Generalized Douglas-Weyl Finsler Metrics

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Abstract. In this paper, we study generalized Douglas-Weyl Finsler metrics. We find some conditions under which the class of generalized Douglas-Weyl \((\alpha, \beta)\)-metric with vanishing S-curvature reduce to the class of Berwald metrics.

Keywords: Generalized Douglas-Weyl metrics, S-curvature.


1. Introduction

Let \((M, F)\) be a Finsler manifold. In local coordinates, a curve \(c(t)\) is a geodesic if and only if its coordinates \((c^i(t))\) satisfy \(\ddot{c}^i + 2G^i(\dot{c}) = 0\), where the local functions \(G^i = G^i(x, y)\) are called the spray coefficients [10]. \(F\) is called a Berwald metric, if \(G^i\) are quadratic in \(y \in T_x M\) for any \(x \in M\) or equivalently \(G^i = \frac{1}{2} \Gamma^i_{jk}(x) y_j y_k\). As a generalization of Berwald curvature, Bācső-Matsumoto introduced the notion of Douglas metrics which are projective invariants in Finsler geometry [2]. \(F\) is called a Douglas metric if \(G^i = \frac{1}{2} \Gamma^i_{jk}(x) y_j y_k + P(x, y) y^i\).

A Finsler metric \(F\) is called generalized Douglas-Weyl metric (briefly, GDW-metric) if \(D^i_{jkl} y^m = T_{jkl} y^i\) holds for some tensor \(T_{jkl}\), where \(D^i_{jkl}||m\) denotes the horizontal covariant derivatives of \(D^i_{jkl}\) with respect to the Berwald
connection of $F$ [8][18]. For a manifold $M$, let $\mathcal{GDW}(M)$ denotes the class of all Finsler metrics satisfying in above relation for some tensor $T_{jkl}$. In [3], Bácsó-Papp showed that $\mathcal{GDW}(M)$ is closed under projective changes. Then, Najafi-Shen-Tayebi characterized generalized Douglas-Weyl Randers metrics [8]. In [18], it is proved that all generalized Douglas-Weyl spaces with vanishing Landsberg curvature have vanishing the quantity $H$. For other works, see [12] and [13].

The notion of S-curvature is originally introduced by Shen for the volume comparison theorem [9]. The Finsler metric $F$ is said to be of isotropic S-curvature if $S = (n + 1)cF$, where $c = c(x)$ is a scalar function on $M$. In [14], it is shown that every isotropic Berwald metric has isotropic S-curvature. In [4], Cheng-Shen show that every $(\alpha, \beta)$-metric with constant Killing 1-form has vanishing S-curvature. Then, Bácsó-Cheng-Shen proved that a Finsler metric $F = \alpha \pm \beta^2/\alpha + \epsilon\beta$ has vanishing S-curvature if and only if $\beta$ is a constant Killing 1-form [1]. Therefore, the Finsler metrics with vanishing S-curvature are of some important geometric structures which deserve to be studied deeply.

An $(\alpha, \beta)$-metric is a Finsler metric on $M$ defined by $F := \alpha\phi(s), s = \beta/\alpha$, where $\phi = \phi(s)$ is a $C^\infty$ function on the $(-b_0, b_0)$ with certain regularity, $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta(y) = b_i(x)y^i$ is a 1-form on $M$ [6]. In this paper, we are going to study generalized Douglas-Weyl $(\alpha, \beta)$-metrics with vanishing S-curvature.

**Theorem 1.1.** Let $F = \alpha\phi(s), s = \beta/\alpha$, be an $(\alpha, \beta)$-metric on a manifold $M$ of dimension $n \geq 3$. Suppose that

$$F \neq \frac{c_3\alpha}{\beta} \left( \frac{\beta}{\alpha} \right)^{\frac{2}{n+2}} \left( c_1\frac{\beta}{\alpha} + c_2 + 1 \right)^{\frac{1}{1+n}}$$

and

$$F \neq d_1\sqrt{\alpha^2 + d_2\beta^2 + d_3\beta},$$

where $c_1, c_2, c_3, d_1, d_2$ and $d_3$ are real constants. Let $F$ has vanishing S-curvature. Then $F$ is a GDW-metric if and only if it is a Berwald metric.

