Generalized Douglas-Weyl Finsler Metrics

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Abstract. In this paper, we study generalized Douglas-Weyl Finsler metrics. We find some conditions under which the class of generalized Douglas-Weyl \((\alpha, \beta)\)-metric with vanishing S-curvature reduce to the class of Berwald metrics.

Keywords: Generalized Douglas-Weyl metrics, S-curvature.


1. Introduction

Let \((M, F)\) be a Finsler manifold. In local coordinates, a curve \(c(t)\) is a geodesic if and only if its coordinates \((c^i(t))\) satisfy \(\ddot{c}^i + 2G^i(\dot{c}) = 0\), where the local functions \(G^i = G^i(x, y)\) are called the spray coefficients [10]. \(F\) is called a Berwald metric, if \(G^i\) are quadratic in \(y \in T_x M\) for any \(x \in M\) or equivalently \(G^i = \frac{1}{2} \Gamma^i_{jk}(x)y^jy^k\). As a generalization of Berwald curvature, Bácsó-Matsumoto introduced the notion of Douglas metrics which are projective invariants in Finsler geometry [2]. \(F\) is called a Douglas metric if \(G^i = \frac{1}{2} \Gamma^i_{jk}(x)y^jy^k + P(x, y)y^i\).

A Finsler metric \(F\) is called generalized Douglas-Weyl metric (briefly, GDW-metric) if \(D^i_{jkl||m}y^m = T_{jkl}y^i\) holds for some tensor \(T_{jkl}\), where \(D^i_{jkl||m}\) denotes the horizontal covariant derivatives of \(D^i_{jkl}\) with respect to the Berwald
connection of $F$ [8][18]. For a manifold $M$, let $GDW(M)$ denotes the class of all Finsler metrics satisfying in above relation for some tensor $T_{jkl}$. In [3], Bácsó-Papp showed that $GDW(M)$ is closed under projective changes. Then, Najafi-Shen-Tayebi characterized generalized Douglas-Weyl Randers metrics [8]. In [18], it is proved that all generalized Douglas-Weyl spaces with vanishing Landsberg curvature have vanishing the quantity $H$. For other works, see [12] and [13].

The notion of S-curvature is originally introduced by Shen for the volume comparison theorem [9]. The Finsler metric $F$ is a GDW-metric if and only if it is a Berwald metric. Then, Bácsó-Cheng-Shen proved that a Finsler metric $F = \alpha, \beta$ has vanishing S-curvature if and only if $\beta$ is a constant Killing 1-form [1]. Therefore, the Finsler metrics with vanishing S-curvature are of some important geometric structures which deserve to be studied deeply.

An $(\alpha, \beta)$-metric is a Finsler metric on $M$ defined by $F := \alpha \phi(s), s = \beta/\alpha$, where $\phi = \phi(s)$ is a $C^\infty$ function on the $(-b_0, b_0)$ with certain regularity, $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta(y) = b_i(x)y^i$ is a 1-form on $M$ [6]. In this paper, we are going to study generalized Douglas-Weyl $(\alpha, \beta)$-metrics with vanishing S-curvature.

**Theorem 1.1.** Let $F = \alpha \phi(s), s = \beta/\alpha$, be an $(\alpha, \beta)$-metric on a manifold $M$ of dimension $n \geq 3$. Suppose that

$$F \neq c_3\alpha \left(\frac{\beta}{\alpha}\right)^{\frac{n+2}{n+2}} \left(c_1 \frac{\beta}{\alpha} + c_2 + 1\right)^{\frac{1}{n+2}} \text{ and } F \neq d_1 \sqrt{\alpha^2 + d_2 \beta^2} + d_3 \beta,$$

where $c_1, c_2, c_3, d_1, d_2$ and $d_3$ are real constants. Let $F$ has vanishing S-curvature. Then $F$ is a GDW-metric if and only if it is a Berwald metric.

