Generalized Douglas-Weyl Finsler Metrics

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Abstract. In this paper, we study generalized Douglas-Weyl Finsler metrics. We find some conditions under which the class of generalized Douglas-Weyl \((\alpha, \beta)\)-metric with vanishing S-curvature reduce to the class of Berwald metrics.

Keywords: Generalized Douglas-Weyl metrics, S-curvature.


1. Introduction

Let \((M, F)\) be a Finsler manifold. In local coordinates, a curve \(c(t)\) is a geodesic if and only if its coordinates \((c^i(t))\) satisfy \(\ddot{c}^i + 2G^i(\dot{c}) = 0\), where the local functions \(G^i = G^i(x, y)\) are called the spray coefficients \([10]\). \(F\) is called a Berwald metric, if \(G^i\) are quadratic in \(y \in T_xM\) for any \(x \in M\) or equivalently \(G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^j y^k\). As a generalization of Berwald curvature, Bácsó-Matsumoto introduced the notion of Douglas metrics which are projective invariants in Finsler geometry \([2]\). \(F\) is called a Douglas metric if \(G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^j y^k + P(x, y)y^i\).

A Finsler metric \(F\) is called generalized Douglas-Weyl metric (briefly, GDW-metric) if \(D^i_{jk||m}y^m = T_{jkl}y^i\) holds for some tensor \(T_{jkl}\), where \(D^i_{jk||m}\) denotes the horizontal covariant derivatives of \(D^i_{jk}\) with respect to the Berwald

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connection of $F$ [8][18]. For a manifold $M$, let $\mathcal{GDW}(M)$ denotes the class of all Finsler metrics satisfying in above relation for some tensor $T_{jkl}$. In [3], Bácsó-Papp showed that $\mathcal{GDW}(M)$ is closed under projective changes. Then, Najafi-Shen-Tayebi characterized generalized Douglas-Weyl Randers metrics [8]. In [18], it is proved that all generalized Douglas-Weyl spaces with vanishing Landsberg curvature have vanishing the quantity $H$. For other works, see [12] and [13].

The notion of $S$-curvature is originally introduced by Shen for the volume comparison theorem [9]. The Finsler metric $F$ is said to be of isotropic $S$-curvature if $S = (n + 1)cF$, where $c = c(x)$ is a scalar function on $M$. In [14], it is shown that every isotropic Berwald metric has isotropic $S$-curvature. In [4], Cheng-Shen show that every $(\alpha, \beta)$-metric with constant Killing 1-form has vanishing $S$-curvature. Then, Bácsó-Cheng-Shen proved that a Finsler metric $F = \alpha \pm \beta^2/\alpha + \epsilon \beta$ has vanishing $S$-curvature if and only if $\beta$ is a constant Killing 1-form [1]. Therefore, the Finsler metrics with vanishing $S$-curvature are of some important geometric structures which deserve to be studied deeply.

An $(\alpha, \beta)$-metric is a Finsler metric on $M$ defined by $F := \alpha \phi(s), \ s = \beta/\alpha$, where $\phi = \phi(s)$ is a $C^\infty$ function on the $(-b_0, b_0)$ with certain regularity, $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta(y) = b_i(y)y^i$ is a 1-form on $M$ [6]. In this paper, we are going to study generalized Douglas-Weyl $(\alpha, \beta)$-metrics with vanishing $S$-curvature.

**Theorem 1.1.** Let $F = \alpha \phi(s), \ s = \beta/\alpha$, be an $(\alpha, \beta)$-metric on a manifold $M$ of dimension $n \geq 3$. Suppose that

$$F \neq c_3 \alpha \left( \frac{\beta}{\alpha} \right) \left( \frac{c_3}{\alpha} \beta + c_2 + 1 \right) \frac{1}{\alpha^{\frac{c_1}{2}}} \quad \text{and} \quad F \neq d_1 \sqrt{\alpha^2 + d_2 \beta^2 + d_3 \beta},$$

where $c_1, \ c_2, \ c_3, \ d_1, \ d_2 \ and \ d_3$ are real constants. Let $F$ has vanishing $S$-curvature. Then $F$ is a $\mathcal{GDW}$-metric if and only if it is a Berwald metric.