2. Preliminary

Given a Finsler manifold $(M, F)$, then a global vector field $G$ is induced by $F$ on $TM_0$, which in a standard coordinate $(x^i, y^i)$ for $TM_0$ is given by $G = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i}$, where

$$G^i := \frac{1}{4} g^{il} \left\{ [F^2]_{x^j y^k} y^k - [F^2]_{x^l} \right\}, \quad y \in T_x M.$$

The $G$ is called the spray associated to $F$.

Define $B_y : T_x M \otimes T_x M \otimes T_x M \rightarrow T_x M$ and $E_y : T_x M \otimes T_x M \rightarrow \mathbb{R}$ by $B_y(u, v, w) := B_{ijk}(y)u^i v^k w^l \frac{\partial}{\partial x^l}$ and $E_y(u, v) := E_{jk}(y)u^j v^k$ where

$$B_{ijk} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}, \quad E_{jk} := \frac{1}{2} B_{jkm}.$$
\(B\) and \(E\) are called the Berwald curvature and mean Berwald curvature, respectively. \(F\) is called a Berwald and weakly Berwald if \(B = 0\) and \(E = 0\), respectively [5][7].

Let
\[
D_{ij} \triangleq \frac{\partial^3}{\partial y^i \partial y^j \partial y^k} \left( G^i - \frac{1}{n+1} \frac{\partial G^m}{\partial y^m} y^i \right).
\]
It is easy to verify that \(D := D_{ij} dx^i \otimes \partial_j \otimes dx^k \otimes dx^l\) is a well-defined tensor on slit tangent bundle \(TM_0\). We call \(D\) the Douglas tensor. A Finsler metric with \(D = 0\) is called a Douglas metric. The notion of Douglas metrics was proposed by Bacsó-Matsumoto as a generalization of Berwald metrics [2].

The Douglas tensor \(D\) is a non-Riemannian projective invariant, namely, if two Finsler metrics \(F\) and \(\tilde{F}\) are projectively equivalent, \(G^i = \tilde{G}^i + P y^i\), where \(P = P(x, y)\) is positively \(y\)-homogeneous of degree one, then the Douglas tensor of \(F\) is same as that of \(\tilde{F}\). Finsler metrics with vanishing Douglas tensor are called Douglas metrics [11].

For a Finsler metric \(F\) on an \(n\)-dimensional manifold \(M\), the Busemann-Hausdorff volume form \(dV_F = \sigma_F(x) dx^1 \cdots dx^n\) is defined by
\[
\sigma_F(x) \triangleq \frac{\text{Vol}(\mathbb{B}^n(1))}{\text{Vol}\left(\{y^i \in \mathbb{R}^n \mid F\left(\frac{\partial}{\partial y^i} \big|_x\right) < 1\}\right)}.
\]
Let \(G^i\) denote the geodesic coefficients of \(F\) in the same local coordinate system. The S-curvature is defined by
\[
S(y) := \frac{\partial G^i}{\partial y^i}(x, y) - y^j \frac{\partial}{\partial x^j} \left[ \ln \sigma_F(x) \right],
\]
where \(y = y^j \frac{\partial}{\partial y^j} \big|_x \in T_x M\). \(S\) is said to be isotropic if there is a scalar functions \(c = c(x)\) on \(M\) such that \(S = (n + 1)cF\).

For an \((\alpha, \beta)\)-metric \(F = \alpha \phi(s), s = \beta/\alpha\), put
\[
\Phi := -(q - sq')[n\Delta + 1 + sq] - (b^2 - s^2)(1 + sq)q'',
\]
where
\[
q := \frac{\phi'}{\phi - sq'}, \quad \Delta := 1 + sq + (b^2 - s^2)q'.
\]
In [4], Cheng-Shen characterize \((\alpha, \beta)\)-metrics with isotropic S-curvature.

