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Linear Functions Preserving Sut-Majorization on $\mathbb{R}^n$

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Abstract. Suppose $\mathbf{M}_n$ is the vector space of all n-by-n real matrices, and let $\mathbb{R}^n$ be the set of all n-by-1 real vectors. A matrix $R \in \mathbf{M}_n$ is said to be *row substochastic* if it has nonnegative entries and each row sum is at most 1. For x, $y \in \mathbb{R}^n$, it is said that x is *sut-majorized* by y (denoted by $x \prec_{sut} y$) if there exists an n-by-n upper triangular row substochastic matrix R such that $x = Ry$. In this note, we characterize the linear functions $T : \mathbb{R}^n \to \mathbb{R}^n$ preserving (resp. strongly preserving) $\prec_{sut}$ with additional condition $Te_1 \neq 0$ (resp. no additional conditions).

Keywords:(strong) Linear preserver, Row substochastic matrix, Sut-majorization.

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1. Introduction

Over the years, the theory of majorization as a powerful tool has widely been applied to the related research areas of pure mathematics and the applied mathematics (see [19]). A good survey on the theory of majorization was given by Marshall, Olkin, and Arnold [17]. Recently, the concept of generalized stochastic matrices has been attended specially and many papers have been published in this topic [1-8] and [10-15]. The triangular matrices play an important role in the matrix analysis and its application. So, in this work, we pay attention to a new kind of majorization which has been defined by a

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special type of the triangular matrices. Some kinds of majorization with theirs linear preservers can be found in [9], [16], and [18].

Throughout the article,
$\mathbf{M}_n$ denotes the set of all n-by-n real matrices.
$\mathbb{R}^n$ denotes the set of all n-by-1 real vectors.
$\mathcal{RS}_n^{ut}$ denotes the collection of all n-by-n upper triangular row substochastic matrices.
$\{e_1, \ldots, e_n\}$ denotes the standard basis of $\mathbb{R}^n$.
$A(n_1, \ldots, n_l|m_1, \ldots, m_k)$ denotes the submatrix of A obtained from A by deleting rows $n_1, \ldots, n_l$ and columns $m_1, \ldots, m_k$.
$A(n_1, \ldots, n_l)$ denotes the abbreviation of $A(n_1, \ldots, n_l|n_1, \ldots, n_l)$.
$\mathbb{N}_k$ denotes the set $\{1, \ldots, k\} \subset \mathbb{N}$.
A^t denotes the transpose of a given matrix $A \in \mathbf{M}_n$.
$[T]$ denotes the matrix representation of a linear function $T : \mathbb{R}^n \to \mathbb{R}^n$ with respect to the standard basis.
$\mathcal{C}(\mathcal{A})$ denotes the set $\{\sum_{i=1}^m \lambda_i a_i \mid m \in \mathbb{N},\ \lambda_i \geq 0,\ \sum_{i=1}^m \lambda_i \leq 1,\ a_i \in A,\ \forall i \in \mathbb{N}_m\}$, where $A \subseteq \mathbb{R}^n$.
$x^\downarrow = (x_1^\downarrow, \ldots, x_n^\downarrow)^t$ denotes the decreasing rearrangment of a vector $x = (x_1, \ldots, x_n)^t \in \mathbb{R}^n$. This means $x_1 \geq \ldots \geq x_n$.
$x^\uparrow = (x_1^\uparrow, \ldots, x_n^\uparrow)^t$ denotes the increasing rearrangment of a vector $x = (x_1, \ldots, x_n)^t \in \mathbb{R}^n$. This means $x_1 \leq \ldots \leq x_n$.

Definition 1.1. Let $\mathcal{R}$ be a relation on $\mathbb{R}^n$. A linear function $T : \mathbb{R}^n \to \mathbb{R}^n$ is said to be a linear preserver of $\mathcal{R}$ if for all x, $y \in \mathbb{R}^n$

$$x\mathcal{R}y \Rightarrow Tx\mathcal{R}Ty.$$

If T is a linear preserver of $\mathcal{R}$ and $Tx\mathcal{R}Ty$ implies that $x\mathcal{R}y$, then T is called a strong linear preserver of $\mathcal{R}$.

