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# Generalized Douglas-Weyl Finsler Metrics

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Abstract. In this paper, we study generalized Douglas-Weyl Finsler metrics. We find some conditions under which the class of generalized Douglas-Weyl $(\alpha, \beta)$-metric with vanishing S-curvature reduce to the class of Berwald metrics.

**Keywords:** Generalized Douglas-Weyl metrics, S-curvature.

**2000 Mathematics subject classification:** 53C60, 53C25.



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## 1. Introduction

Let $(M, F)$ be a Finsler manifold. In local coordinates, a curve $c(t)$ is a geodesic if and only if its coordinates $(c^i(t))$ satisfy $\ddot{c}^i + 2G^i(\dot{c}) = 0$, where the local functions $G^i = G^i(x, y)$ are called the spray coefficients [10]. $F$ is called a Berwald metric, if $G^i$ are quadratic in $y \in T_xM$ for any $x \in M$ or equivalently $G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^jy^k$. As a generalization of Berwald curvature, Bácsó-Matsumoto introduced the notion of Douglas metrics which are projective invariants in Finsler geometry [2]. $F$ is called a Douglas metric if $G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^jy^k + P(x, y)y^i$.

A Finsler metric $F$ is called generalized Douglas-Weyl metric (briefly, GDW-metric) if ${D^i}_{jkl||m}y^m = T_{jkl}y^i$ holds for some tensor $T_{jkl}$, where ${D^i}_{jkl||m}$ denotes the horizontal covariant derivatives of ${D^i}_{jkl}$ with respect to the Berwald

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connection of $F$ [8][18]. For a manifold $M$, let $\mathcal{G}DW(M)$ denotes the class of all Finsler metrics satisfying in above relation for some tensor $T_{jkl}$. In [3], Bácsó-Papp showed that $\mathcal{G}DW(M)$ is closed under projective changes. Then, Najafi-Shen-Tayebi characterized generalized Douglas-Weyl Randers metrics [8]. In [18], it is proved that all generalized Douglas-Weyl spaces with vanishing Landsberg curvature have vanishing the quantity $\mathbf{H}$. For other works, see [12] and [13].

The notion of S-curvature is originally introduced by Shen for the volume comparison theorem [9]. The Finsler metric $F$ is said to be of isotropic S-curvature if $\mathbf{S} = (n+1)cF$, where $c = c(x)$ is a scalar function on $M$. In [14], it is shown that every isotropic Berwald metric has isotropic S-curvature. In [4], Cheng-Shen show that every $(\alpha, \beta)$-metric with constant Killing 1-form has vanishing $S$-curvature. Then, Bácsó-Cheng-Shen proved that a Finsler metric $F = \alpha \pm \beta^2/\alpha + \epsilon\beta$ has vanishing S-curvature if and only if $\beta$ is a constant Killing 1-form [1]. Therefore, the Finsler metrics with vanishing S-curvature are of some important geometric structures which deserve to be studied deeply.

An $(\alpha, \beta)$-metric is a Finsler metric on $M$ defined by $F := \alpha\phi(s)$, $s = \beta/\alpha$, where $\phi = \phi(s)$ is a $C^\infty$ function on the $(-b_0, b_0)$ with certain regularity, $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta(y) = b_i(x)y^i$ is a 1-form on $M$ [6]. In this paper, we are going to study generalized Douglas-Weyl $(\alpha, \beta)$-metrics with vanishing S-curvature.