2. Preliminary

Given a Finsler manifold $(M, F)$, then a global vector field $G$ is induced by $F$ on $TM_0$, which in a standard coordinate $(x^i, y^j)$ for $TM_0$ is given by $G = y^j \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i}$, where

$$G^i := \frac{1}{4} y^i \left\{ [F^2]_{x^i y^j} y^k - [F^2]_{x^i} \right\}, \quad y \in T_x M.$$  

The $G$ is called the spray associated to $F$.

Define $B_y : T_x M \otimes T_x M \rightarrow T_x M$ and $E_y : T_x M \otimes T_x M \rightarrow \mathbb{R}$ by $B_y(u, v, w) := B^i_{jkl}(y)u^j v^k w^l |_{x}$ and $E_y(u, v) := E_{jk}(y)u^j v^k$ where

$$B^i_{jkl} := \frac{\partial^3 G^i}{\partial y^k \partial y^l \partial y^j}, \quad E_{jk} := \frac{1}{2} B^m_{jkm}.$$
\( B \) and \( E \) are called the Berwald curvature and mean Berwald curvature, respectively. \( F \) is called a Berwald and weakly Berwald if \( B = 0 \) and \( E = 0 \), respectively [5][7].

Let
\[
D^i_{jkl} := \frac{\partial^3}{\partial y^j \partial y^k \partial y^l} \left( G^i - \frac{1}{n+1} \frac{\partial G^m}{\partial y^m} y^i \right).
\]
It is easy to verify that \( D := D^i_{jkl} dx^j \otimes \partial_i \otimes dx^k \otimes dx^l \) is a well-defined tensor on slit tangent bundle \( TM_0 \). We call \( D \) the Douglas tensor. A Finsler metric with \( D = 0 \) is called a Douglas metric. The notion of Douglas metrics was proposed by Bácsó-Matsumoto as a generalization of Berwald metrics [2].

The Douglas tensor \( D \) is a non-Riemannian projective invariant, namely, if two Finsler metrics \( F \) and \( \bar{F} \) are projectively equivalent, \( G^i = \bar{G}^i + Py^i \), where \( P = P(x, y) \) is positively \( y \)-homogeneous of degree one, then the Douglas tensor of \( F \) is same as that of \( \bar{F} \). Finsler metrics with vanishing Douglas tensor are called Douglas metrics [11].

For a Finsler metric \( F \) on an \( n \)-dimensional manifold \( M \), the Busemann-Hausdorff volume form \( dV_F = \sigma_F(x) dx^1 \cdots dx^n \) is defined by
\[
\sigma_F(x) := \frac{\text{Vol}(B^n(1))}{\text{Vol}\left[ (y^i) \in \mathbb{R}^n \mid F\left( y^i \frac{\partial}{\partial x^i} \big|_x \right) < 1 \right]}.
\]
Let \( G^i \) denote the geodesic coefficients of \( F \) in the same local coordinate system. The S-curvature is defined by
\[
S(y) := \frac{\partial G^i}{\partial y^i}(x, y) - y^i \frac{\partial}{\partial x^i} \left[ \ln \sigma_F(x) \right],
\]
where \( y = y^i \frac{\partial}{\partial x^i} |_x \in T_x M \). \( S \) is said to be isotropic if there is a scalar functions \( c = c(x) \) on \( M \) such that \( S = (n+1)cF \).

For an \((\alpha, \beta)\)-metric \( F = \alpha \phi(s) \), \( s = \beta/\alpha \), put
\[
\Phi := -(q - sq')[n\Delta + 1 + sq] - (b^2 - s^2)(1 + sq)q'',
\]
where
\[
q := \frac{\phi'}{\phi - s\phi'}, \quad \Delta := 1 + sq + (b^2 - s^2)q'.
\]
In [4], Cheng-Shen characterize \((\alpha, \beta)\)-metrics with isotropic S-curvature.