2. **Preliminary**

Given a Finsler manifold $(M, F)$, then a global vector field $G$ is induced by $F$ on $TM_0$, which in a standard coordinate $(x^i, y^i)$ for $TM_0$ is given by $G = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i}$, where

$$G^i := \frac{1}{4} y^j ln \left\{ [F^2]_{x^j y^k} y^k - [F^2]_{x^i} \right\}, \quad y \in T_x M.$$

The G is called the spray associated to $F$.

Define $B_y : T_x M \otimes T_x M \otimes T_x M \rightarrow T_x M$ and $E_y : T_x M \otimes T_x M \rightarrow \mathbb{R}$ by $B_y(u, v, w) := B^i_{jkl}(y)u^j v^k w^l \frac{\partial}{\partial x^i}$ and $E_y(u, v) := E_{jkl}(y)u^j v^k$ where

$$B^i_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}, \quad E_{jkl} := \frac{1}{2} B_{jkm}.$$
B and E are called the Berwald curvature and mean Berwald curvature, respectively. F is called a Berwald and weakly Berwald if B = 0 and E = 0, respectively [5][7].

Let
\[ D_{ij} := \frac{\partial G^i}{\partial y^j} (G^i - \frac{1}{n+1} \frac{\partial G^m}{\partial y^m} y^i). \]
It is easy to verify that \( D := D_{ij} dx^j \otimes \partial_i \otimes dx^k \otimes dx^l \) is a well-defined tensor on slit tangent bundle \( TM_0 \). We call \( D \) the Douglas tensor. A Finsler metric with \( D = 0 \) is called a Douglas metric. The notion of Douglas metrics was proposed by Bácsó-Matsumoto as a generalization of Berwald metrics [2].

The Douglas tensor \( D \) is a non-Riemannian projective invariant, namely, if two Finsler metrics \( F \) and \( \bar{F} \) are projectively equivalent, \( G^i = \bar{G}^i + P y^i \), where \( P = P(x,y) \) is positively \( y \)-homogeneous of degree one, then the Douglas tensor of \( F \) is same as that of \( \bar{F} \). Finsler metrics with vanishing Douglas tensor are called Douglas metrics [11].

For a Finsler metric \( F \) on an \( n \)-dimensional manifold \( M \), the Busemann-Hausdorff volume form \( dV_F = \sigma_F(x) dx^1 \cdots dx^n \) is defined by
\[ \sigma_F(x) := \frac{\text{Vol}(\mathbb{B}^n(1))}{\text{Vol}\left(\left\{ y^i \in \mathbb{R}^n \mid F\left(y^i \frac{\partial}{\partial x^i}|x\right) < 1\right\}\right)}. \]
Let \( G^i \) denote the geodesic coefficients of \( F \) in the same local coordinate system. The S-curvature is defined by
\[ S(y) := \frac{\partial G^i}{\partial y^i}(x,y) - y^i \frac{\partial}{\partial x^i}\left[\ln \sigma_F(x)\right], \]
where \( y = y^i \frac{\partial}{\partial x^i}|x \in T_x M \). S is said to be isotropic if there is a scalar function \( c = c(x) \) on \( M \) such that \( S = (n+1)cF \).

For an \((\alpha, \beta)\)-metric \( F = \alpha \phi(s) \), \( s = \beta/\alpha \), put
\[ \Phi := -(q - sq')(n\Delta + 1 + sq) - (b^2 - s^2)(1 + sq)q'', \]
where
\[ q := \frac{\phi'}{\phi - sq'}, \quad \Delta := 1 + sq + (b^2 - s^2)q'. \]
In [4], Cheng-Shen characterize \((\alpha, \beta)\)-metrics with isotropic S-curvature.