**Theorem 1.1.** *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an $(\alpha, \beta)$-metric on a manifold $M$ of dimension $n \geq 3$. Suppose that*

$$F \neq c_3\alpha\left(\frac{\beta}{\alpha}\right)^{\frac{c_2}{1+c_2}}\left(c_1\frac{\beta}{\alpha} + c_2 + 1\right)^{\frac{1}{1+c_2}} \qquad \textit{and} \qquad F \neq d_1\sqrt{\alpha^2 + d_2\beta^2} + d_3\beta.$$

*where $c_1$, $c_2$ ,$c_3$ ,$d_1$, $d_2$ and $d_3$ are real constants. Let $F$ has vanishing S-curvature. Then $F$ is a GDW-metric if and only if it is a Berwald metric.*

## 2. Preliminary

Given a Finsler manifold $(M, F)$, then a global vector field $\mathbf{G}$ is induced by $F$ on $TM_0$, which in a standard coordinate $(x^i, y^i)$ for $TM_0$ is given by $\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i}$, where

$$G^i := \frac{1}{4}g^{il}\Big\{[F^2]_{x^k y^l}y^k - [F^2]_{x^l}\Big\}, \qquad y \in T_xM.$$

The $\mathbf{G}$ is called the spray associated to $F$.

Define $\mathbf{B}_y : T_xM \otimes T_xM \otimes T_xM \to T_xM$ and $\mathbf{E}_y : T_xM \otimes T_xM \to \mathbb{R}$ by $\mathbf{B}_y(u, v, w) := B^i{}_{jkl}(y)u^j v^k w^l \frac{\partial}{\partial x^i}|_x$ and $\mathbf{E}_y(u, v) := E_{jk}(y)u^j v^k$ where

$$B^i{}_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}, \qquad E_{jk} := \frac{1}{2}B^m{}_{jkm}.$$



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**B** and **E** are called the Berwald curvature and mean Berwald curvature, respectively. $F$ is called a Berwald and weakly Berwald if $\mathbf{B} = \mathbf{0}$ and $\mathbf{E} = 0$, respectively [5][7].

Let

$$D^i_{j\ kl} := \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\big(G^i - \frac{1}{n+1}\frac{\partial G^m}{\partial y^m} y^i\big).$$

It is easy to verify that $\mathcal{D} := D^i_{j\ kl} dx^j \otimes \partial_i \otimes dx^k \otimes dx^l$ is a well-defined tensor on slit tangent bundle $TM_0$. We call $\mathcal{D}$ the Douglas tensor. A Finsler metric with $\mathcal{D} = 0$ is called a Douglas metric. The notion of Douglas metrics was proposed by B*ác*s*ó*-Matsumoto as a generalization of Berwald metrics [2]. The Douglas tensor $\mathcal{D}$ is a non-Riemannian projective invariant, namely, if two Finsler metrics $F$ and $\bar{F}$ are projectively equivalent, $G^i = \bar{G}^i + Py^i$, where $P = P(x, y)$ is positively $y$-homogeneous of degree one, then the Douglas tensor of $F$ is same as that of $\bar{F}$. Finsler metrics with vanishing Douglas tensor are called Douglas metrics [11].

For a Finsler metric $F$ on an $n$-dimensional manifold $M$, the Busemann-Hausdorff volume form $dV_F = \sigma_F(x)dx^1 \cdots dx^n$ is defined by

$$\sigma_F(x) := \frac{\mathrm{Vol}(\mathbb{B}^n(1))}{\mathrm{Vol}\Big[(y^i) \in R^n \ \Big|\ F\Big(y^i \frac{\partial}{\partial x^i}|_x\Big) < 1\Big]}.$$

Let $G^i$ denote the geodesic coefficients of $F$ in the same local coordinate system. The S-curvature is defined by

$$\mathbf{S}(\mathbf{y}) := \frac{\partial G^i}{\partial y^i}(x, y) - y^i \frac{\partial}{\partial x^i}\Big[\ln \sigma_F(x)\Big],$$

where $\mathbf{y} = y^i \frac{\partial}{\partial x^i}|_x \in T_xM$. **S** is said to be isotropic if there is a scalar functions $c = c(x)$ on $M$ such that $\mathbf{S} = (n + 1)cF$.