**Lemma 2.1.** ([4]) Let \( F = \alpha \phi(s) \), \( s = \beta/\alpha \), be a non-Riemannian \((\alpha, \beta)\)-metric on a manifold \( M \) of dimension \( n \geq 3 \). Suppose that \( \phi \neq c_1 \sqrt{1 + c_2 s^2} + c_3 s \) for any constant \( c_1 > 0 \), \( c_2 \) and \( c_3 \). Then \( F \) is of isotropic S-curvature \( S = (n+1)cF \) if and only if one of the following holds
(a) \( \beta \) satisfies
\[
r_{ij} = \varepsilon(b^2 a_{ij} - b_i b_j), \quad s_j = 0,
\]
where $\varepsilon = \varepsilon(x)$ is a scalar function, $b := \|\beta x\|^{\alpha}$ and $\phi = \phi(s)$ satisfies

$$\Phi = -2(n+1)k \phi^2$$

where $k$ is a constant. In this case, $S = (n+1)cF$ with $c = k\varepsilon$.

(b) $\beta$ satisfies

$$r_{ij} = 0, \quad s_j = 0 \quad (2.3)$$

In this case, $S = 0$.

The characterization of Finsler metrics with isotropic S-curvature in Cheng-Shen’s paper is not complete [4]. Their result is correct for dimension $n \geq 3$. For the case dimension $(M) = 2$, see [16].

3. Proof of Main Results

Let $F := \alpha \phi(s)$, $s = \beta/\alpha$, be an $(\alpha, \beta)$-metric on a manifold $M$, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ and $\beta(y) = b_i(x)y^i$. Define $b_{ij}$ by $b_{ij}\theta^i := db^i - b_j \theta^j$, where $\theta^i := dx^i$ and $\theta^j := \Gamma^j_{ik} dx^k$ denote the Levi-Civita connection forms of $\alpha$. Let

$$r_{ij} := \frac{1}{2} [b_{ij} + b_{ji}], \quad s_{ij} := \frac{1}{2} [b_{ij} - b_{ji}],$$

$$r_{00} := r_{ij} y^i, \quad r_{00} := r_{ij} y^i y^j, \quad r_j := b^i r_{ij}, \quad t^i_j := s^i m^j s^m_j$$

Then $\beta = b_i(x)y^i$ is a constant Killing one-form on $M$ if $r_{ij} = s_j = 0$ hold. By definition, we have

$$b_{ij} = s_{ij} + r_{ij}.$$ 

Since $y^i|_s = 0$, then for a constant Killing 1-form $\beta$ we have

$$r_{00} = 0, \quad r_i + s_i = 0.$$

For an $(\alpha, \beta)$-metric $F = \alpha \phi(s)$, $s = \beta/\alpha$, the following hold.

**Proposition 3.1.** Let $F = \alpha \phi(s)$, $s = \beta/\alpha$, be an $(\alpha, \beta)$-metric on an $n$-dimensional manifold $M$ of dimension $n \geq 3$, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a one-form on $M$. Suppose that $F$ is of vanishing $S$-curvature. Then $F$ is a GDW-metric if and only if the following holds

$$C_1 s_{j0} y^i - (C_2 y_j + C_3 b_j) y^i t_{00} = C_4 y_j s^i_{0j0} + C_5 (b_j s^i_{0j0} + s_{j0} s^i_0)$$

