Generalized Douglas-Weyl Finsler Metrics

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Abstract. In this paper, we study generalized Douglas-Weyl Finsler metrics. We find some conditions under which the class of generalized Douglas-Weyl \((\alpha, \beta)\)-metric with vanishing S-curvature reduce to the class of Berwald metrics.

Keywords: Generalized Douglas-Weyl metrics, S-curvature.


1. Introduction

Let \((M, F)\) be a Finsler manifold. In local coordinates, a curve \(c(t)\) is a geodesic if and only if its coordinates \((c^i(t))\) satisfy \(\ddot{c}^i + 2G^i(\dot{c}) = 0\), where the local functions \(G^i = G^i(x, y)\) are called the spray coefficients [10]. \(F\) is called a Berwald metric, if \(G^i\) are quadratic in \(y \in T_xM\) for any \(x \in M\) or equivalently \(G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^jy^k\). As a generalization of Berwald curvature, Bácsó-Matsumoto introduced the notion of Douglas metrics which are projective invariants in Finsler geometry [2]. \(F\) is called a Douglas metric if \(G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^jy^k + P(x, y)y^i\).

A Finsler metric \(F\) is called generalized Douglas-Weyl metric (briefly, GDW-metric) if \(D^i_{jkl}[m]y^m = T_{jkl}y^i\) holds for some tensor \(T_{jkl}\), where \(D^i_{jkl}[m]\) denotes the horizontal covariant derivatives of \(D^i_{jkl}\) with respect to the Berwald
connection of $F$ [8][18]. For a manifold $M$, let $\mathcal{GDW}(M)$ denotes the class of all Finsler metrics satisfying in above relation for some tensor $T_{jkl}$. In [3], Bácsó-Papp showed that $\mathcal{GDW}(M)$ is closed under projective changes. Then, Najafi-Shen-Tayebi characterized generalized Douglas-Weyl Randers metrics [8]. In [18], it is proved that all generalized Douglas-Weyl spaces with vanishing Landsberg curvature have vanishing the quantity $H$. For other works, see [12] and [13].

The notion of S-curvature is originally introduced by Shen for the volume comparison theorem [9]. The Finsler metric $F$ is said to be of isotropic S-curvature if $S = (n+1)cF$, where $c = c(x)$ is a scalar function on $M$. In [14], it is shown that every isotropic Berwald metric has isotropic S-curvature. In [4], Cheng-Shen show that every $(\alpha, \beta)$-metric with constant Killing 1-form has vanishing S-curvature. Then, Bácsó-Cheng-Shen proved that a Finsler metric $F = \alpha \pm \beta^2/\alpha + \epsilon \beta$ has vanishing S-curvature if and only if $\beta$ is a constant Killing 1-form [1]. Therefore, the Finsler metrics with vanishing S-curvature are of some important geometric structures which deserve to be studied deeply.

An $(\alpha, \beta)$-metric is a Finsler metric on $M$ defined by $F := \alpha \phi(s), s = \beta/\alpha$, where $\phi = \phi(s)$ is a $C^\infty$ function on the $(-b_0, b_0)$ with certain regularity, $\alpha = (a(x)y^jy^k)$ is a Riemannian metric and $\beta(y) = b_i(x)y^i$ is a 1-form on $M$ [6]. In this paper, we are going to study generalized Douglas-Weyl $(\alpha, \beta)$-metrics with vanishing S-curvature.

**Theorem 1.1.** Let $F = \alpha \phi(s), s = \beta/\alpha$, be an $(\alpha, \beta)$-metric on a manifold $M$ of dimension $n \geq 3$. Suppose that

$$F \neq c_3\alpha \left( \frac{\beta}{\alpha} \right)^{\frac{c_2+1}{c_1}} \left( c_1 \beta + c_2 + 1 \right)^{\frac{1}{c_2}} \quad \text{and} \quad F \neq d_1\sqrt{\alpha^2 + d_2\beta^2} + d_3\beta.$$ 

where $c_1, c_2, c_3, d_1, d_2$ and $d_3$ are real constants. Let $F$ has vanishing S-curvature. Then $F$ is a GDW-metric if and only if it is a Berwald metric.

2. **Preliminary**

Given a Finsler manifold $(M, F)$, then a global vector field $G$ is induced by $F$ on $TM_0$, which in a standard coordinate $(x^i, y^i)$ for $TM_0$ is given by $G = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i}$, where

$$G^i := \frac{1}{4} y^{il} \left\{ [F^2]_{x^l y^i y^k} - [F^2]_{x^i} \right\}, \quad y \in T_x M.$$

The $G$ is called the spray associated to $F$.

