Generalized Douglas-Weyl Finsler Metrics

Mohammad Hosein Emamian, Akbar Tayebi∗
Department of Mathematics, Faculty of Science University of Qom, Qom, Iran.
E-mail: hosein.emamian@gmail.com
E-mail: akbar.tayebi@gmail.com

Abstract. In this paper, we study generalized Douglas-Weyl Finsler metrics. We find some conditions under which the class of generalized Douglas-Weyl \((\alpha, \beta)\)-metric with vanishing S-curvature reduce to the class of Berwald metrics.

Keywords: Generalized Douglas-Weyl metrics, S-curvature.


1. Introduction

Let \((M, F)\) be a Finsler manifold. In local coordinates, a curve \(c(t)\) is a geodesic if and only if its coordinates \((c^i(t))\) satisfy \(\dddot{c}^i + 2G^i(\dot{c}) = 0\), where the local functions \(G^i = G^i(x, y)\) are called the spray coefficients [10]. \(F\) is called a Berwald metric, if \(G^i\) are quadratic in \(y \in T_xM\) for any \(x \in M\) or equivalently \(G^i = \frac{1}{2} \Gamma^i_{jk}(x)y^jy^k\). As a generalization of Berwald curvature, Bácsó-Matsumoto introduced the notion of Douglas metrics which are projective invariants in Finsler geometry [2]. \(F\) is called a Douglas metric if \(G^i = \frac{1}{2} \Gamma^i_{jk}(x)y^jy^k + P(x, y)y^i\).

A Finsler metric \(F\) is called generalized Douglas-Weyl metric (briefly, GDW-metric) if \(D^i_{jkl||m}y^m = T_{jkl}y^i\) holds for some tensor \(T_{jkl}\), where \(D^i_{jkl||m}\) denotes the horizontal covariant derivatives of \(D^i_{jkl}\) with respect to the Berwald
connection of $F$ [8][18]. For a manifold $M$, let $\mathcal{GDW}(M)$ denotes the class of all Finsler metrics satisfying in above relation for some tensor $T_{jkl}$. In [3], Bácsó-Papp showed that $\mathcal{GDW}(M)$ is closed under projective changes. Then, Najafi-Shen-Tayebi characterized generalized Douglas-Weyl Randers metrics [8]. In [18], it is proved that all generalized Douglas-Weyl spaces with vanishing Landsberg curvature have vanishing the quantity $H$. For other works, see [12] and [13].

The notion of S-curvature is originally introduced by Shen for the volume comparison theorem [9]. The Finsler metric $F$ vanishing $\alpha,\beta$ are of some important geometric structures which deserve to be studied deeply.

An $(\alpha,\beta)$-metric is a Finsler metric on $M$ defined by $F := \alpha \phi(s)$, $s = \beta/\alpha$, where $\phi = \phi(s)$ is a $C^\infty$ function on the $(-b_0, b_0)$ with certain regularity, $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta(y) = b_i(x)y^i$ is a 1-form on $M [6]$. In this paper, we are going to study generalized Douglas-Weyl $(\alpha,\beta)$-metrics with vanishing S-curvature.

**Theorem 1.1.** Let $F = \alpha \phi(s)$, $s = \beta/\alpha$, be an $(\alpha,\beta)$-metric on a manifold $M$ of dimension $n \geq 3$. Suppose that

$$F \neq c_3\alpha \left( \frac{\beta}{\alpha} \right)^{\frac{n+2}{n+3}} \left( c_1 \frac{\beta}{\alpha} + c_2 + 1 \right)^{\frac{n+3}{n+2}}$$

and

$$F \neq d_1 \sqrt{\alpha^2 + d_2 \beta^2} + d_3 \beta.$$ 

where $c_1$, $c_2$, $c_3$, $d_1$, $d_2$ and $d_3$ are real constants. Let $F$ has vanishing S-curvature. Then $F$ is a GDW-metric if and only if it is a Berwald metric.

2. Preliminary

Given a Finsler manifold $(M, F)$, then a global vector field $G$ is induced by $F$ on $TM_0$, which in a standard coordinate $(x^i, y^i)$ for $TM_0$ is given by $G = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i}$, where

$$G^i := \frac{1}{4} g^{ij} \left\{ [F^2]_{x^jy^k} y^k - [F^2]_{x^j} \right\}, \quad y \in T_xM.$$ 

The $G$ is called the spray associated to $F$.