**Lemma 2.1.** ([4]) Let \( F = \alpha \phi(s) \), \( s = \beta/\alpha \), be an non-Riemannian \((\alpha, \beta)\)-metric on a manifold \( M \) of dimension \( n \geq 3 \). Suppose that \( \phi \neq c_1 \sqrt{1 + c_2 s^2} + c_3 s \) for any constant \( c_1 > 0 \), \( c_2 \) and \( c_3 \). Then \( F \) is of isotropic S-curvature \( S = (n+1)cF \) if and only if one of the following holds
(a) \( \beta \) satisfies
\[ r_{ij} = \varepsilon(b^2 a_{ij} - b_i b_j), \quad s_j = 0, \quad (2.1) \]
where $\varepsilon = \varepsilon(x)$ is a scalar function, $b := \|\beta x\|^{\alpha}$ and $\phi = \phi(s)$ satisfies

$$\Phi = -2(n+1)k \phi \Delta^2 b^2 + C_3 b^j y_{i00} = C_4 y_{j0} s_{i0} + C_5$$

$$(b_{j0} s_{i0} + s_{j0} s_{i0}) + C_6 s^j _{j0} + C_7(y_{j0} t^i_0 + s_{j0} s^i_0) + C_8 b^j t^{i0}.$$  

In this case, $\mathbf{S} = (n+1)cF$ with $c = k\varepsilon$. (b) $\beta$ satisfies

$$r_{ij} = 0, \quad s_j = 0$$  

In this case, $\mathbf{S} = 0$.

The characterization of Finsler metrics with isotropic S-curvature in Cheng-Shen’s paper is not complete [4]. Their result is correct for dimension $n \geq 3$. For the case $\text{dimension}(M) = 2$, see [16].

3. PROOF OF MAIN RESULTS

Let $F := \alpha \phi(s)$, $s = \beta/\alpha$, be an $(\alpha, \beta)$-metric on a manifold $M$, where $\alpha = \sqrt{\alpha_{ij}(x)y^iy^j}$ and $\beta(y) = b_i(x)y^i$. Define $b_{ij}$ by $b_{ij} \theta^i := db_i - b_j \theta^j$, where $\theta^i := dx^i$ and $\theta^j := \tilde{\Gamma}^j_{ik}dx^k$ denote the Levi-Civita connection forms of $\alpha$. Let

$$r_{ij} := \frac{1}{2} [b_{ij} + b_{ji}], \quad s_{ij} := \frac{1}{2} [b_{ij} - b_{ji}],$$

$$r_{i0} := r_{ij} y^j, \quad r_{00} := r_{ij} y^i y^j, \quad r_j := b^i r_{ij}, \quad t^i_j := s^i_m s^m_j$$

Then $\beta = b_i(x)y^i$ is a constant Killing one-form on $M$ if $r_{ij} = s_j = 0$ hold. By definition, we have

$$b_{ij} = s_{ij} + r_{ij}.$$  

Since $y^i |_s = 0$, then for a constant Killing 1-form $\beta$ we have

$$r_{00} = 0, \quad r_i + s_i = 0.$$  

For an $(\alpha, \beta)$-metric $F = \alpha \phi(s)$, $s = \beta/\alpha$, the following hold.

**Proposition 3.1.** Let $F = \alpha \phi(s)$, $s = \beta/\alpha$, be an $(\alpha, \beta)$-metric on an $n$-dimensional manifold $M$ of dimension $n \geq 3$, where $\alpha = \sqrt{\alpha_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a one-form on $M$. Suppose that $F$ is of vanishing S-curvature. Then $F$ is a GDW-metric if and only if the following holds

$$C_1 s_{j0} y^j - (C_2 y_j + C_3 b_j) y^i t_{i00} = C_4 y_j s^i_{0j0} + C_5 (b_j s^i_{0j0} + s_{j0} s^i_0)$$