**Lemma 2.1.** ([4]) Let \(F = \alpha \phi(s), s = \beta/\alpha\), be a non-Riemannian \((\alpha, \beta)\)-metric on a manifold \(M\) of dimension \(n \geq 3\). Suppose that \(\phi \neq c_1 \sqrt{1 + c_2 s^2} + c_3 s\) for any constant \(c_1 > 0, c_2\) and \(c_3\). Then \(F\) is of isotropic S-curvature \(S = (n + 1)cF\) if and only if one of the following holds

(a) \(\beta\) satisfies
\[
 r_{ij} = \varepsilon(b^2 a_{ij} - b_i b_j), \quad s_j = 0,
\]
where $\varepsilon = \varepsilon(x)$ is a scalar function, $b := \|\beta x\|_\alpha$ and $\phi = \phi(s)$ satisfies
\[
\Phi = -2(n+1)k\phi \Delta^2 b^2 \ldots + C_3 b^j y_{i00} = C_4 y_{jsi}^0 + C_5 \left( b_{jsi}^0 + s_j s_i^0 \right) + C_6 s_j t_i^0,
\]

(b) $\beta$ satisfies
\[
r_{ij} = 0, \quad s_j = 0
\]

In this case, $S = 0$.

The characterization of Finsler metrics with isotropic S-curvature in Cheng-Shen’s paper is not complete [4]. Their result is correct for dimension $n \geq 3$. For the case $\text{dim}(M) = 2$, see [16].

3. Proof of Main Results

Let $F := \alpha \phi(s)$, $s = \beta/\alpha$, be an $(\alpha, \beta)$-metric on a manifold $M$, where $\alpha = \sqrt{a_{ij}(x) y^i y^j}$ and $\beta(y) = b_i(x) y^i$. Define $b_{ij}$ by $b_{ij} \theta^i := db_i - b_j \theta^j$, where $\theta^i := dx^i$ and $\theta_i^j := \tilde{\Gamma}_{ik}^j dx^k$ denote the Levi-Civita connection forms of $\alpha$. Let
\[
\begin{align*}
    r_{ij} &:= \frac{1}{2} \left[ b_{ij} + b_{ji} \right], \quad s_{ij} := \frac{1}{2} \left[ b_{ij} - b_{ji} \right], \\
    r_{00} &:= r_{ij} y^i, \quad r_{00} := r_{ij} y^j, \quad r_j := b^i r_{ij}, \quad t_j^i := s^i_m s^m_j \\
    s_{00} &:= s_{ij} y^i, \quad s_j := b^i s_{ij}, \quad r_0 := r_{ij} y^j, \quad s_0 := s_j y^j.
\end{align*}
\]

Then $\beta = b_i(x) y^i$ is a constant Killing one-form on $M$ if $r_{ij} = s_j = 0$ hold. By definition, we have
\[
b_{ij} = s_{ij} + r_{ij}.
\]

Since $y^i |_s = 0$, then for a constant Killing 1-form $\beta$ we have
\[
r_{00} = 0, \quad r_i + s_i = 0.
\]

For an $(\alpha, \beta)$-metric $F = \alpha \phi(s)$, $s = \beta/\alpha$, the following hold.

**Proposition 3.1.** Let $F = \alpha \phi(s)$, $s = \beta/\alpha$, be an $(\alpha, \beta)$-metric on an $n$-dimensional manifold $M$ of dimension $n \geq 3$, where $\alpha = \sqrt{a_{ij}(x) y^i y^j}$ is a Riemannian metric and $\beta = b_i(x) y^i$ is a one-form on $M$. Suppose that $F$ is of vanishing S-curvature. Then $F$ is a GDW-metric if and only if the following holds
\[
C_1 s_{j0} y^i + (C_2 y_j + C_3 b_j) y^i t_{00} = C_4 y_j s^i_{0j0} + C_5 \left( b_j s^i_{0j0} + s_{j0} s^i_0 \right) + C_6 s^i_{j0j} + C_7 (y_j t_i^0 + s_{j0} s^i_0) + C_8 b_j t_i^0,
\]
where