Define $B_y : T_xM \otimes T_xM \otimes T_xM \to T_xM$ and $E_y : T_xM \otimes T_xM \to \mathbb{R}$ by $B_y(u, v, w) := B^{ijkl}(y)u^iv^kw^l \frac{\partial}{\partial x^i}l$ and $E_y(u, v) := E_{jk}(y)u^iv^k$ where

$$B^{ijkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}, \quad E_{jk} := \frac{1}{2} B^{ijkl}.$$
\(B\) and \(E\) are called the Berwald curvature and mean Berwald curvature, respectively. \(F\) is called a Berwald and weakly Berwald if \(B = 0\) and \(E = 0\), respectively [5][7].

Let
\[
D^j_{\ k l} := \frac{\partial^3}{\partial y^j \partial y^k \partial y^l} \left( G^i - \frac{1}{n+1} \frac{\partial G^m}{\partial y^m} y^i \right).
\]
It is easy to verify that \(D := D^j_{\ k l} dx^j \otimes \partial_k \otimes dx^l \otimes dx^i\) is a well-defined tensor on slit tangent bundle \(TM_0\). We call \(D\) the Douglas tensor. A Finsler metric with \(D = 0\) is called a Douglas metric. The notion of Douglas metrics was proposed by Bacsó-Matsumoto as a generalization of Berwald metrics [2]. The Douglas tensor \(D\) is a non-Riemannian projective invariant, namely, if two Finsler metrics \(F\) and \(\bar{F}\) are projectively equivalent, \(G^i = \bar{G}^i + P y^i\), where \(P = P(x, y)\) is positively \(y\)-homogeneous of degree one, then the Douglas tensor of \(F\) is same as that of \(\bar{F}\). Finsler metrics with vanishing Douglas tensor are called Douglas metrics [11].

For a Finsler metric \(F\) on an \(n\)-dimensional manifold \(M\), the Busemann-Hausdorff volume form \(dV_F = \sigma_F(x) dx^1 \cdots dx^n\) is defined by
\[
\sigma_F(x) := \frac{\operatorname{Vol}(\mathbb{B}^n(1))}{\operatorname{Vol}\left(\{y^i \in \mathbb{R}^n \mid F\left(y^i \frac{\partial}{\partial x^i} \bigg| x\right) < 1\}\right)}.
\]
Let \(G^i\) denote the geodesic coefficients of \(F\) in the same local coordinate system. The S-curvature is defined by
\[
S(y) := \frac{\partial G^i}{\partial y^i}(x, y) - y^i \frac{\partial}{\partial x^i}\left[ \ln \sigma_F(x) \right],
\]
where \(y = y^i \frac{\partial}{\partial x^i} \big| x \in T_x M\). \(S\) is said to be isotropic if there is a scalar function \(c = c(x)\) on \(M\) such that \(S = (n+1)cF\).

For an \((\alpha, \beta)\)-metric \(F = \alpha \phi(s)\), \(s = \beta/\alpha\), put
\[
\Phi := -(q - sq')[n\Delta + 1 + sq] - (b^2 - s^2)(1 + sq)q'',
\]
where
\[
q := \frac{\phi'}{\phi - sq'}, \quad \Delta := 1 + sq + (b^2 - s^2)q'.
\]
In [4], Cheng-Shen characterize \((\alpha, \beta)\)-metrics with isotropic S-curvature.

**Lemma 2.1.** ([4]) Let \(F = \alpha \phi(s), s = \beta/\alpha\), be a non-Riemannian \((\alpha, \beta)\)-metric on a manifold \(M\) of dimension \(n \geq 3\). Suppose that \(\phi \neq c_1 \sqrt{1 + c_2 s^2} + c_3 s\) for any constant \(c_1 > 0, c_2\) and \(c_3\). Then \(F\) is of isotropic S-curvature \(S = (n+1)cF\) if and only if one of the following holds
\[(a) \ \beta \ satisfies
r_{ij} = \varepsilon(b^2a_{ij} - b_i b_j), \quad s_j = 0, \quad (2.1)\]
where $\varepsilon = \varepsilon(x)$ is a scalar function, $b := \|\beta_x\|_\alpha$ and $\phi = \phi(s)$ satisfies

$$\Phi = -2(n+1)k \phi \Delta^2 b^2 \ldots + C_3 b_j y_{i00} = C_4 y_{jsi}^0 + C_5 \left( b_j s_{i00}^j + s_{j0} s_{i0}^j \right) + C_6 s_{j0} y_{i0} + C_7 (y_j t_{i0}^j + s_{j0} s_{i0}^j) + C_8 b_j t_{i0}^j, \quad (3.1)$$

where $k$ is a constant. In this case, $S = (n+1)cF$ with $c = k\varepsilon$.