$$+ C_6 s^i_{j0} + C_7 (y_j t^i_0 + s_{j0} s^i_0) + C_8 b_j t^i_0, \quad (3.1)$$
where
\[ C_1 := -\left( n + 1 \right) Q_\alpha + 2bQ_\alpha \alpha - \left[ Q_\alpha + b^2Q_\beta + Q_\alpha + b^2Q_\beta + Q_\alpha \alpha \right] \alpha^{-3} - \left[ Q_\alpha + b^2Q_\beta + Q_\alpha + b^2Q_\beta + Q_\alpha \alpha \right] \alpha^{-2}, \]
\[ C_2 := (n + 1) \left[ Q_\alpha^2 + Q_\alpha + \alpha^{-1}Q_\alpha \right] \alpha^{-4} - 2 \left[ Q_\alpha Q_\beta + Q_\alpha + b^2Q_\beta + Q_\alpha \alpha \right] \beta \alpha^{-5} + \left[ 2Q_\alpha Q_\beta + Q_\alpha + b^2Q_\beta + Q_\alpha \alpha \right] \alpha^{-3}, \]
\[ C_3 := (n + 3) \left[ Q_\alpha Q_\beta + Q_\alpha \alpha \right] \alpha^{-3} + 2 \left[ Q_\alpha Q_\beta + Q_\alpha + b^2Q_\beta + Q_\alpha \alpha \right] \alpha^{-2} + \left[ b^2Q_\alpha + Q_\alpha Q_\alpha \right] \alpha^{-1} \]
\[ C_4 := -\left( n + 1 \right) Q_\alpha + 2bQ_\alpha \alpha - 2 \left[ Q_\alpha Q_\beta + Q_\alpha + b^2Q_\beta + Q_\alpha \alpha \right] \alpha^{-2} + \left[ b^2Q_\alpha + Q_\alpha Q_\alpha \right] \alpha^{-1}, \]
\[ C_5 := (n + 3) \alpha^{-1}Q_\alpha + Q_\alpha + b^2Q_\beta + Q_\alpha + b^2Q_\beta + Q_\alpha \alpha, \]
\[ C_6 := (n + 1) \alpha^{-1}Q_\alpha + Q_\alpha + b^2Q_\beta + Q_\alpha + b^2Q_\beta + Q_\alpha \alpha, \]
\[ C_7 := (n + 1) \alpha^{-3}Q_\alpha - (n + 1) \alpha^{-2}Q_\alpha^2 + Q_\alpha Q_\alpha - 2b^2Q_\alpha \alpha \]
\[ = -\left( (n + 3) \right) \left[ Q_\alpha Q_\beta + Q_\alpha + Q_\alpha \alpha \right] \alpha^{-1} - 2 \left[ Q_\alpha Q_\beta + Q_\alpha + Q_\alpha \alpha \right] \alpha^{-2} + \left[ b^2Q_\alpha + Q_\alpha Q_\alpha \right] \alpha^{-1}, \]
\[ C_8 := -\left( n + 3 \right) \left[ Q_\alpha + Q_\alpha Q_\beta + Q_\alpha + Q_\alpha \alpha \right] \alpha^{-1} - 2 \left[ Q_\alpha Q_\beta + Q_\alpha + Q_\alpha \alpha \right] \alpha^{-2} + \left[ b^2Q_\alpha + Q_\alpha Q_\alpha \right] \alpha^{-1}. \]

**Proof.** Let \( G^i \) and \( G^i_\alpha \) denote the spray coefficients of \( F \) and \( \alpha \), respectively, in the same coordinate system. Then, we have
\[ G^i = G^i_\alpha + Py^i + Q^i, \tag{3.2} \]
where
\[ Q := \alpha q = \frac{\alpha \phi'}{\phi - s \phi'}, \]
\[ P := \alpha^{-1} \Theta (r_0 - 2Qs_0), \quad Q^i := Qs^i_0 + \Psi (r_0 - 2Qs_0)b^i, \]
\[ \Theta := \frac{q - sq'}{2\Delta} = \frac{\phi \phi' - s(\phi \phi'' + \phi' \phi')}{2\phi \left( \phi - s \phi' \right) + \left( b^2 - s^2 \right) \phi''}, \]
\[ \Psi := \frac{q'}{2\Delta} = \frac{1}{2} \frac{\phi''}{\left( \phi - s \phi' \right) + \left( b^2 - s^2 \right) \phi''}. \]
By Lemma 2.1, we have \( r_{00} = s_0 = 0 \). Then (3.2) reduces to following
\[
G^i = G^i_\alpha + Qs^i_0. 
\]  
(3.3)