$$+ C_6 s^i_{j0} + C_7 (y_j t^i_0 + s_{j0} s^i_0) + C_8 b_j t^{i0},$$  

(3.1)
where

\[ C_1 := - \left[ (n+1)Q_\alpha + 2\beta Q_{\alpha\beta} \right] \alpha^{-3} - \left[ Q_{\alpha\alpha} + b^2 Q_{\beta\beta} \right] \alpha^{-2}, \]

\[ C_2 := (n+1) \left[ Q_\alpha^2 + Q_{\alpha\alpha} - \alpha^{-1} Q_{\alpha} \right] \alpha^{-4} - 2 \left[ Q_{\alpha\beta} + Q_{\alpha\alpha\beta} \right] \beta \alpha^{-5} \]

\[ + 2 \left[ 2Q_{\alpha\alpha\beta} + Q_{\alpha\alpha\alpha} + Q_{\alpha\alpha\beta} \right] \beta \alpha^{-4} - b^2 \left[ 2Q_{\alpha\beta} + Q_{\alpha\alpha\beta} \right] \alpha^{-3} \]

\[ + \left[ b^2 Q_{\alpha\beta\beta} + 3Q_{\alpha\alpha\alpha} + Q_{\alpha\alpha\beta} \right] \alpha^{-3}, \]

\[ C_3 := (n+3) \left[ Q_\alpha Q_{\beta} + Q Q_{\alpha} \right] \alpha^{-3} - 2 \left[ Q_\alpha Q_{\beta\beta} + Q Q_{\alpha\beta\beta} \right] \beta \alpha^{-3} \]

\[ + 2 \left[ 2Q_{\alpha\beta} + Q_{\beta\alpha} + Q_{\alpha\alpha\beta} + 4\beta^{-1} Q_{\beta\beta} Q_{\beta\alpha} \right] \alpha^{-2} \]

\[ + b^2 \left[ 3Q_{\beta\beta\beta} + Q_{\beta\beta\alpha} + Q_{\beta\alpha\beta} \right] \alpha^{-2}, \]

\[ C_4 := - \left[ (n+1)Q_\alpha + 2\beta Q_{\alpha\beta} \right] \alpha^{-3} + 2 \left[ \beta Q_{\alpha\alpha\beta} + Q_{\alpha\alpha} \right] \alpha^{-2} \]

\[ + \left[ b^2 Q_{\alpha\beta\beta} + Q_{\alpha\alpha\alpha} \right] \alpha^{-1}, \]

\[ C_5 := (n+3) \alpha^{-1} Q_\alpha + Q_{\alpha\alpha} + 2\beta^{-1} Q_{\alpha\beta\beta} + b^2 Q_{\beta\beta\beta}, \]

\[ C_6 := (n+1) \alpha^{-1} Q_\alpha + Q_{\alpha\alpha} + 2\beta^{-1} Q_{\alpha\beta\beta} + b^2 Q_{\beta\beta\beta}, \]

\[ C_7 := (n+1) \alpha^{-3} Q_\alpha - (n+1) \alpha^{-2} \left[ Q_\alpha^2 + Q_{\alpha\alpha} \right] - 2\beta^{-2} Q Q_{\alpha\beta} \]

\[ + 2 \left[ Q_{\alpha\alpha\beta} + Q_{\beta\alpha} \right] \beta \alpha^{-3} - b^2 \left[ Q_{\alpha\beta\beta} + 2Q_{\alpha\beta} Q_{\beta} \right] \alpha^{-1} \]

\[ - 2 \left[ 2Q_{\alpha\beta} + Q_{\beta\alpha} \right] \beta \alpha^{-2} \]

\[ - b^2 \alpha^{-1} Q_\alpha Q_{\beta\beta} - 3\alpha^{-1} Q_{\alpha\alpha\alpha} - 2\alpha^{-1} Q_{\alpha\alpha\beta}, \]

\[ C_8 := - (n+3) \left[ Q_\alpha Q_{\beta} + Q_{\alpha} Q_{\beta} \right] \alpha^{-1} - 2 \left[ 2Q_{\beta\beta} Q_{\alpha\beta} + Q_{\alpha\beta\beta} + Q_{\alpha} Q_{\beta\beta} \right] \beta \alpha^{-1} \]

\[ - b^2 \left[ Q_{\beta\beta\beta} + 3Q_{\beta\beta\alpha} + Q_{\beta\alpha\beta} - Q_{\beta\alpha\alpha} - Q_{\alpha\alpha\beta} - 2Q_{\alpha} Q_{\beta} \right]. \]