Define $B_y : T_x M \otimes T_x M \otimes T_x M \rightarrow T_x M$ and $E_y : T_x M \otimes T_x M \rightarrow \mathbb{R}$ by

$$B_{yjkl}(u, v, w) := B^i_{jkl}(y)u^i v^j w^k \frac{\partial}{\partial x^i} \quad \text{and} \quad E_y(u, v) := E_{jkl}(y)u^j v^k$$

where

$$B^i_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}, \quad E_{jkl} := \frac{1}{2} B^m_{jkl}.$$
B and E are called the Berwald curvature and mean Berwald curvature, respectively. F is called a Berwald and weakly Berwald if B = 0 and E = 0, respectively [5][7].

Let
\[ D^i_{jk} := \frac{\partial G^i}{\partial y^j} \delta_y^k \delta_y^l - \left( \frac{1}{n+1} \frac{\partial G^m}{\partial y^n} y^i \right). \]

It is easy to verify that \( \mathcal{D} := D^i_{jk} \delta x^j \otimes \delta y^k \otimes \delta x^l \) is a well-defined tensor on slit tangent bundle \( TM_0 \). We call \( \mathcal{D} \) the Douglas tensor. A Finsler metric with \( \mathcal{D} = 0 \) is called a Douglas metric. The notion of Douglas metrics was proposed by Bácsó-Matsumoto as a generalization of Berwald metrics [2].

The Douglas tensor \( \mathcal{D} \) is a non-Riemannian projective invariant, namely, if two Finsler metrics \( F \) and \( \bar{F} \) are projectively equivalent, \( G^i = \bar{G}^i + Py^i \), where \( P = P(x,y) \) is positively \( y \)-homogeneous of degree one, then the Douglas tensor of \( F \) is same as that of \( \bar{F} \). Finsler metrics with vanishing Douglas tensor are called Douglas metrics [11].

For a Finsler metric \( F \) on an \( n \)-dimensional manifold \( M \), the Busemann-Hausdorff volume form \( dV_F = \sigma_F(x) dx^1 \cdots dx^n \) is defined by
\[ \sigma_F(x) := \frac{\text{Vol}(\mathbb{B}^n(1))}{\text{Vol}\left[ \left( y^i \in \mathbb{R}^n \mid F\left( y^i \frac{\partial}{\partial x^i} \big|_x \right) < 1 \right) \right]}. \]

Let \( G^i \) denote the geodesic coefficients of \( F \) in the same local coordinate system. The S-curvature is defined by
\[ S(y) := \frac{\partial G^i}{\partial y^i}(x,y) - y^i \frac{\partial}{\partial x^i}\left[ \ln \sigma_F(x) \right], \]
where \( y = y^i \frac{\partial}{\partial x^i} \big|_x \in T_x M \). \( S \) is said to be isotropic if there is a scalar function \( c = c(x) \) on \( M \) such that \( S = (n+1)cF \).

For an \((\alpha,\beta)\)-metric \( F = \alpha \phi(s) \), \( s = \beta / \alpha \), put
\[ \Phi := -(q - sq')[n\Delta + 1 + sq] - (b^2 - s^2)(1 + sq)q'', \]
where
\[ q := \frac{\phi'}{\phi - sq'}, \quad \Delta := 1 + sq + (b^2 - s^2)q'. \]

In [4], Cheng-Shen characterize \((\alpha,\beta)\)-metrics with isotropic S-curvature.

**Lemma 2.1.** ([4]) Let \( F = \alpha \phi(s) \), \( s = \beta / \alpha \), be a non-Riemannian \((\alpha,\beta)\)-metric on a manifold \( M \) of dimension \( n \geq 3 \). Suppose that \( \phi \neq c_1 \sqrt{1 + c_2 s^2} + c_3 s \) for any constant \( c_1 > 0, c_2 \) and \( c_3 \). Then \( F \) is of isotropic S-curvature \( S = (n+1)cF \) if and only if one of the following holds
(a) \( \beta \) satisfies
\[ r_{ij} = \varepsilon(b^2 a_{ij} - b_i b_j), \quad s_j = 0, \quad (2.1) \]
where \( \varepsilon = \varepsilon(x) \) is a scalar function, \( b := \|\beta x\|_\alpha \) and \( \phi = \phi(s) \) satisfies

\[
\Phi = -2(n+1)k \phi \Delta^2 b^2 + C_3 b j y i t_00 = C_4 y j s^i_0 + C_5 (b_j s^i_0 + s_j 0 s^i_0) + C_6 s^i_0 + C_7 (y_j t^i_0 + s_j 0 s^i_0) + C_8 b_j t^i_0, \quad (3.1)
\]

where \( k \) is a constant. In this case, \( S = (n+1)cF \) with \( c = k\varepsilon \).