For an $(\alpha, \beta)$-metric $F = \alpha\phi(s)$, $s = \beta/\alpha$, put

$$\Phi := -(q - sq')[n\Delta + 1 + sq] - (b^2 - s^2)(1 + sq)q'',$$

where

$$q := \frac{\phi'}{\phi - s\phi'}, \qquad \Delta := 1 + sq + (b^2 - s^2)q'.$$

In [4], Cheng-Shen characterize $(\alpha, \beta)$-metrics with isotropic S-curvature.

**Lemma 2.1.** ([4]) Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an non-Riemannian $(\alpha, \beta)$-metric on a manifold $M$ of dimension $n \geq 3$. Suppose that $\phi \neq c_1\sqrt{1 + c_2 s^2} + c_3 s$ for any constant $c_1 > 0$, $c_2$ and $c_3$. Then $F$ is of isotropic S-curvature $\mathbf{S} = (n + 1)cF$ if and only if one of the following holds
(a) $\beta$ satisfies

$$r_{ij} = \varepsilon(b^2 a_{ij} - b_i b_j), \quad s_j = 0, \tag{2.1}$$


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where $\varepsilon = \varepsilon(x)$ is a scalar function, $b := \|\beta_x\|_\alpha$ and $\phi = \phi(s)$ satisfies

$$\Phi = -2(n+1)k\frac{\phi\Delta^2}{b^2 - s^2}, \tag{2.2}$$

where $k$ is a constant. In this case, $\mathbf{S} = (n+1)cF$ with $c = k\varepsilon$.
(b) $\beta$ satisfies

$$r_{ij} = 0, \qquad s_j = 0 \tag{2.3}$$

In this case, $\mathbf{S} = 0$.

The characterization of Finsler metrics with isotropic S-curvature in Cheng-Shen's paper is not complete [4]. Their result is correct for dimension $n \geq 3$. For the case $dimension(M) = 2$, see [16].

## 3. Proof of Main Results

Let $F := \alpha\phi(s)$, $s = \beta/\alpha$, be an $(\alpha,\beta)$-metric on a manifold $M$, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ and $\beta(y) = b_i(x)y^i$. Define $b_{i|j}$ by $b_{i|j}\theta^j := db_i - b_j\theta_i{}^j$, where $\theta^i := dx^i$ and $\theta_i{}^j := \tilde{\Gamma}^j_{ik}dx^k$ denote the Levi-Civita connection forms of $\alpha$. Let

$$\begin{aligned}
&r_{ij} := \frac{1}{2}\Big[b_{i|j} + b_{j|i}\Big], \qquad s_{ij} := \frac{1}{2}\Big[b_{i|j} - b_{j|i}\Big],\\
&r_{i0} := r_{ij}y^j, \quad r_{00} := r_{ij}y^iy^j, \quad r_j := b^i r_{ij}, \quad t^i_j := s^i{}_m s^m_j\\
&s_{i0} := s_{ij}y^j, \quad s_j := b^i s_{ij}, \quad r_0 := r_j y^j, \quad s_0 := s_j y^j.
\end{aligned}$$

Then $\beta = b_i(x)y^i$ is a constant Killing one-form on $M$ if $r_{ij} = s_j = 0$ hold. By definition, we have

$$b_{i|j} = s_{ij} + r_{ij}.$$

Since $y^i{}_{|s} = 0$, then for a constant Killing 1-form $\beta$ we have

$$r_{00} = 0, \quad r_i + s_i = 0.$$

For an $(\alpha,\beta)$-metric $F = \alpha\phi(s)$, $s = \beta/\alpha$, the following hold.