\[ C_1 := - \left[ (n + 1)Q_\alpha + 2\beta Q_{\alpha\beta} \right] \alpha^{-3} - \left[ Q_{\alpha\alpha} + b^2 Q_{\beta\beta} \right] \alpha^{-2}, \]
\[ C_2 := (n + 1) \left[ \left( Q_\alpha^2 + Q_{\alpha\alpha} - \alpha^{-1} Q_{Q_\alpha} \right) \alpha^{-4} - 2 \left( Q_{a\beta} + Q_{Q_{a\alpha}} \right) \beta \alpha^{-5} \right. \]
\[ + \left. 2 \left[ 2Q_{a\alpha} Q_{\alpha\beta} + Q_{\alpha\alpha} Q_{\beta} + Q_{Q_{\alpha\alpha}} \right] \beta \alpha^{-4} + b^2 \left[ 2Q_{a\beta} Q_{\beta} + Q_{a} Q_{\beta\beta} \right] \alpha^{-3} \right. \]
\[ + \left. b^2 Q_{\alpha\beta} + 3Q_{a\alpha} + Q_{Q_{\alpha\alpha}} \right] \alpha^{-3}, \]
\[ C_3 := (n + 3) \left[ Q_{a\alpha} Q_{\alpha\beta} + Q Q_{a\beta} \right] \alpha^{-3} + 2 \left[ Q_{a\alpha} Q_{\alpha\beta} + Q Q_{a\beta} \right] \beta \alpha^{-3} \]
\[ + \left[ 2Q_{a\alpha} Q_{\alpha\beta} + Q_{\alpha\alpha} + Q_{Q_{a\alpha}} + 4\beta^{-1} Q_{Q_{a\beta}} \right] \alpha^{-2} \]
\[ + b^2 \left[ 3Q_{Q_{a\beta}} + Q_{Q_{a\alpha}} \right] \alpha^{-1}, \]
\[ C_4 := - \left[ (n + 1)Q_\alpha + 2\beta Q_{\alpha\beta} \right] \alpha^{-3} + 2 \left[ \beta Q_{\alpha\alpha} + Q_{\alpha} \right] \alpha^{-2} \]
\[ + \left[ b^2 Q_{\alpha\beta} + Q_{\alpha\alpha} \right] \alpha^{-1}, \]
\[ C_5 := (n + 3) \alpha^{-1} Q_{a\beta} + Q_{\alpha\alpha} + 2\beta \alpha^{-1} Q_{a\alpha} + b^2 Q_{\beta\beta}, \]
\[ C_6 := (n + 1) \alpha^{-1} Q_{a\beta} + Q_{\alpha\alpha} + 2\beta \alpha^{-1} Q_{a\alpha} + b^2 Q_{\beta\beta}, \]
\[ C_7 := (n + 1) \alpha^{-3} Q_{\alpha} - (n + 1) \alpha^{-2} \left( Q_\alpha^2 + Q_{Q_{\alpha\alpha}} \right) - 2\beta \alpha^{-2} Q_{Q_{\alpha\alpha}} \]
\[ + 2 \left[ Q_{Q_{\alpha\beta}} + Q_{Q_{\beta}} \right] \beta \alpha^{-3} - b^2 \left[ Q_{Q_{a\beta}} + 2Q_{a\beta} Q_{\beta} \right] \alpha^{-1} \]
\[ - 2 \left[ 2Q_{a\alpha} Q_{\alpha\beta} + Q_{\alpha} Q_{\alpha} \right] \beta \alpha^{-2} \]
\[ - b^2 \alpha^{-1} Q_{a\alpha} Q_{\beta} - 3\alpha^{-1} Q_{a\alpha} Q_{\alpha} - 2\alpha^{-1} Q_{Q_{\alpha\alpha}}, \]
\[ C_8 := - (n + 3) \left[ Q_{Q_{\alpha\beta}} + Q_{Q_{\beta}} \right] \alpha^{-1} - 2 \left[ 2Q_{a\beta} Q_{a\beta} + Q_{Q_{a\beta}} + Q_{a} Q_{\beta\beta} \right] \beta \alpha^{-1} \]
\[ - b^2 \left[ Q_{Q_{\alpha\beta}} + 3Q_{Q_{\beta}} \right] - Q_{\beta} Q_{\alpha} - Q_{Q_{\alpha\beta}} - 2Q_{a} Q_{a\beta}. \]