(b) \( \beta \) satisfies

\[
\begin{align*}
  r_{ij} &= 0, \quad s_j = 0 \\
\end{align*}
\]

(2.3)

In this case, \( S = 0 \).

The characterization of Finsler metrics with isotropic S-curvature in Cheng-Shen’s paper is not complete [4]. Their result is correct for dimension \( n \geq 3 \).

For the case dimension \( (M) = 2 \), see [16].

3. PROOF OF MAIN RESULTS

Let \( F := \alpha \phi(s) \), \( s = \beta/\alpha \), be an \( (\alpha, \beta) \)-metric on a manifold \( M \), where

\[ \alpha = \sqrt{a_{ij}(x)y^i y^j} \] and \( \beta(y) = b_i(x)y^i \). Define \( b_{ij} \) by \( b_{ij} \theta^i := db_i - b_j \theta^j \), where \( \theta^i := dx^i \) and \( \theta^i := \Gamma^i_{jk} dx^k \) denote the Levi-Civita connection forms of \( \alpha \). Let

\[
\begin{align*}
  r_{ij} &:= \frac{1}{2} [b_{ij} + b_{ji}], \quad s_{ij} := \frac{1}{2} [b_{ij} - b_{ji}], \\
  r_{io} &:= r_{ij} y^i, \quad r_{00} := r_{ij} y^i y^j, \quad r_j := b^r r_{ij}, \quad t^i_j := s^i_m s^m_j, \\
  s_{io} &:= s_{ij} y^i, \quad s_j := b^i s_{ij}, \quad r_0 := r_{ij} y^i, \quad s_0 := s_j y^i.
\end{align*}
\]

Then \( \beta = b_i(x)y^i \) is a constant Killing one-form on \( M \) if \( r_{ij} = s_j = 0 \) hold. By definition, we have

\[
  b_{ij} = s_{ij} + r_{ij}.
\]

Since \( y^i |_a = 0 \), then for a constant Killing 1-form \( \beta \) we have

\[
  r_{00} = 0, \quad r_i + s_i = 0.
\]

For an \( (\alpha, \beta) \)-metric \( F = \alpha \phi(s) \), \( s = \beta/\alpha \), the following hold.

**Proposition 3.1.** Let \( F = \alpha \phi(s) \), \( s = \beta/\alpha \), be an \((\alpha, \beta)\)-metric on an \( n\)-dimensional manifold \( M \) of dimension \( n \geq 3 \), where \( \alpha = \sqrt{a_{ij}(x)y^i y^j} \) is a Riemannian metric and \( \beta = b_i(x)y^i \) is a one-form on \( M \). Suppose that \( F \) is of vanishing S-curvature. Then \( F \) is a GDW-metric if and only if the following holds

\[
\begin{align*}
  C_1 s_j 0 y^i |_0 - (C_2 y_j + C_3 b_j) y^i t_{00} &= C_4 y_j s^i_0 + C_5 (b_j s^i_0 + s_j 0 s^i_0) + C_6 s^i_0 + C_7 (y_j t^i_0 + s_j 0 s^i_0) + C_8 b_j t^i_0, \quad (3.1)
\end{align*}
\]
where

\[ C_1 := -\left[ (n+1)Q_{\alpha} + 2\beta Q_{\alpha\beta} \right]^{\alpha^{-3}} - \left[ Q_{\alpha\alpha} + b^2 Q_{\beta\beta} \right]^{\alpha^{-2}}, \]

\[ C_2 := (n+1) \left[ Q_{\alpha}^2 + Q_{\alpha\alpha} - \alpha^{-1} Q_{\alpha} \right]^{\alpha^{-4}} - 2 \left[ Q_{\alpha\beta} + Q_{\alpha\alpha\beta} \right]^{\beta\alpha^{-5}} + 2 \left[ 2Q_{\alpha\beta} + Q_{\alpha\alpha\beta} + Q_{\alpha\alpha} \right]^{\beta\alpha^{-4}} + b^2 \left[ 2Q_{\alpha\beta} + Q_{\alpha\alpha\beta} \right]^{\alpha^{-3}}, \]