**Proposition 3.1.** *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an $(\alpha,\beta)$-metric on an $n$-dimensional manifold $M$ of dimension $n \geq 3$, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a one-form on $M$. Suppose that $F$ is of vanishing S-curvature. Then $F$ is a GDW-metric if and only if the following holds*

$$\begin{aligned}
C_1\, s_{j0|0}y^i - (C_2\, y_j + C_3\, b_j)y^i t_{00} = {}& C_4\, y_j s^i{}_{0|0} + C_5\big(b_j s^i{}_{0|0} + s_{j0}s^i{}_0\big)\\
&+C_6\, s^i{}_{j|0} + C_7(y_j t^i{}_0 + s_{j0}s^i{}_0) + C_8\, b_j t^i{}_0,
\end{aligned} \tag{3.1}$$



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*where*

$$
\begin{aligned}
C_1 &:= -\Big[(n+1)Q_\alpha + 2\beta Q_{\alpha\beta}\Big]\alpha^{-3} - \Big[Q_{\alpha\alpha} + b^2 Q_{\beta\beta}\Big]\alpha^{-2},\\
C_2 &:= (n+1)\Big[Q_\alpha^2 + QQ_{\alpha\alpha} - \alpha^{-1}QQ_\alpha\Big]\alpha^{-4} - 2\Big[Q_\alpha Q_\beta + QQ_{\alpha\beta}\Big]\beta\alpha^{-5}\\
&+ \ 2\Big[2Q_\alpha Q_{\alpha\beta} + Q_{\alpha\alpha}Q_\beta + QQ_{\alpha\alpha\beta}\Big]\beta\alpha^{-4} + b^2\Big[2Q_{\alpha\beta}Q_\beta + Q_\alpha Q_{\beta\beta}\Big]\alpha^{-3}\\
&+ \ \Big[b^2 QQ_{\alpha\beta\beta} + 3Q_\alpha Q_{\alpha\alpha} + QQ_{\alpha\alpha\alpha}\Big]\alpha^{-3},\\
C_3 &:= (n+3)\Big[Q_\alpha Q_\beta + QQ_{\alpha\beta}\Big]\alpha^{-3} + 2\Big[Q_\alpha Q_{\beta\beta} + QQ_{\alpha\beta\beta}\Big]\beta\alpha^{-3}\\
&+ \ \Big[2Q_\alpha Q_{\alpha\beta} + Q_\beta Q_{\alpha\alpha} + QQ_{\alpha\alpha\beta} + 4\beta\alpha^{-1}Q_\beta Q_{\alpha\beta}\Big]\alpha^{-2}\\
&+ \ b^2\Big[3Q_\beta Q_{\beta\beta} + QQ_{\beta\beta\beta}\Big]\alpha^{-2},\\
C_4 &:= -\Big[(n+1)Q_\alpha + 2\beta Q_{\alpha\beta}\Big]\alpha^{-3} + 2\Big[\beta Q_{\alpha\alpha\beta} + Q_{\alpha\alpha}\Big]\alpha^{-2}\\
&+ \ \Big[b^2 Q_{\alpha\beta\beta} + Q_{\alpha\alpha\alpha}\Big]\alpha^{-1},\\
C_5 &:= (n+3)\alpha^{-1}Q_{\alpha\beta} + Q_{\alpha\alpha\beta} + 2\beta\alpha^{-1}Q_{\alpha\beta\beta} + b^2 Q_{\beta\beta\beta},\\
C_6 &:= (n+1)\alpha^{-1}Q_\alpha + Q_{\alpha\alpha} + 2\beta\alpha^{-1}Q_{\alpha\beta} + b^2 Q_{\beta\beta},\\
C_7 &:= (n+1)\alpha^{-3}QQ_\alpha - (n+1)\alpha^{-2}(Q_\alpha^2 + QQ_{\alpha\alpha}) - 2\beta\alpha^{-2}\,QQ_{\alpha\alpha\beta}\\
&+ \ 2\Big[QQ_{\alpha\beta} + Q_\alpha Q_\beta\Big]\beta\alpha^{-3} - b^2\Big[QQ_{\alpha\beta\beta} + 2Q_{\alpha\beta}Q_\beta\Big]\alpha^{-1}\\
&- \ 2\Big[2Q_\alpha Q_{\alpha\beta} + Q_\beta Q_{\alpha\alpha}\Big]\beta\alpha^{-2}\\
&- \ b^2\alpha^{-1}Q_\alpha Q_{\beta\beta} - 3\alpha^{-1}Q_\alpha Q_{\alpha\alpha} - 2\alpha^{-1}QQ_{\alpha\alpha\alpha},\\
C_8 &:= -(n+3)\Big[QQ_{\alpha\beta} + Q_\alpha Q_\beta\Big]\alpha^{-1} - 2\Big[2Q_\beta Q_{\alpha\beta} + QQ_{\alpha\beta\beta} + Q_\alpha Q_{\beta\beta}\Big]\beta\alpha^{-1}\\
&- \ b^2\Big[QQ_{\beta\beta\beta} + 3Q_\beta Q_{\beta\beta}\Big] - Q_\beta Q_{\alpha\alpha} - QQ_{\alpha\alpha\beta} - 2Q_\alpha Q_{\alpha\beta}.
\end{aligned}
$$