A matrix $R \in \mathbf{M}_n$ with nonnegative entries is called row stochastic if $Re = e$, where $e = (1, \ldots, 1)^t \in \mathbb{R}^n$. Let x, $y \in \mathbb{R}^n$. We say that x is ut-majorized by y, written $x \prec_{ut} y$, if $x = Ry$ for some upper triangular row stochastic matrix R. In [15], the authors found all linear functions $T : \mathbb{R}^n \to \mathbb{R}^n$ preserving ut-majorization with additional condition $Te_1 \neq 0$ and strong preserving ut-majorization as follow.

Theorem 1.2. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear function. Assume $[T] = [a_{ij}]$, and $Te_1 \neq 0$. Then T preserves $\prec_{ut}$ if and only if*

$$[T] = \begin{pmatrix} a_{11} & 0 & 0 & \dots & 0 & 0 & a_{1n} \\ 0 & a_{22} & 0 & \dots & 0 & 0 & a_{2n} \\ 0 & 0 & a_{33} & \dots & 0 & 0 & a_{3n} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & a_{n-1n-1} & a_{n-1n} \\ 0 & 0 & 0 & \dots & 0 & 0 & a_{nn} \end{pmatrix},$$

$a_{11} + a_{1n} = a_{22} + a_{2n} = \cdots = a_{n-1n-1} + a_{n-1n} = a_{nn}$, *and the finite sequence* $(0, a_{11}, a_{22}, \dots, a_{n-1n-1})^t$ *is monotone.*

Theorem 1.3. *A linear function $T : \mathbb{R}^n \to \mathbb{R}^n$ strongly preserves $\prec_{ut}$ if and only if there exist a, $b \in \mathbb{R}$ such that a, $a + b \neq 0$, and*

$$[T] = \begin{pmatrix} a & 0 & 0 & \dots & 0 & 0 & b \\ 0 & a & 0 & \dots & 0 & 0 & b \\ 0 & 0 & a & \dots & 0 & 0 & b \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & a & b \\ 0 & 0 & 0 & \dots & 0 & 0 & a+b \end{pmatrix}.$$

In this paper, we introduce the relation $\prec_{sut}$ on $\mathbb{R}^n$ and we obtain all linear functions $T : \mathbb{R}^n \to \mathbb{R}^n$ preserving sut-majorization with additional condition $Te_1 \neq 0$ and strongly preserving sut-majorization.

2. Sut-Majorization on $\mathbb{R}^n$

In this section, we focus on the upper triangular row substochastic matrices and introduce a new type of majorization. Then we characterize the structure of (resp. strong) linear preservers of sut-majorization on $\mathbb{R}^n$ (resp. no additional conditions) with additional condition $Te_1 \neq 0$.

Definition 2.1. A matrix R with nonnegative entries is called *row substochastic* if all its row sums is less than or equal to one.

Definition 2.2. Let $x, y \in \mathbb{R}^n$. We say that x sut-majorized by y (in symbol $x \prec_{sut} y$) if $x = Ry$, for some $R \in \mathcal{RS}_n^{ut}$.

Let $x = Ry$, for some $R \in \mathcal{RS}_n^{ut}$. Then

$$R = \begin{pmatrix} r_{11} & r_{12} & \dots & r_{1n-1} & r_{1n} \\ 0 & r_{22} & \dots & r_{2n-1} & r_{2n} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & \dots & 0 & r_{n-1n-1} & r_{n-1n} \\ 0 & \dots & 0 & 0 & r_{nn} \end{pmatrix},$$