\[ C_3 := (n+3) \left[ Q_{\alpha\beta} + Q_{\alpha\alpha\beta} \right]^{\alpha^{-3}} + 2 \left[ Q_{\alpha\beta} + Q_{\alpha\alpha\beta} \right]^{\beta\alpha^{-3}} + b^2 \left[ 3Q_{\alpha\beta} + Q_{\alpha\alpha\beta} \right]^{\alpha^{-2}}, \]

\[ C_4 := -\left[ (n+1)Q_{\alpha} + 2\beta Q_{\alpha\beta} \right]^{\alpha^{-3}} + 2 \left[ \beta Q_{\alpha\alpha\beta} + Q_{\alpha\alpha} \right]^{\alpha^{-2}} + b^2 \left[ 3Q_{\alpha\beta} + Q_{\alpha\alpha\beta} \right]^{\alpha^{-1}}, \]

\[ C_5 := (n+3)\alpha^{-1} Q_{\alpha\beta} + Q_{\alpha\alpha\beta} + 2\beta\alpha^{-1} Q_{\alpha\beta} + b^2 Q_{\beta\beta}, \]

\[ C_6 := (n+1)\alpha^{-1} Q_{\alpha} + Q_{\alpha\alpha} + 2\beta\alpha^{-1} Q_{\alpha\beta} + b^2 Q_{\beta\beta}, \]

\[ C_7 := (n+1)\alpha^{-3} Q_{\alpha} - (n+1)\alpha^{-2} (Q_{\alpha}^2 + Q_{\alpha\alpha}) - 2\beta\alpha^{-2} Q_{\alpha\alpha\beta}, \]

\[ + 2 \left[ 2Q_{\alpha\beta} + Q_{\alpha\alpha\beta} \right]^{\beta\alpha^{-3}} - b^2 \left[ 2Q_{\alpha\beta} + Q_{\alpha\alpha\beta} \right]^{\alpha^{-1}} - 2 \left[ 2Q_{\alpha\beta} + Q_{\alpha\alpha\beta} \right]^{\beta\alpha^{-2}} - b^2 \alpha^{-1} Q_{\alpha\beta} - 2\alpha^{-1} Q_{\alpha\alpha\beta} - 2\alpha^{-1} Q_{\alpha\alpha}, \]

\[ C_8 := -(n+3) \left[ Q_{\alpha\beta} + Q_{\alpha\alpha\beta} \right]^{\alpha^{-1}} - 2 \left[ 2Q_{\alpha\beta} + Q_{\alpha\alpha\beta} + Q_{\alpha\alpha} \right]^{\beta\alpha^{-1}} - b^2 \left[ 3Q_{\alpha\beta} + 3Q_{\alpha\alpha\beta} \right]^{\alpha^{-2}} - Q_{\alpha\beta} - Q_{\alpha\alpha\beta} - 2Q_{\alpha\alpha} \]

\[ \text{Proof. Let } G^i \text{ and } G^i_0 \text{ denote the spray coefficients of } F \text{ and } \alpha, \text{ respectively, in the same coordinate system. Then, we have} \]

\[ G^i = G^i_0 + Py^i + Q^i, \tag{3.2} \]

where

\[ Q := \alpha q = \frac{\alpha \phi'}{\phi - s\phi'}, \]

\[ P := \alpha^{-1} \Theta(r_{00} - 2Qs_0), \quad Q^i := Qs^i_0 + \Psi(r_{00} - 2Qs_0)b^i, \]

\[ \Theta = \frac{q - sq'}{2\Delta} = \frac{\phi' - s(\phi'' + \phi'\phi')}{2\phi(\phi - s\phi') + (b^2 - s^2)\phi''} \]