*Proof.* Let $G^i$ and $G^i_\alpha$ denote the spray coefficients of $F$ and $\alpha$, respectively, in the same coordinate system. Then, we have

$$G^i = G^i_\alpha + Py^i + Q^i, \tag{3.2}$$

where

$$
\begin{aligned}
Q &:= \alpha q = \frac{\alpha\phi'}{\phi - s\phi'},\\
P &:= \alpha^{-1}\Theta(r_{00} - 2Qs_0), \quad Q^i := Qs^i{}_0 + \Psi(r_{00} - 2Qs_0)b^i,\\
\Theta &= \frac{q - sq'}{2\Delta} = \frac{\phi\phi' - s(\phi\phi'' + \phi'\phi')}{2\phi\Big[(\phi - s\phi') + (b^2 - s^2)\phi''\Big]}\\
\Psi &:= \frac{q'}{2\Delta} = \frac{1}{2}\frac{\phi''}{(\phi - s\phi') + (b^2 - s^2)\phi''}.
\end{aligned}
$$



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By Lemma 2.1, we have $r_{00} = s_0 = 0$. Then (3.2) reduces to following

$$G^i = G^i_\alpha + Qs^i_0. \tag{3.3}$$

Let "$\|$" and "$|$" denote the covariant differentiations with respect to $G^i$ and $G^i_\alpha$ respectively. Then by (3.3), we have

$$D^i_{jkl\|m}y^m = D^i_{jkl|m}y^m - 2Qs^p_0\frac{\partial D^i_{jkl}}{\partial y^p} + D^p_{jkl}\tilde{N}^i_p - D^i_{pkl}\tilde{N}^p_j - D^i_{jpl}\tilde{N}^p_k - D^i_{jkp}\tilde{N}^p_l, \tag{3.4}$$