Proof. Let \( G^i \) and \( G^i_0 \) denote the spray coefficients of \( F \) and \( \alpha \), respectively, in the same coordinate system. Then, we have

\[ G^i = G^i_0 + Py^i + Q^i, \quad (3.2) \]

where

\[ Q := \alpha \phi = \frac{\alpha \phi'}{\phi - s \phi''}, \]

\[ P := \alpha^{-1} \Theta(r_{00} - 2Qs_0), \quad Q^i := Qs_0^i + \Psi(r_{00} - 2Qs_0)b^i, \]

\[ \Theta = \frac{q - sq'}{2\Delta} = \frac{\phi \phi' - s(\phi \phi'' + \phi' \phi')}{2\phi \left( \phi - s \phi'' \right) + (b^2 - s^2) \phi''} \]

\[ \Psi := \frac{q'}{2\Delta} = \frac{1}{2} \frac{\phi''}{\phi - s \phi'' + (b^2 - s^2) \phi''}. \]
By Lemma 2.1, we have \( r_{00} = s_0 = 0 \). Then (3.2) reduces to following
\[
G^i = G^i_\alpha + Q s^i_{00}. \tag{3.3}
\]
Let “\( \partial \)” and “\( \parallel \)” denote the covariant differentiations with respect to \( G^i \) and \( G^i_\alpha \) respectively. Then by (3.3), we have
\[
D^i_{jklt||mn} y^m = D^i_{jklt||nm} y^m - 2Q s^i_{00} \frac{\partial D^i_{jklt}}{\partial y^p} - D^p_{jklt} \tilde{N}^i_p - D^i_{jklt} \tilde{N}^j_p - D^i_{jpl} \tilde{N}^j_k - D^i_{jkp} \tilde{N}^p_j, \tag{3.4}
\]
where
\[
D^i_{jklt||mn} y^m = \alpha^{i-4} Q_{\alpha\alpha} - \alpha^{-1} Q_{\alpha}(A_{jkly} + A_{klyj} + A_{jlyk}) s^i_{00} + \alpha^{-3} Q_{\alpha}(A_{jk}s^i_{10} + A_{kls}^i s^i_{0j} + A_{jls}^i s^i_{0j}) + \alpha^{-3} Q_{\alpha\beta}(A_{jk}b^i + A_{kbl} s^i_{0j} + A_{jls}^i s^i_{0j}) + (A_{jkl} s^i_{0j} + A_{kl}s_{j0} + A_{jkl} s^i_{0j}) + \alpha^{-2} Q_{\alpha\beta\alpha}(y_j y_k b^i + y_k y_j b^i + y_l y_k b^i) s^i_{00} + (y_j y_k s^i_{0j} + y_k y_l s^i_{0j}) + \alpha^{-1} Q_{\alpha\beta\alpha\beta}(y_j y_k b^i + y_k y_l s^i_{0j} + y_l y_k s^i_{0j} + y_j y_k s^i_{0j}) + Q_{\alpha\beta\alpha\beta}(y_j y_k s^i_{0j} + y_k y_l s^i_{0j} + y_l y_k s^i_{0j}) + \alpha^{-3} Q_{\alpha\alpha\alpha}(y_j y_k s^i_{0j} + y_k y_l s^i_{0j} + y_l y_k s^i_{0j}) + \alpha^{-3} Q_{\alpha\alpha\alpha\beta}(y_j y_k s^i_{0j} + y_k y_l s^i_{0j} + y_l y_k s^i_{0j}) + \alpha^{-1} Q_{\alpha\alpha\alpha\beta}(y_j y_k s^i_{0j} + y_k y_l s^i_{0j} + y_l y_k s^i_{0j}) + Q_{\alpha\beta\alpha\beta}(y_j y_k s^i_{0j} + y_k y_l s^i_{0j} + y_l y_k s^i_{0j}) + \alpha^{-1} Q_{\alpha\alpha\alpha\beta}(y_j y_k s^i_{0j} + y_k y_l s^i_{0j} + y_l y_k s^i_{0j}) + Q_{\alpha\beta\alpha\beta}(y_j y_k s^i_{0j} + y_k y_l s^i_{0j} + y_l y_k s^i_{0j}) + \alpha^{-3} Q_{\alpha\alpha\alpha\beta}(y_j y_k s^i_{0j} + y_k y_l s^i_{0j} + y_l y_k s^i_{0j}) + \alpha^{-3} Q_{\alpha\alpha\alpha\beta}(y_j y_k s^i_{0j} + y_k y_l s^i_{0j} + y_l y_k s^i_{0j}) + \alpha^{-1} Q_{\alpha\alpha\alpha\beta}(y_j y_k s^i_{0j} + y_k y_l s^i_{0j} + y_l y_k s^i_{0j}) + Q_{\alpha\beta\alpha\beta}(y_j y_k s^i_{0j} + y_k y_l s^i_{0j} + y_l y_k s^i_{0j}) \tag{3.5}
\]
and
\[
A_{ij} = \alpha^2 a_{ij} - y_i y_j, \tag{3.6}
\]
\[
\tilde{N}^i_p = Q s^i_{p} + \left[ \alpha^{-1} Q_{\alpha\alpha y_p} + Q_{\beta\beta} b^i \right] s^i_0, \tag{3.7}
\]
\[
\frac{\partial D^i_{jklt}}{\partial y^p} = Q_{jklt} s^i_{0} + Q_{jklt} s^i_{0} + Q_{jklp} s^i_{0} + Q_{jklp} s^i_{0} + Q_{jklp} s^i_{0} + Q_{jklp} s^i_{0}. \tag{3.8}
\]
Let \( F \) is a \( GDW \)-metric. Then there exists a tensor \( D^i_{jklt} \) such that
\[
D^i_{jklt||mn} y^m = D^i_{jklt} y^i. \]
By (3.4), we have