$\sum_{j=i}^{n} r_{ij} \leq 1$, $r_{ij} \geq 0$, and $x_i = \sum_{j=i}^{n} r_{ij} y_j$, for each $i \in \mathbb{N}_n$. So $x_i \in \mathcal{C}\{y_i, \ldots, y_n\}$, for each $i \in \mathbb{N}_n$.
Also, if $x_i \in \mathcal{C}\{y_i, \ldots, y_n\}$, for each $i \in \mathbb{N}_n$, then there exist $r_{ij} \geq 0$ such that $\sum_{j=i}^{n} r_{ij} \leq 1$ and $x_i = \sum_{j=i}^{n} r_{ij} y_j$, for each $i \in \mathbb{N}_n$ and for each $j \in \mathbb{N}_i$. Let $r_{ij} = 0$ for each $1 \leq i < j$ and put $R = (r_{ij})$. It is clear that $R \in \mathcal{RS}_n^{ut}$ and $x = Ry$. Therefore, $x \prec_{sut} y$.

We summarize the foregoing discussion in the following proposition. This proposition provides a criterion for sut-majorization on $\mathbb{R}^n$.

Proposition 2.3. *Let $x = (x_1, \ldots, x_n)^t$, $y = (y_1, \ldots, y_n)^t \in \mathbb{R}^n$. Then $x \prec_{sut} y$ if and only if $x_i \in \mathcal{C}\{y_i, \ldots, y_n\}$, for all $i \in \mathbb{N}_n$.*

Now, we assert some prerequisites for introducing the main results of this section.

Lemma 2.4. *Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear preserver of $\prec_{sut}$. Assume that $S : \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$ is a linear function with $[S] = [T](1, \ldots, k)$. Then S preserves $\prec_{sut}$ on $\mathbb{R}^{n-k}$.*

Proof. Let $x' = (x_{k+1}, \ldots, x_n)^t$, $y' = (y_{k+1}, \ldots, y_n)^t \in \mathbb{R}_{n-k}$, and let $x' \prec_{sut} y'$. By Proposition 2.3, we obtain
$x := (0, \ldots, 0, x_{k+1}, \ldots, x_n)^t \prec_{sut} y := (0, \ldots, 0, y_{k+1}, \ldots, y_n)^t$, where x, $y \in \mathbb{R}^n$, and hence $Tx \prec_{sut} Ty$. This shows that $Sx' \prec_{sut} Sy'$. Therefore, S preserves $\prec_{sut}$, as desired. □

Lemma 2.5. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear preserver of $\prec_{sut}$. Then $[T]$ is upper triangular.*

Proof. Let $[T] = [a_{ij}]$. We proceed by induction. There is nothing to prove for $n = 1$. Suppose that $n \geq 2$ and that the assertion has been established for all linear preservers of $\prec_{sut}$ on $\mathbb{R}^{n-1}$. Let $S : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ be the linear function with $[S] = [T](1)$. By Lemma 2.4, S preserves $\prec_{sut}$ on $\mathbb{R}^{n-1}$. The induction hypothesis insures that $[S]$ is an $n-1$-by-$n-1$ upper triangular matrix. So it is enough to show that $a_{21} = \cdots = a_{n1} = 0$. As $e_1 \prec_{sut} e_2$, we see that $Te_1 \prec_{sut} Te_2$ and hence $(a_{11}, \ldots, a_{n1})^t \prec_{sut} (a_{12}, a_{22}, 0, \ldots, 0)^t$. It implies that $a_{31} = \cdots = a_{n1} = 0$. So it remains to prove that $a_{21} = 0$. Assume, if possible, that $a_{21} \neq 0$. Set $x = e_1$ and $y = (\frac{-a_{22}}{a_{21}}, 1, 0, \ldots, 0)^t$. So $x \prec_{sut} y$, and then $Tx \prec_{sut} Ty$. This follows that $a_{21} = 0$, which is a contradiction. Thus $a_{21} = 0$ and we observe that the induction argument is completed. Therefore, $[T]$ is an upper triangular matrix. □

The following theorem characterizes structure of the linear functions $T : \mathbb{R}^n \to \mathbb{R}^n$ preserving sut-majorization with additional condition $Te_1 \neq 0$. Note that the vector $x = (x_1, \ldots, x_n)^t$ is monotone if $x = (x_1^{\uparrow}, \ldots, x_n^{\uparrow})^t$ or $x = (x_1^{\downarrow}, \ldots, x_n^{\downarrow})^t$.