where

$$\begin{aligned}
D^i_{jkl|m}y^m = \ & \alpha^{-4}(Q_{\alpha\alpha} - \alpha^{-1}Q_\alpha)(A_{jk}y_l + A_{kl}y_j + A_{jl}y_k){s^i}_{0|0} \\
& + \alpha^{-3}Q_\alpha(A_{jk}{s^i}_{l|0} + A_{kl}{s^i}_{j|0} + A_{jl}{s^i}_{k|0}) \\
& + \alpha^{-3}Q_{\alpha\beta}\Big[(A_{jk}b_l + A_{kl}b_j + A_{jl}b_k){s^i}_{0|0} \\
& + (A_{jk}s_{l0} + A_{kl}s_{j0} + A_{jl}s_{k0}){s^i}_0\Big] \\
& + \alpha^{-2}Q_{\alpha\alpha\beta}\Big[(y_jy_kb_l + y_ky_lb_j + y_jy_lb_k){s^i}_{0|0} \\
& + (y_jy_ks_{l0} + y_ky_ls_{j0} + y_jy_ls_{k0}){s^i}_0\Big] \\
& + \alpha^{-1}Q_{\alpha\beta\beta}\Big[(y_jb_kb_l + y_kb_jb_l + y_lb_kb_j){s^i}_{0|0} \\
& + \big((y_jb_l + y_lb_j)s_{k0} + (y_jb_k + y_kb_j)s_{l0} \\
& + (y_kb_l + y_lb_k)s_{j0}\big){s^i}_0\Big] + \alpha^{-2}Q_{\alpha\alpha}(y_jy_k{s^i}_{l|0} + y_ky_l{s^i}_{j|0} + y_jy_l{s^i}_{k|0}) \\
& + Q_{\beta\beta\beta}(b_kb_ls_{j0} + b_jb_ls_{k0} + b_jb_ks_{l0})s^i_0 + \alpha^{-3}Q_{\alpha\alpha\alpha}y_jy_ky_l{s^i}_{0|0} \\
& + \alpha^{-1}Q_{\alpha\beta}\Big[(y_jb_k + y_kb_j){s^i}_{l|0} + (y_kb_l + y_lb_k){s^i}_{j|0} + (y_lb_j + y_jb_l){s^i}_{k|0} \\
& + (y_js_{k0} + y_ks_{j0}){s^i}_l + (y_ks_{l0} + y_ls_{k0}){s^i}_j + (y_ls_{j0} + y_js_{l0}){s^i}_k\Big] \\
& + Q_{\beta\beta}\Big[b_jb_k{s^i}_{l|0} + b_kb_l{s^i}_{j|0} + b_jb_l{s^i}_{k|0} + (s_{j0}b_k + b_js_{k0}){s^i}_l \\
& + (s_{k0}b_l + b_ks_{l0}){s^i}_j + (b_ls_{j0} + b_js_{l0}){s^i}_k\Big] + Q_{\beta\beta\beta}b_jb_kb_l{s^i}_{0|0}
\end{aligned} \tag{3.5}$$

and

$$A_{ij} = \alpha^2 a_{ij} - y_iy_j, \tag{3.6}$$

$$\tilde{N}^i_p = Q{s^i}_p + \Big[\alpha^{-1}Q_\alpha y_p + Q_\beta b_p\Big]{s^i}_0, \tag{3.7}$$

$$\frac{\partial D^i_{jkl}}{\partial y^p} = Q_{jklp}{s^i}_0 + Q_{jkl}{s^i}_p + Q_{jkp}{s^i}_l + Q_{jlp}{s^i}_k + Q_{klp}{s^i}_j. \tag{3.8}$$

Let $F$ is a $GDW$-metric. Then there exists a tensor $D_{jkl}$ such that

$$D^i_{jkl\|m}y^m = D_{jkl}y^i.$$



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By (3.4), we have

$$D_{jkl}y^i = D^i_{jkl|m}y^m - 2Q\ \frac{\partial D^i_{jkl}}{\partial y^p}s^p{}_0 + D^p_{jkl}\tilde{N}^i_p - D^i_{pkl}\tilde{N}^p_j - D^i_{jpl}\tilde{N}^p_k - D^i_{jkp}\tilde{N}^p_l. \tag{3.9}$$

By contracting (3.9) with $y_i$ and using (3.5), (3.7) and (3.8) we get the following

$$\begin{aligned}
D_{jkl} = \quad & D_1\Big[A_{jk}s_{l0|0} + A_{kl}s_{j0|0} + A_{jl}s_{k0|0}\Big] \\
& + D_2\Big[y_jy_ks_{l0|0} + y_ky_ls_{j0|0} + y_jy_ls_{k0|0}\Big] \\
& + D_3\Big[(y_jb_k + y_kb_j)s_{l0|0} + (y_kb_l + y_lb_k)s_{j0|0} + (y_jb_l + y_lb_j)s_{k0|0}\Big] \\
& + D_4\Big[b_jb_ks_{l0|0} + b_kb_ls_{j0|0} + b_jb_ls_{k0|0}\Big] \\
& + D_5\Big[A_{jk}y_l + A_{kl}y_j + A_{jl}y_k\Big]t_{00} \\
& + D_6\Big[A_{jk}b_l + A_{kl}b_j + A_{jl}b_k\Big]t_{00} \\
& + D_7\Big[y_jy_kb_l + y_ky_lb_j + y_jy_lb_k\Big]t_{00} \\
& + D_8\Big[y_lb_jb_k + y_jb_kb_l + y_kb_jb_l\Big]t_{00} \\
& + D_9\, y_jy_ky_lt_{00} + D_{10}\, b_jb_kb_lt_{00} \\
& + D_{11}\Big[y_ls_{j0}s_{k0} + y_js_{k0}s_{l0} + y_ks_{j0}s_{l0}\Big] \\
& + D_{12}\Big[b_ls_{j0}s_{k0} + b_js_{k0}s_{l0} + b_ks_{j0}s_{l0}\Big],
\end{aligned} \tag{3.10}$$