\[
D_{jkl} y^i = D^p_{jkl|m} y^m - 2Q \frac{\partial D^p_{jkl}}{\partial y_p} s^0 + D^p_{jkl} \hat{N}_p^i - D^p_{jkl} \hat{N}_p^0.
\]  

(3.9)

By contracting (3.9) with \( y_i \) and using (3.5), (3.7) and (3.8) we get the following

\[
D_{jkl} = D_1 \left[ A_{jk}s_{l0}^{00} + A_{kl}s_{0j}^{00} + A_{jl}s_{k0}^{00} \right]
+ D_2 \left[ y_j y_k s_{l0}^{00} + y_k y_j s_{j0}^{00} + y_j y_i s_{k0}^{00} \right]
+ D_3 \left[ (y_j b_k + y_k b_j) s_{l0}^{00} + (y_k b_i + y_i b_k) s_{j0}^{00} + (y_j b_l + y_l b_j) s_{k0}^{00} \right]
+ D_4 \left[ b_j b_k s_{l0}^{00} + b_k b_j s_{j0}^{00} + b_j b_k s_{k0}^{00} \right]
+ D_5 \left[ A_{jk} y_l + A_{kl} y_j + A_{jl} y_k \right] t_{00}
+ D_6 \left[ A_{jk} b_l + A_{kl} b_j + A_{jl} b_k \right] t_{00}
+ D_7 \left[ y_j y_k b_l + y_k y_j b_j + y_j y_i b_k \right] t_{00}
+ D_8 \left[ y_k b_j b_k + y_j b_k b_l + y_j b_l b_j \right] t_{00}
+ D_9 \left[ y_j y_k y_l t_{00} + D_{10} b_j b_k b_l t_{00} \right]
+ D_{11} \left[ b_j s_{j0}^{00} b_{k0}^{00} + y_j s_{k0}^{00} + y_j s_{j0}^{00} \right]
+ D_{12} \left[ b_k s_{j0}^{00} b_{k0}^{00} + b_j s_{k0}^{00} + b_k s_{j0}^{00} \right],
\]  