\[ \Psi := \frac{q'}{2\Delta} = \frac{1}{2} \frac{\phi''}{(\phi - s\phi') + (b^2 - s^2)\phi''}. \]
By Lemma 2.1, we have $r_{00} = s_0 = 0$. Then (3.2) reduces to following
\[ G_i = G^i_\alpha + Qs^i_\alpha. \]  
Let “$\parallel$” and “$\prime$” denote the covariant differentiations with respect to $G^i_\alpha$ and $G^i_\alpha$ respectively. Then by (3.3), we have
\[
D^i_{jkl\|m}y^m = D^i_{jkl\|m}y^m - 2Qs^i_0 \frac{\partial D^i_{jkl}}{\partial y^p} + D^p_{jkl} \tilde{N}^i_p - D^i_{jkl} \tilde{N}^p_k,
\]
(3.4)
where
\[
D^i_{jkl\|m}y^m = \alpha^{-4}(Q_{\alpha \alpha} - \alpha^{-1}Q_{\alpha})(A_{jk}y_l + A_{kl}y_j + A_{jl}y_k)s^{i\|0}_0 \\
+ \alpha^{-3}Q_{\alpha}(A_{jk}s^{i\|0}_0 + A_{kl}s^{i\|0}_0 + A_{jl}s^{i\|0}_0) \\
+ \alpha^{-3}Q_{\alpha \beta}[(A_{jk}b_l + A_{kl}b_j + A_{jl}b_k)s^{i\|0}_0 \\
+ (A_{jk}s_0 + A_{kl}s_0 + A_{jl}s_0)s^{i\|0}_0] \\
+ \alpha^{-2}Q_{\alpha \beta \gamma}[(y_jy_ky_l + y_ky_jy_l + y_jy_ly_k)s^{i\|0}_0 \\
+ (y_jy_ky_0 + y_ky_jy_0 + y_jy_ly_0)s^{i\|0}_0] \\
+ \alpha^{-1}Q_{\alpha \beta \gamma}[(y_jb_ky_l + y_kb_jy_l + y_kb_jy_k)s^{i\|0}_0 \\
+ (y_jb_ky_0 + y_kb_jy_0 + y_kb_jy_0)s^{i\|0}_0] \\
+ \alpha^{-2}Q_{\alpha \beta \gamma}[(y_jy_ky_0 + y_ky_jy_0 + y_jy_ly_0)s^{i\|0}_0 \\
+ Q_{\beta \gamma \gamma}(b_kb_jy_0 + b_kb_jy_0 + b_kb_jy_0)s^{i\|0}_0 + \alpha^{-3}Q_{\alpha \alpha \alpha \gamma}y_jy_ky_0] \\
+ \alpha^{-1}Q_{\alpha \beta \gamma}[(y_jb_ky_l + y_kb_jy_l + y_kb_jy_k)s^{i\|0}_0 \\
+ (y_jy_ky_0 + y_ky_jy_0 + y_jy_ly_0)s^{i\|0}_0] \\
+ Q_{\beta \gamma \gamma}(b_kb_jy_0 + b_kb_jy_0 + b_kb_jy_0)s^{i\|0}_0 + \alpha^{-3}Q_{\alpha \alpha \alpha \gamma}y_jy_ky_0 \\
+ (y_jy_ky_0 + y_ky_jy_0 + y_jy_ly_0)s^{i\|0}_0) \\
+ Q_{\beta \gamma \gamma}(b_kb_jy_0 + b_kb_jy_0 + b_kb_jy_0)s^{i\|0}_0 + \alpha^{-3}Q_{\alpha \alpha \alpha \gamma}y_jy_ky_0 \\
+ (y_jy_ky_0 + y_ky_jy_0 + y_jy_ly_0)s^{i\|0}_0) \\
+ Q_{\beta \gamma \gamma}(b_kb_jy_0 + b_kb_jy_0 + b_kb_jy_0)s^{i\|0}_0 + \alpha^{-3}Q_{\alpha \alpha \alpha \gamma}y_jy_ky_0)
\]
and
\[
A_{ij} = \alpha^2a_{ij} - y_{yi}y_j, \quad \tilde{N}^i_p = Qs^i_p + \left[\alpha^{-1}Q_{\alpha}y_p + Q_{\beta}b_p\right]s^i_0, \\
\frac{\partial D^i_{jkl}}{\partial y^p} = Q_{jklp}s^i_0 + Q_{jklp}s^i_1 + Q_{jlp}s^i_k + Q_{klp}s^i_j.
\]
(3.7)
Let $F$ is a GDW-metric. Then there exists a tensor $D_{jkl}$ such that
\[
D^i_{jkl\|m}y^m = D_{jkl}y^i.
\]
By (3.4), we have
\[ D_{jkl} y^i = D_{jkl}^i + 2Q \frac{\partial D_{jkl}^i}{\partial y^p} s^0 + D_{jkl}^p \tilde{N}_p^i - D_{jkl}^p \tilde{N}_p^i, \] (3.9)