Theorem 2.6. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear function. Assume that $[T] = [a_{ij}]$ and $Te_1 \neq 0$. Then T preserves $\prec_{sut}$ if and only if*

$$[T] = \begin{pmatrix} a_{11} & 0 & 0 & \dots & 0 \\ 0 & a_{22} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 & a_{nn} \end{pmatrix},$$

and the vector $(0, a_{11}, \dots, a_{nn})^t$ is monotone.

Proof. First, suppose that T preserves $\prec_{sut}$. It is clear that T preserves $\prec_{sut}$ if and only if αT preserves $\prec_{sut}$ for all $\alpha \in \mathbb{R} \setminus \{0\}$. So we can assume without loss of generality that $a_{11} = 1$. By Lemma 2.5, $[T]$ is upper triangular. We prove the statement by induction. The result is trivial for $n = 1$. Assume that our claim has been proved for all linear preservers of $\prec_{sut}$ on $\mathbb{R}^{n-1}$.
We claim that $a_{22} \neq 0$. If $a_{22} = 0$, we consider the following two cases.
First, let $a_{12} = -1$. Then $e_1 \prec_{sut} e_2$, but $Te_1 \not\prec_{sut} Te_2$, which is a contradiction.
Next, let $a_{12} \neq -1$. Put $x = e_1 + e_2$ and $y = -a_{12}e_1 + e_2$. We see that $x \prec_{sut} y$, but $Tx \not\prec_{sut} Ty$. This means T does not preserve $\prec_{sut}$.

Thus $a_{22} \neq 0$.
Let $S : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ be the linear function with $[S] = [T](1)$. By Lemma 2.4, S preserves $\prec_{sut}$ on $\mathbb{R}^{n-1}$. Since $a_{22} \neq 0$, the induction hypothesis ensures that

$$[S] = \begin{pmatrix} a_{22} & 0 & 0 & \dots & 0 \\ 0 & a_{33} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 & a_{nn} \end{pmatrix},$$

and the vector $(0, a_{22}, \dots, a_{nn})^t$ is monotone. So it is enough to show that $a_{12} = \dots = a_{1n} = 0$ and $1 \leq a_{22}$. Assume that there is some j $(2 \leq j \leq n)$ such that $a_{1j} \neq 0$. Choose $x = -a_{1j}e_1$ and $y = -a_{1j}e_1 + e_j$. The proof is divided into two steps.
Step 1. If $a_{jj} > 0$; We consider two cases.
Case 1. $a_{1j} > 0$. Since $x \prec_{sut} y$, but $Tx \not\prec_{sut} Ty$, a contradiction.
Case 2. $a_{1j} < 0$. As $e_j \prec_{sut} y$, but $Te_j \not\prec_{sut} Ty$, we conclude T does not preserve $\prec_{sut}$.
Step 2. If $a_{jj} < 0$; We have two cases.
Case 1. $a_{1j} > 0$. One can see that $e_j \prec_{sut} y$, but $Te_j \not\prec_{sut} Ty$, which is a contradiction.
Case 2. $a_{1j} < 0$. It is clear that $x \prec_{sut} y$, but $Tx \not\prec_{sut} Ty$. It implies that T does not preserve $\prec_{sut}$.

Hence $a_{1j} = 0$ for each j $(2 \leq j \leq n)$, and so

$$[T] = \begin{pmatrix} a_{11} & 0 & 0 & \dots & 0 \\ 0 & a_{22} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 & a_{nn} \end{pmatrix}.$$

Since $e_1 \prec_{sut} e_2$, we have $Te_1 \prec_{sut} Te_2$. This means that $1 \leq a_{22}$. So the vector $(0, 1, a_{22}, \dots, a_{nn})^t$ is monotone.

