gt_y,(t(y))^*]) \\
&= (\theta(g)t_y)((t(y))^*) \\
&= (\theta(g)[y])((t(y))^*) \\
&= \begin{cases} (\theta(g))[y][\overline{y}](t(y)) & if\, o(y) \in V(T) \\ (\theta(g))[y](t(y)) & if\, o(y) \notin V(T) \end{cases} \\
&= \begin{cases} t((\theta(g))(y)) & if\, o(y) \in V(T) \\ t((\theta(g))[y](y) & if\, o(y) \notin V(T) \end{cases} \\
&= t(\sigma[g,y]),
\end{aligned}$$

$$\begin{aligned}
\sigma(\overline{[g,y]}) &= \begin{cases} [gt_y,\bar{y}] & if\ y \in E(T)\cup +Y \cup \overline{+Y} \\ [gt_y,y] & if\ y \in -Y \end{cases} \\
&= \begin{cases} (\theta(gt_y))[\bar{y}](\bar{y}) & if\ y \in E(T)\cup +Y \\ (\theta(gt_y))(\bar{y}) & if\ y \in E(T)\cup \overline{+Y} \\ (\theta(gt_y))(y) & if\ y \in -Y \end{cases} \\
&= \begin{cases} (\theta(g[y]))[\bar{y}](\bar{y}) & if\ y \in E(T)\cup +Y \\ (\theta(g[y]))(\bar{y}) & if\ y \in E(T)\cup \overline{+Y} \\ (\theta(g[y]))(y) & if\ y \in -Y \end{cases} \\
&= \begin{cases} (\theta(g))(\bar{y}) & if\ y \in E(T)\cup +Y \\ (\theta(g[y]))(\bar{y}) & if\ y \in E(T)\cup \overline{+Y} \\ (\theta(g))(\bar{y}) & if\ y \in -Y \end{cases} \\
&= \begin{cases} \overline{(\theta(g))(y)} & if\ y \in E(T)\cup +Y \\ \overline{(\theta(g[y]))(y)} & if\ y \in E(T)\cup \overline{+Y} \\ \overline{(\theta(g))(y)} & if\ y \in -Y \end{cases} \\
&= \overline{\sigma[g,\,y]}.
\end{aligned}$$

Thus σ is a well-defined morphism. Since θ is onto, therefore σ is onto. This completes the proof.

The following concept is needed in order to show that $\sigma : \widetilde{(T,Y)} \to X$ is an isomorphism.

If Γ_1 and Γ_2 are two graphs, and $f : \Gamma_1 \to \Gamma_2$ is a morphism, then locally injective if for every two edges e_1 and e_2 of Γ_1 such that $o(e_1) = o(e_2)$, and $f(e_1) = f(e_2)$, then $e_1 = e_2$.

The following proposition is essential to prove the main theorem of this section.

Proposition 3.5. *σ is locally injective.*

Proof. Let $[a_1, e_1]$ and $[a_2, e_2]$ be two edges of $\widetilde{(T,Y)}$ such that $o[a_1, e_1] = o[a_2, e_2]$ and $\sigma[a_1, e_2] = \sigma[a_2, e_2]$. We need to show that $e_1 = e_2$, and $a_1^{-1}a_2 \in G_{e_1}$. It is clear that $(o(e_1))^* = (o(e_2))^*$, $a_1^{-1}a_2 \in G_{(o(e_1))^*}$, and $\theta(a_1^{-1}a_2)(e_1) = e_2$. This implies that e_1 and e_2 are in the same $G-$edge orbit on X. Since e_1 and e_2 are in Y, therefore the properties of Y imply that $e_1 = e_2$, or $e_1 = \bar{e}_2$. If $e_1 = e_2$, then it is clear that $\theta(a_1^{-1}a_2) \in G_{e_1}$. Since $G_{e_1} \le G_{(o(e_1))^*}$, and θ is the identity on $G_{(o(e_1))*}$, therefore θ is the identity on G_{e_1}. This implies that $a_1^{-1}a_2 \in G_{e_1}$. Consequently $[a_1, e_1] = [a_2, e_2]$.
If $e_1 = \bar{e}_2$, then $G(e_1, \bar{e}_1) \ne \emptyset$. This implies that e_2 is in $\overline{-Y_1}$. This contradicts the fact that the edges of $\widetilde{(T,Y)}$ are of the forms $[g, e]$, where $g \in \pi(T, Y)$, and $y \in E(T) \cup +Y \cup \overline{+Y} \cup -Y$. This completes the proof.
For the proof of the following lemma we refer the readers to [6, Lemma 5, p 39].

Lemma 3.6. *If Γ_1 is a connected graph, Γ_2 is a tree, and $f : \Gamma_1 \to \Gamma_2$ is locally injective, then f is injective.*

Now we state the main result of this section.

Theorem 3.7. *Let G, X, T, Y, σ, and θ be as above such that X is a tree. Then (1) $\sigma : \widetilde{(T, Y)} \to X$ is an isomorphism. (2) $\theta : \pi(T, Y) \to G$ is an isomorphism.*

Proof. (i) By Proposition 3.4, σ is an onto morphism, and by Proposition 3.5, σ is locally injective. Since X is a tree, therefore by Lemma 3.6.σ is an isomorphism.
(ii) Suppose that $h \in ker(\theta)$, and v be any vertex of T. Then $\sigma[1, v] = (\theta(1))(v) = v$, because θ is a homomorphism, and $\sigma[h, v] = (\theta(h))(v) = v$, since $h \in ker(\theta)$. Then we have $\sigma[h, v] = \sigma[1, v]$. Since σ is an isomorphism, therefore $[h, v] = [1, v]$. This implies that $h \in G_v$. Since θ restricted to G_v is an isomorphism, therefore $h = 1$. Since by Proposition 3.2, θ is an onto homomorphism, therefore θ is an isomorphism. This completes the proof.

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