By contracting (3.9) with \( y_i \) and using (3.5), (3.7) and (3.8) we get the following
\[ D_{jkl} = D_1 + D_2 \left[ y_j y_k s^0_{|0} + y_k y_l s^0_{|0} + y_j y_l s^0_{|0} \right] + D_3 \left[ (y_j b_k + y_l b_j) s^0_{|0} + (y_k b_l + y_l b_k) s^0_{|0} + (y_j b_l + y_l b_j) s^0_{|0} \right] + D_4 \left[ b_j b_k s^0_{|0} + b_k b_l s^0_{|0} + b_j b_l s^0_{|0} \right] + D_5 \left[ A_j k y_i + A_k l y_j + A_j k y_i \right] t_00 + D_6 \left[ A_j k b_i + A_k l b_j + A_j l b_k \right] t_00 + D_7 \left[ y_j y_k b_i + y_k y_l b_j + y_j y_l b_k \right] t_00 + D_8 \left[ y_j b_k b_i + y_k b_l b_j + y_l b_j b_i \right] t_00 + D_9 y_j y_k y_l t_00 + D_{10} b_j b_k b_l t_00 + D_{11} \left[ y_j s^0_{|0} s^0_{|0} + y_k s^0_{|0} s^0_{|0} + y_k s^0_{|0} s^0_{|0} \right] + D_{12} \left[ b_j s^0_{|0} s^0_{|0} + b_k s^0_{|0} s^0_{|0} + b_k s^0_{|0} s^0_{|0} \right], \] (3.10)

where
\[ D_1 := -\alpha^{-5} Q_\alpha, \]
\[ D_2 := -\alpha^{-4} Q_{\alpha \alpha}, \]
\[ D_3 := -\alpha^{-3} Q_{\alpha \beta}, \]
\[ D_4 := -\alpha^{-2} Q_{\beta \beta}, \]
\[ D_5 := -\alpha^{-6} Q_{\alpha} - \alpha^{-6} Q_{\alpha \alpha} + \alpha^{-7} Q_{\alpha \beta}, \]
\[ D_6 := -\alpha^{-5} Q_{\alpha} Q_\beta - \alpha^{-5} Q_{\alpha \beta}, \]
\[ D_7 := -\alpha^{-4} Q_{\alpha \alpha} Q_\beta - 2\alpha^{-4} Q_{\alpha \beta} Q_\alpha - \alpha^{-4} Q_{\alpha \alpha} Q_\beta, \]
\[ D_8 := -\alpha^{-3} Q_{\beta \beta} Q_\alpha - 2\alpha^{-3} Q_{\alpha \beta} Q_\beta - \alpha^{-3} Q_{\alpha \beta} Q_\beta, \]
\[ D_9 := -3\alpha^{-5} Q_{\alpha \alpha} Q_\alpha - \alpha^{-5} Q_{\alpha \alpha} Q_\alpha, \]
\[ D_{10} := -3\alpha^{-2} Q_{\beta \beta} B_{\beta} - \alpha^{-2} Q_{\beta \beta}, \]
\[ D_{11} := -2\alpha^{-3} Q_{\alpha \beta} + 2\alpha^{-3} Q_{\alpha}^2 + 2\alpha^{-4} Q_{\alpha \alpha} - 2\alpha^{-5} Q_{\alpha \beta}, \]
\[ D_{12} := -2\alpha^{-2} Q_{\beta \beta} + 2\alpha^{-3} Q_{\alpha \beta} + 2\alpha^{-3} Q_{\alpha \beta}. \]
Now, by plugging (3.10) into (3.9), and contracting the obtained result with $a_{kl}$, we get (3.1). □

Proof of Theorem 1.1: Let $F = \alpha \phi(s)$, $s = \beta/\alpha$, be an $(\alpha, \beta)$-metric on an $n$-dimensional manifold $M$. By multiplying (3.1) with $y_i$ and $y_j$, we get

$$-\alpha QQ_{\alpha\alpha}t_{00} = 0.$$  

(3.11)

If $Q_{\alpha\alpha} = 0$ then

$$Q = c_1 \alpha + c_2 \frac{\alpha^2}{\beta},$$

where $c_1$ and $c_2$ are real constants. Thus, we get

$$F = c_3 \alpha \left( \frac{\beta}{\alpha} \right)^{\frac{c_1}{c_2}} \left( c_1 \frac{\beta}{\alpha} + c_2 + 1 \right)^{\frac{1}{1+c_2}},$$

where $c_3$ is a real constant. This is a contradiction with our assumption. Then by (3.11), we get $t_{00} = 0$ which results that $s_{i0} = 0$. This means that $\beta$ is a closed one-form. By assumption, $\beta$ is parallel one-form and then $F$ reduces to a Berwald metric. □

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