(b) $\beta$ satisfies

$$r_{ij} = 0, \quad s_j = 0 \quad (2.3)$$

In this case, $S = 0$.

The characterization of Finsler metrics with isotropic $S$-curvature in Cheng-Shen’s paper is not complete [4]. Their result is correct for dimension $n \geq 3$.

For the case $\text{dimension}(M) = 2$, see [16].

3. Proof of Main Results

Let $F := \alpha \phi(s)$, $s = \beta/\alpha$, be an $(\alpha, \beta)$-metric on a manifold $M$, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ and $\beta(y) = b_i(x)y^i$. Define $b_{ij}$ by $b_{ij}\theta^i := db_i - b_j \theta^i_j$, where $\theta^i := dx^i$ and $\theta^i_j := \tilde{\Gamma}^i_{jk} dx^k$ denote the Levi-Civita connection forms of $\alpha$. Let

$$r_{ij} := \frac{1}{2} \left[ b_{ij} + b_{ji} \right], \quad s_{ij} := \frac{1}{2} \left[ b_{ij} - b_{ji} \right],$$

$$r_{i0} := r_{ij} y^j_i, \quad r_{00} := r_{ij} y^i_j, \quad r_j := b^i r_{ij}, \quad t^i_j := s^i_m s^m_j,$$

$$s_{i0} := s_{ij} y^j_i, \quad s_j := b^i s_{ij}, \quad r_0 := r_j y^j_i, \quad s_0 := s_j y^j_i.$$

Then $\beta = b_i(x)y^i$ is a constant Killing one-form on $M$ if $r_{ij} = s_{ij} = 0$ hold. By definition, we have

$$b_{ij} = s_{ij} + r_{ij}.$$ 

Since $y^i_j 0 = 0$, then for a constant Killing 1-form $\beta$ we have

$$r_{00} = 0, \quad r_i + s_i = 0.$$

For an $(\alpha, \beta)$-metric $F = \alpha \phi(s)$, $s = \beta/\alpha$, the following hold.

**Proposition 3.1.** Let $F = \alpha \phi(s)$, $s = \beta/\alpha$, be an $(\alpha, \beta)$-metric on an $n$-dimensional manifold $M$ of dimension $n \geq 3$, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a one-form on $M$. Suppose that $F$ is of vanishing $S$-curvature. Then $F$ is a GDW-metric if and only if the following holds