where

$$\begin{aligned}
D_1 &:= -\alpha^{-5}Q_\alpha, \\
D_2 &:= -\alpha^{-4}Q_{\alpha\alpha}, \\
D_3 &:= -\alpha^{-3}Q_{\alpha\beta}, \\
D_4 &:= -\alpha^{-2}Q_{\beta\beta}, \\
D_5 &:= -\alpha^{-6}Q^2_\alpha - \alpha^{-6}QQ_{\alpha\alpha} + \alpha^{-7}QQ_\alpha, \\
D_6 &:= -\alpha^{-5}Q_\alpha Q_\beta - \alpha^{-5}QQ_{\alpha\beta}, \\
D_7 &:= -\alpha^{-4}Q_{\alpha\alpha}Q_\beta - 2\alpha^{-4}Q_{\alpha\beta}Q_\alpha - \alpha^{-4}QQ_{\alpha\alpha\beta}, \\
D_8 &:= -\alpha^{-3}Q_{\beta\beta}Q_\alpha - 2\alpha^{-3}Q_{\alpha\beta}Q_\beta - \alpha^{-3}QQ_{\alpha\beta\beta}, \\
D_9 &:= -3\alpha^{-5}Q_{\alpha\alpha}Q_\alpha - \alpha^{-5}QQ_{\alpha\alpha\alpha}, \\
D_{10} &:= -3\alpha^{-2}Q_{\beta\beta}Q_\beta - \alpha^{-2}QQ_{\beta\beta\beta}, \\
D_{11} &:= -2\alpha^{-3}Q_{\alpha\beta} + 2\alpha^{-4}Q^2_\alpha + 2\alpha^{-4}QQ_{\alpha\alpha} - 2\alpha^{-5}QQ_\alpha, \\
D_{12} &:= -2\alpha^{-2}Q_{\beta\beta} + 2\alpha^{-3}QQ_{\alpha\beta} + 2\alpha^{-3}Q_\alpha Q_\beta.
\end{aligned}$$



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Now, by plugging (3.10) into (3.9), and contracting the obtained result with $a^{kl}$, we get (3.1). □

**Proof of Theorem 1.1:** Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an $(\alpha, \beta)$-metric on an $n$-dimensional manifold $M$. By multiplying (3.1) with $y_i$ and $y^j$, we get

$$-\alpha Q Q_{\alpha\alpha\alpha} t_{00} = 0. \tag{3.11}$$

If $Q_{\alpha\alpha\alpha} = 0$ then

$$Q = c_1\alpha + c_2\frac{\alpha^2}{\beta},$$

where $c_1$ and $c_2$ are real constants. Thus, we get

$$F = c_3\alpha\left(\frac{\beta}{\alpha}\right)^{\frac{c_2}{1+c_2}}\left(c_1\frac{\beta}{\alpha} + c_2 + 1\right)^{\frac{1}{1+c_2}},$$

where $c_3$ is a real constant. This is a contradiction with our assumption. Then by (3.11), we get $t_{00} = 0$ which results that $s_{i0} = 0$. This means that $\beta$ is a closed one-form. By assumption, $\beta$ is parallel one-form and then $F$ reduces to a Berwald metric. □

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