**Proof.** Let \( G^i \) and \( G^i_\alpha \) denote the spray coefficients of \( F \) and \( \alpha \), respectively, in the same coordinate system. Then, we have

\[ G^i = G^i_\alpha + Py^i + Q^i, \quad (3.2) \]

where

\[ Q := \Theta, \]
\[ P := \alpha^{-1} \Theta(r_{00} - 2Qs_0), \]
\[ Q^i := Qs^i_0 + \Psi(r_{00} - 2Qs_0)b^i, \]
\[ \Theta := \frac{q - sq^0}{2\Delta} = \frac{\phi' - s(\phi'' + \phi' \phi')}{2\phi[(\phi - s\phi') + (b^2 - s^2)\phi'']} \]
\[ \Psi := \frac{q'}{2\Delta} = \frac{\phi'}{2(\phi - s\phi') + (b^2 - s^2)\phi''}. \]
By Lemma 2.1, we have $r_{00} = s_0 = 0$. Then (3.2) reduces to following
\[ G^i = G^i_\alpha + Qs^i_0. \]  
(3.3)

Let "∥" and "\textquoteleft\textquoteleft" denote the covariant differentiations with respect to $G^i$ and $G^i_\alpha$ respectively. Then by (3.3), we have
\[ D^i_{jkl\|} y^m = D^i_{jkl\|} y^m - 2Qs^i_0 \frac{\partial D^i_{jkl}}{\partial y^p} + D^p_{jkl} \tilde{N}^i_p - D^i_{jkl} \tilde{N}^p_k \]
(3.4)

where
\[ D^i_{jkl\|} y^m = \alpha^{-4}(Q_{\alpha\alpha} - \alpha^{-1}Q_{\alpha})(A_{jkl}y_l + A_{klj}y_j + A_{jyk}y_k)s^i_{0j0} 
+ \alpha^{-3}Q_{\alpha}(A_{jki}y_i + A_{klj}y_j + A_{jyk}y_k)s^i_{0j0} 
+ \alpha^{-3}Q_{\alpha\beta}(A_{jkl}b_l + A_{klj}b_j + A_{jyk}b_k)s^i_{0j0} 
+ (A_{jkl}s_{00} + A_{klj}s_{0j0} + A_{jyk}s_{i00})s^i_{00} 
+ \alpha^{-2}Q_{\alpha\beta\gamma}(y_{jkl}y_{b0} + y_{jykl}y_{bj} + y_{kjy}b_k)s^i_{0j0} 
+ (y_{jkl}y_{b0} + y_{jykl}y_{bj} + y_{kjy}b_k)s^i_{0j0} 
+ \alpha^{-1}Q_{\alpha\beta\gamma}(y_{jkl}b_l + y_{jykl}b_j + y_{kjy}b_k)s^i_{0j0} 
+ (y_{jkl}b_l + y_{jykl}b_j + y_{kjy}b_k)s^i_{0j0} \]
(3.5)

and
\[ A_{ij} = \alpha^2 a_{ij} - y_{iyj}, \]
(3.6)
\[ \tilde{N}^i_p = Qs^i_p + \left[ \alpha^{-1}Q_{\alpha\gamma}y_p + \alpha^{-1}Q_{\beta\gamma}b_p \right] s^i_0, \]
(3.7)
\[ \frac{\partial D^i_{jkl}}{\partial y^p} = Q_{ijkl} s^i_{00} + Q_{jkl} s^i_{p0} + Q_{jkl} s^i_{p0} + Q_{jkl} s^i_{k0} + Q_{jkl} s^i_{k0} + Q_{jkl} s^i_{k0}. \]
(3.8)