$$C_1 s_{j0} y^i_j - (C_2 y_j + C_3 b_j) y^i t_{00} = C_4 y_j s_{i0}^i + C_5 (b_j s_{i0}^j + s_{j0} s_{i0}^j) + C_6 s_{j0}^i + C_7 (y_j t_{i0}^j + s_{j0} s_{i0}^j) + C_8 b_j t_{i0}^j, \quad (3.1)$$
where
\[ C_1 := -\left[(n+1)Q_\alpha + 2\beta Q_{\alpha\beta}\right]^{\alpha^{-3}} - \left[Q_{\alpha\alpha} + b^2 Q_{\beta\beta}\right]^{\alpha^{-2}}, \]
\[ C_2 := (n+1)\left[Q_\alpha^2 + Q_{\alpha\alpha} - \alpha^{-1}Q_{\alpha\alpha}\right]^{\alpha^{-4}} - 2\left[Q_{\alpha\beta} + Q_{\alpha\alpha\beta}\right]^{\beta\alpha^{-5}} \]
\[ + 2\left[2Q_{\alpha\beta} + Q_{\alpha\alpha}Q_{\beta} + Q_{\alpha\alpha\beta}\right]^{\beta\alpha^{-4}} + b^2\left[2Q_{\alpha\beta} + Q_{\alpha\beta}\right]^{\beta\alpha^{-3}} \]
\[ + \left[b^2Q_{\alpha\beta} + 3Q_{\alpha\alpha} + QQ_{\alpha\alpha\alpha}\right]^{\alpha^{-3}}, \]
\[ C_3 := (n+3)\left[Q_\alpha Q_{\beta} + Q_{\alpha\beta}\right]^{\alpha^{-3}} + 2\left[Q_{\alpha} Q_{\beta\beta} + Q_{\alpha\beta}\right]^{\beta\alpha^{-3}} \]
\[ + \left[2Q_{\alpha\beta} + Q_{\beta\alpha} + QQ_{\alpha\alpha\beta} + 4\beta\alpha^{-1}Q_{\beta\beta}\right]^{\alpha^{-2}} \]
\[ + b^2\left[3Q_{\beta\beta} + QQ_{\beta\beta}\right]^{\alpha^{-2}}, \]
\[ C_4 := -\left[(n+1)Q_\alpha + 2\beta Q_{\alpha\beta}\right]^{\alpha^{-3}} + 2\left[Q_{\alpha\alpha} + Q_{\alpha\alpha}\right]^{\alpha^{-2}} \]
\[ + \left[b^2Q_{\alpha\beta} + Q_{\alpha\alpha}\right]^{\alpha^{-1}}, \]
\[ C_5 := (n+3)\alpha^{-1}Q_{\alpha\beta} + Q_{\alpha\alpha\beta} + 2\beta\alpha^{-1}Q_{\beta\beta}, \]
\[ C_6 := (n+1)\alpha^{-1}Q_{\alpha} + Q_{\alpha\alpha} + 2\beta\alpha^{-1}Q_{\alpha\beta} + b^2 Q_{\beta\beta}, \]
\[ C_7 := (n+1)\alpha^{-2}Q_\alpha - (n+1)\alpha^{-2}(Q_\alpha^2 + Q_{\alpha\alpha}) - 2\beta\alpha^{-2}Q_{\alpha\beta} \]
\[ + 2\left[QQ_{\alpha\beta} + Q_{\alpha\beta}\right]^{\beta\alpha^{-3} - b^2\left[QQ_{\alpha\beta} + QQ_{\beta\beta}\right]^{\alpha^{-1}} \]
\[ - 2\left[2Q_{\alpha\beta} + Q_{\beta\alpha}\right]^{\beta\alpha^{-2}} \]
\[ - b^2\alpha^{-1}Q_{\alpha\beta} - 3\alpha^{-1}Q_{\alpha\alpha\alpha} - 2\alpha^{-1}Q_{\alpha\alpha\alpha} \]
\[ C_8 := -(n+3)\left[Q_\alpha Q_{\beta} + Q_{\alpha\alpha}\right]^{\alpha^{-1}} - 2\left[2Q_{\beta\alpha} + Q_{\alpha\beta}\right]^{\beta\alpha^{-1}} \]
\[ - b^2\left[QQ_{\beta\beta} + QQ_{\beta\beta}\right]^{\alpha^{-1}} - Q_{\beta\alpha} - QQ_{\alpha\alpha\beta} - 2Q_{\alpha\beta}. \]

**Proof.** Let \( G^i \) and \( G'_i \) denote the spray coefficients of \( F \) and \( \alpha \), respectively, in the same coordinate system. Then, we have
\[ G^i = G'_i + Py^i + Q^i, \] (3.2)

where
\[ Q := \alpha q = \frac{\alpha \phi'}{\phi - s \phi'}, \]
\[ P := \alpha^{-1} \Theta(r_{00} - 2Qs_0), \quad Q^i := Qs^i_0 + \Psi(r_{00} - 2Qs_0) b^i, \]
\[ \Theta = \frac{q - sq'}{2\Delta} = \frac{\phi' - s(\phi'' + \phi'')}{2\phi(\phi - s \phi') + (b^2 - s^2) \phi''} \]
\[ \Psi := \frac{q'}{2\Delta} = \frac{1}{2} \frac{\phi''}{(\phi - s \phi') + (b^2 - s^2) \phi''}. \]
By Lemma 2.1, we have \( r_{00} = s_0 = 0 \). Then (3.2) reduces to following

\[
G^i = G^i_\alpha + Qs^i_0. \tag{3.3}
\]