(3.10)

where

\[
D_1 := -\alpha^{-5} Q_\alpha,
D_2 := -\alpha^{-4} Q_{\alpha\alpha},
D_3 := -\alpha^{-3} Q_{\alpha\beta},
D_4 := -\alpha^{-2} Q_{\beta\beta},
D_5 := -\alpha^{-6} Q_\alpha^2 - \alpha^{-6} Q_{\alpha\alpha} + \alpha^{-7} Q Q_\alpha,
D_6 := -\alpha^{-5} Q_\alpha Q_\beta - \alpha^{-5} Q Q_\alpha \beta,
D_7 := -\alpha^{-4} Q_{\alpha\alpha} Q_\beta - 2\alpha^{-4} Q_{\alpha\beta} Q_\alpha - \alpha^{-4} Q Q_{\alpha\alpha} \beta,
D_8 := -\alpha^{-3} Q_{\beta\beta} Q_\alpha - 2\alpha^{-3} Q_{\alpha\beta} Q_\beta - \alpha^{-3} Q Q_{\alpha\beta} \beta,
D_9 := -3\alpha^{-5} Q_{\alpha\alpha} Q_\alpha - \alpha^{-5} Q Q_{\alpha\alpha} \alpha,
D_{10} := -3\alpha^{-2} Q_{\beta\beta} Q_\beta - \alpha^{-2} Q Q_{\beta\beta} \beta,
D_{11} := -2\alpha^{-3} Q_{\alpha\beta} + 2\alpha^{-3} Q_\alpha^2 + 2\alpha^{-4} Q Q_{\alpha\alpha} - 2\alpha^{-5} Q Q_\alpha,
D_{12} := -2\alpha^{-2} Q_{\beta\beta} + 2\alpha^{-3} Q Q_{\alpha\beta} + 2\alpha^{-3} Q_\alpha Q_\beta.
\]
Now, by plugging (3.10) into (3.9), and contracting the obtained result with $a_{kl}$, we get (3.1).

\[\square\]

**Proof of Theorem 1.1:** Let $F = \alpha \phi(s), s = \beta/\alpha$, be an $(\alpha, \beta)$-metric on an $n$-dimensional manifold $M$. By multiplying (3.1) with $y_i$ and $y^j$, we get

\[-\alpha Q_{\alpha\alpha} t_{00} = 0.\]

(3.11)

If $Q_{\alpha\alpha} = 0$ then

\[Q = c_1 \alpha + c_2 \frac{\alpha^2}{\beta},\]

where $c_1$ and $c_2$ are real constants. Thus, we get

\[F = c_3 \alpha \left( \frac{\beta}{\alpha} \right)^\frac{c_1^2}{c_2} \left( c_1 \frac{\beta}{\alpha} + c_2 + 1 \right)^\frac{c_1^2}{c_2},\]

where $c_3$ is a real constant. This is a contradiction with our assumption. Then by (3.11), we get $t_{00} = 0$ which results that $s_{i0} = 0$. This means that $\beta$ is a closed one-form. By assumption, $\beta$ is parallel one-form and then $F$ reduces to a Berwald metric. \[\square\]

**Acknowledgments**

The authors are very grateful to the anonymous referee for his or her comments and suggestions.

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