To prove the sufficiency, let $x = (x_1, \dots, x_n)^t, y = (y_1, \dots, y_n)^t \in \mathbb{R}^n$ and let $x \prec_{sut} y$. Then

$$Tx = (a_{11}x_1, a_{22}x_2, \dots, a_{nn}x_n)^t$$

and

$$Ty = (a_{11}y_1, a_{22}y_2, \dots, a_{nn}y_n)^t.$$

We prove $(Tx)_i \in \mathcal{C}\{(Ty)_i, \dots, (Ty)_n\}$, for all $i \in \mathbb{N}_n$. Let $i \in \mathbb{N}_n$. Since $x_i \in \mathcal{C}\{y_i, \dots, y_n\}$, then there exist $0 \leq \alpha_i, \dots, \alpha_n \leq 1$, $\sum_{k=i}^n \alpha_k \leq 1$, and $x_i = \sum_{k=i}^n \alpha_k y_k$. As $a_{ii}, \dots, a_{nn} \neq 0$, we conclude that $(Tx)_i = \sum_{k=i}^n (\frac{a_{ii}\alpha_k}{a_{kk}})(Ty)_k$. Clearly, $(Tx)_i \in \mathcal{C}\{(Ty)_i, \dots, (Ty)_n\}$. This implies that $Tx \prec_{sut} Ty$. Therefore, T preserves $\prec_{sut}$. □

Corollary 2.7. *If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear preserver of $\prec_{sut}$ such that $Te_1 \neq 0$, then $rank[T] = n$.*

We observe from Theorem 1.2 that if $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear preserver of $\prec_{ut}$ such that $Te_1 \neq 0$, then $rank[T] \geq n - 1$. We need the following lemma in the rest of this paper.

Lemma 2.8. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear function that strongly preserves $\prec_{sut}$. Then T is invertible.*

Proof. Let $x \in \mathbb{R}^n$, and let $Tx = 0$. Since $Tx = T0$ and T strongly preserves $\prec_{sut}$, we have $x \prec_{sut} 0$. So $x = 0$. Therefore, T is invertible. □

The following theorem characterizes the linear functions $T : \mathbb{R}^n \to \mathbb{R}^n$ which strongly preserves sut-majorization. We close this paper with this theorem.

Theorem 2.9. *A linear function $T : \mathbb{R}^n \to \mathbb{R}^n$ strongly preserves $\prec_{sut}$ if and only if $[T] = \alpha I_n$, for some $\alpha \in \mathbb{R} \setminus \{0\}$.*

Proof. First, suppose that T strongly preserves $\prec_{sut}$. Lemma 2.8 ensures that T is invertible and hence $Te_1 \neq 0$. So by Theorem 2.6,

$$[T] = \begin{pmatrix} a_{11} & 0 & 0 & \dots & 0 \\ 0 & a_{22} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 & a_{nn} \end{pmatrix},$$

and the vector $(0, a_{11}, \ldots, a_{nn})^t$ is monotone.
By a simple calculation, we obtain

$$[T]^{-1} = \begin{pmatrix} \frac{1}{a_{11}} & 0 & 0 & \ldots & 0 \\ 0 & \frac{1}{a_{22}} & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & 0 & \frac{1}{a_{nn}} \end{pmatrix}.$$

Since T strongly preserves $\prec_{sut}$, we conclude T^{-1} is a linear preserver of $\prec_{sut}$, and hence the vector $(0, \frac{1}{a_{11}}, \ldots, \frac{1}{a_{nn}})^t$ is monotone. Thus $a_{11} = \cdots = a_{nn}$, as desired.

For the converse, assume that there exists $\alpha \in \mathbb{R}$ such that $\alpha \neq 0$ and $[T] = \alpha I_n$. Thus $[T]^{-1} = \frac{1}{\alpha} I_n$. It follows from Theorem 2.6, T and T^{-1} preserve $\prec_{sut}$, therefore, T strongly preserves $\prec_{sut}$. □

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