Let “\( \nabla \)" and “\( \partial \)" denote the covariant differentiations with respect to \( G^i \) and \( G^i_\alpha \) respectively. Then by (3.3), we have

\[
\begin{align*}
D^i_{jk\|m} y^m &= D^i_{jk\|m} y^m - 2Q s^0_p \frac{\partial D^i_{jkl}}{\partial y^p} + D^p_{jk\|l} \tilde{N}^i_p - D^i_{jk\|l} \tilde{N}^p_j \\
&
- D^i_{jkl} \tilde{N}^p_k - D^i_{jkl} \tilde{N}^p_j, \tag{3.4}
\end{align*}
\]

where

\[
D^i_{jk\|l} y^m = \alpha^{-4}(Q_{\alpha\alpha} - \alpha^{-1} Q_{\alpha}) (A_{jkl} y_l + A_{kjl} y_j + A_{ljk} y_k) s^i_{00} \right.
+ \alpha^{-3} Q_{\alpha}(A_{jkl} s^i_{l0} + A_{kjl} s^i_{j0} + A_{ljk} s^i_{k0})
+ \alpha^{-3} Q_{\alpha\beta} \left[ (A_{jkl} b_l + A_{kjl} b_j + A_{ljk} b_k) s^i_{00} \right.
+ (A_{jkl} s_{l0} + A_{kjl} s_{j0} + A_{ljk} s_{k0}) s^i_0 \\
+ \alpha^{-2} Q_{\alpha\beta\gamma} \left[ (y_j y_k b_l + y_k y_l b_j + y_l y_j b_k) s^i_{00} \right.
+ (y_j y_k s_{l0} + y_k y_l s_{j0} + y_l y_j s_{k0}) s^i_0 \\
+ (y_j b_l + y_k b_j) s_{k0} \right.
+ (y_j b_l + y_k b_j) s_{k0} \\
+ (y_k b_l + y_j b_k) s_{j0} + (y_j b_k + y_k b_j) s_{j0} \right]
+ \alpha^{-2} Q_{\alpha\gamma} \left[ (y_j b_k + y_k b_j) s^i_{10} \right.
+ (y_k b_l + y_j b_k) s^i_{j0} + (y_l b_k + y_j b_l) s^i_{k0} \right]
+ \alpha^{-1} Q_{\alpha\beta\gamma} \left[ (y_j b_k + y_k b_j) s^i_{10} \right.
+ (y_k b_l + y_j b_k) s^i_{j0} + (y_l b_k + y_j b_l) s^i_{k0} \right]
+ (y_j s_{k0} + y_k s_{j0}) s^i_1 + (y_k s_{l0} + y_l s_{k0}) s^i_2 + (y_l s_{j0} + y_j s_{l0}) s^i_3 \\
+ Q_{\beta\gamma} \left[ b_k b_s s^i_{10} \right.
+ b_k b_s s^i_{j0} + b_j b_s s^i_{k0} \right]
+ (s_{j0} b_k + b_j s_{k0}) s^i_0
+ (s_{k0} b_l + b_k s_{l0}) s^i_0 + (b_j s_{j0} + b_s s_{k0}) s^i_0 \tag{3.5}
\]

and

\[
A_{ij} = \alpha^2 a_{ij} - y_i y_j, \tag{3.6}
\]

\[
\tilde{N}^i_p = Q s^i_p + \left[ \alpha^{-1} Q_{\alpha} y_p + Q_{\beta} b_p \right] s^i_0, \tag{3.7}
\]

\[
\frac{\partial D^i_{jk\|l}}{\partial y^p} = Q_{jk\|l} s^i_0 + Q_{jk\|l} s^i_p + Q_{jkl} s^i_1 + Q_{jlp} s^i_k + Q_{jlp} s^i_j. \tag{3.8}
\]

Let \( F \) is a GDW-metric. Then there exists a tensor \( D_{jk\|l} \) such that

\[
D^i_{jk\|l} y^m = D_{jk\|l} y^i. \]
By (3.4), we have

\begin{align*}
D_{jkl} y^i &= D_{jkl|m} y^m - 2Q \frac{\partial D_{jkl}}{\partial y^p} s_0^p + D_{jkl} \tilde{N}_p^i - D_{jkl} \tilde{N}_p^i - D_{jkl} \tilde{N}_p^i, \\
&= -3\alpha^{-2}Q\beta\beta - \alpha^{-2}QQ\beta\beta,
\end{align*}