Let $F$ is a GDW-metric. Then there exists a tensor $D^i_{jkl}$ such that
\[ D^i_{jkl\|} y^m = D^i_{jkl} y^i. \]
By (3.4), we have
\[
D_{jkl} y^i = D_{jkl|m} y^m - 2Q \frac{\partial D_{jkl}^i}{\partial y^p} s^p_0 + D_{jkl}^p \tilde{N}^i_p - D_{pkl}^p \tilde{N}^i_p \quad (3.9)
\]
By contracting (3.9) with \( y_i \) and using (3.5), (3.7) and (3.8) we get the following
\[
D_{jkl} = D_{jkl} = D_{jkl} + D_{klj} + D_{kjl} + D_{jlk} + D_{ljk} + D_{kjl} + D_{ljk} + D_{kjl} + D_{ljk} + D_{kjl} + D_{ljk} + D_{kjl}
\]
where
\[
D_1 := -\alpha^{-5} Q \alpha, \\
D_2 := -\alpha^{-4} Q \alpha \alpha, \\
D_3 := -\alpha^{-3} Q \alpha \beta, \\
D_4 := -\alpha^{-2} Q \beta \beta, \\
D_5 := -\alpha^{-6} Q \alpha^2 - \alpha^{-6} Q Q \alpha \alpha + \alpha^{-7} Q Q \alpha, \\
D_6 := -\alpha^{-5} Q \alpha Q \beta - \alpha^{-5} Q Q \alpha \beta, \\
D_7 := -\alpha^{-4} Q \alpha \alpha Q \beta - 2\alpha^{-4} Q \alpha \beta Q \alpha - \alpha^{-4} Q Q \alpha \alpha \beta, \\
D_8 := -\alpha^{-3} Q \beta \beta Q \alpha - 2\alpha^{-3} Q \alpha \beta Q \beta - \alpha^{-3} Q Q \alpha \beta \beta, \\
D_9 := -3\alpha^{-5} Q \alpha \alpha Q \alpha - \alpha^{-5} Q Q \alpha \alpha \alpha, \\
D_{10} := -3\alpha^{-2} Q \beta \beta Q \beta - \alpha^{-2} Q Q \beta \beta \beta, \\
D_{11} := -2\alpha^{-3} Q \alpha \beta + 2\alpha^{-3} Q \alpha^2 + 2\alpha^{-4} Q Q \alpha \alpha - 2\alpha^{-5} Q Q \alpha, \\
D_{12} := -2\alpha^{-2} Q \beta \beta + 2\alpha^{-3} Q Q \alpha \beta + 2\alpha^{-3} Q Q \beta \beta.
\]
Now, by plugging (3.10) into (3.9), and contracting the obtained result with $a^k l$, we get (3.1).

**Proof of Theorem 1.1:** Let $F = \alpha \phi(s), s = \beta/\alpha$, be an $(\alpha, \beta)$-metric on an $n$-dimensional manifold $M$. By multiplying (3.1) with $y_i$ and $y^j$, we get

$$-\alpha Q_{\alpha \alpha} t_{00} = 0.$$  

(3.11)

If $Q_{\alpha \alpha} = 0$ then

$$Q = c_1 \alpha + c_2 \alpha^2,$$

where $c_1$ and $c_2$ are real constants. Thus, we get

$$F = c_3 \alpha \left( \frac{\beta}{\alpha} \right)^{\frac{c_2}{1+c_2}} \left( c_1 \frac{\beta}{\alpha} + c_2 + 1 \right)^{\frac{1}{1+c_2}},$$

where $c_3$ is a real constant. This is a contradiction with our assumption. Then by (3.11), we get $t_{00} = 0$ which results that $s_{i0} = 0$. This means that $\beta$ is a closed one-form. By assumption, $\beta$ is parallel one-form and then $F$ reduces to a Berwald metric. □

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**References**