By contracting (3.9) with \( y_i \) and using (3.5), (3.7) and (3.8) we get the following

\begin{align*}
D_{jkl} &= D_1 \left[ A_{jk}s_{i0|0} + A_{kl}s_{j0|0} + A_{jl}s_{k0|0} \right] \\
&+ D_2 \left[ y_jy_k s_{i0|0} + y_k y_j s_{j0|0} + y_j y_l s_{k0|0} \right] \\
&+ D_3 \left[ (y_j b_k + y_k b_j) s_{i0|0} + (y_j b_l + y_l b_j) s_{j0|0} + (y_j b_l + y_l b_j) s_{k0|0} \right] \\
&+ D_4 \left[ b_j b_k s_{i0|0} + b_k b_j s_{j0|0} + b_j b_l s_{k0|0} \right] \\
&+ D_5 \left[ A_{jky_i} + A_{kly_j} + A_{jlly_k} \right] t_{00} \\
&+ D_6 \left[ A_{jkb_l} + A_{kblj} + A_{jlkb} \right] t_{00} \\
&+ D_7 \left[ y_j y_k b_l + y_k y_l b_j + y_j y_l b_k \right] t_{00} \\
&+ D_8 \left[ y_j b_k b_l + y_k b_l b_j + y_k b_j b_l \right] t_{00} \\
&+ D_9 \left[ y_j y_k b_l t_{00} + D_{10} b_j b_k b_l t_{00} \right] \\
&+ D_{11} \left[ y_i s_{j0|k0} + y_j s_{k0|i0} + y_k s_{j0|i0} \right] \\
&+ D_{12} \left[ b_i s_{j0|k0} + b_j s_{k0|i0} + b_k s_{j0|i0} \right],
\end{align*}

where

\begin{align*}
D_1 &= -\alpha^{-5}Q\alpha, \\
D_2 &= -\alpha^{-4}Q\alpha\alpha, \\
D_3 &= -\alpha^{-3}Q\alpha\beta, \\
D_4 &= -\alpha^{-2}Q\beta\beta, \\
D_5 &= -\alpha^{-6}Q\alpha^2 - \alpha^{-6}QQ\alpha\alpha + \alpha^{-7}QQ\alpha, \\
D_6 &= -\alpha^{-5}Q\alpha Q\beta - \alpha^{-5}QQ\alpha\beta, \\
D_7 &= -\alpha^{-4}Q\alpha\alpha Q\beta - 2\alpha^{-4}Q\alpha\beta Q\alpha - \alpha^{-4}QQ\alpha\beta, \\
D_8 &= -\alpha^{-3}Q\beta\beta Q\alpha - 2\alpha^{-3}Q\alpha\beta Q\beta - \alpha^{-3}QQ\alpha\beta, \\
D_9 &= -3\alpha^{-5}Q\alpha\alpha Q\alpha - \alpha^{-5}QQ\alpha\alpha, \\
D_{10} &= -3\alpha^{-2}Q\beta\beta\beta - \alpha^{-2}QQ\beta\beta\beta, \\
D_{11} &= -2\alpha^{-3}Q\alpha\beta + 2\alpha^{-3}Q\beta^2 + 2\alpha^{-4}QQ\alpha\alpha - 2\alpha^{-5}QQ\alpha, \\
D_{12} &= -2\alpha^{-2}Q\beta\beta + 2\alpha^{-3}QQ\alpha\beta + 2\alpha^{-3}Q\alpha Q\beta.
\end{align*}
Now, by plugging (3.10) into (3.9), and contracting the obtained result with $a^kl$, we get (3.1). □

**Proof of Theorem 1.1:** Let $F = \alpha \phi(s), s = \beta/\alpha$, be an $(\alpha, \beta)$-metric on an $n$-dimensional manifold $M$. By multiplying (3.1) with $y_i$ and $y^j$, we get

$$-\alpha QQ_{\alpha\alpha}t_{00} = 0.$$ (3.11)

If $Q_{\alpha\alpha} = 0$ then

$$Q = c_1\alpha + c_2\frac{\alpha^2}{\beta},$$

where $c_1$ and $c_2$ are real constants. Thus, we get

$$F = c_3 \alpha \left(\frac{\beta}{\alpha}\right)^{\frac{c_1}{c_2}} \left(c_1\alpha + c_2 + 1\right)^{\frac{c_1}{c_2}},$$

where $c_3$ is a real constant. This is a contradiction with our assumption. Then by (3.11), we get $t_{00} = 0$ which results that $s_{i0} = 0$. This means that $\beta$ is a closed one-form. By assumption, $\beta$ is parallel one-form and then $F$ reduces to a Berwald metric. □

ACKNOWLEDGMENTS

The authors are very grateful to the anonymous referee for his or her comments and suggestions.

REFERENCES