Let “\( \parallel \)” and “\( \| \)” denote the covariant differentiations with respect to \( G^i \) and \( G^i_\alpha \) respectively. Then by (3.3), we have
\[
D^i_{jklt}m y^m = D^i_{jklt}m y^m - 2Qs^0_0 \frac{\partial D^i_{jklt}}{\partial y^p} + D^p_{jklt} \tilde{N}^i_p - D^i_{jktp} \tilde{N}^p_k 
- D^i_{jkpl} \tilde{N}^p_l - D^i_{jklp} \tilde{N}^p_i, 
\]  
(3.4)

where
\[
D^i_{jklt}m y^m = \alpha^{-4}(Q_{\alpha\alpha} - \alpha^{-1}Q_{\alpha} \alpha_{ji}(A_{jkl}y_l + A_{ijkl}y_j + A_{jy}k) s^i_{00}) 
+ \alpha^{-3}Q_{\alpha}(A_{jkl}s^i_{l0} + A_{ijkl}s^i_{j0} + A_{jy}k s^i_{k0}) 
+ \alpha^{-3}Q_{\alpha\beta}(A_{jy}k b_l + A_{kld} b_j + A_{jy}b_k s^i_{00}) 
+ (A_{jy}k s^i_{00} + A_{kl} s^i_{j0} + A_{jly}k s^i_{k0}) s^i_0 
+ \alpha^{-2}Q_{\alpha\alpha\beta}(y_j y_k b_l + y_k y_l b_j + y_k y_b_k s^i_{00}) 
+ (y_j y_k s^i_{00} + y_j y_k s^i_{j0} + y_j y_k s^i_{k0}) 
+ \alpha^{-1}Q_{\alpha\beta\gamma}(y_j y_k b_l + y_k y_l b_j + y_k y_b_k s^i_{00}) 
+ (y_j y_k s^i_{j0} + y_j y_k s^i_{k0}) s^i_0 
+ \alpha^{-2}Q_{\alpha\alpha\beta\gamma}(y_j y_k y_s^i_{j0} + y_k y_l y_s^i_{j0} + y_j y_k y_s^i_{k0}) 
+ Q_{\beta\beta\gamma}(y_j y_k s^i_{j0} + y_k y_l s^i_{j0} + y_j y_k s^i_{k0}) 
+ \alpha^{-1}Q_{\alpha\beta\gamma}(y_j y_k b_l + y_k y_l b_j + y_k y_b_k s^i_{00}) 
+ (y_j y_k s^i_{j0} + y_k y_l s^i_{j0} + y_j y_k s^i_{k0}) s^i_0 
+ Q_{\beta\beta\gamma}[(b_j k s^i_{j0} + b_k b_l s^i_{j0} + b_j k b_k s^i_{j0} + (s_j b_k + b_j b_k s^i_{j0} + b_j b_k s^i_{k0}) s^i_{j0} 
+ (s_j b_k + b_k s^i_{j0} + s_j b_s^i_{j0} + s_j b_k s^i_{k0}) s^i_{j0} + (s_j b_k + b_k s^i_{j0} + s_j b_s^i_{j0} + s_j b_k s^i_{k0}) s^i_{j0} + Q_{\beta\beta\gamma} b_l b_k b_k s^i_{00}) 
(3.5)

and
\[
A_{ij} = \alpha^2 a_{ij} - y_i y_j, \quad (3.6)
\]
\[
\tilde{N}^i_p = Qs^i_p + [\alpha^{-1}Q_{\alpha} y_p + Q_{\beta} b_p] s^i_0, \quad (3.7)
\]
\[
\frac{\partial D^i_{jklt}}{\partial y^p} = Q_{jklt} s^i_0 + Q_{jklt} s^i_{j0} + Q_{jklt} s^i_{k0} + Q_{jklt} s^i_{j0} + Q_{jklt} s^i_{k0}, \quad (3.8)
\]

Let \( F \) is a GDW-metric. Then there exists a tensor \( D_{jklt} \) such that
\[
D^i_{jklt}m y^m = D_{jklt} y^i. 
\]
By (3.4), we have

\[ D_{ijkl} y^i = D_{ijkl|m} b^m - 2Q \frac{\partial D_{ijkl}^p}{\partial y^p} s_0 + D_{ijkl}^p \hat{N}_p^i - D_{pkli}^p \hat{N}_l^i, \]  

where

By contracting (3.9) with \( y_i \) and using (3.5), (3.7) and (3.8) we get the following

\[ D_{ijkl} = D_1 \left[ A_{jk}s_{l0} + A_{kl}s_{j0} + A_{lj}s_{k0} \right] 
+ D_2 \left[ y_jy_k s_{l0} + y_k y_l s_{j0} + y_j y_l s_{k0} \right] 
+ D_3 \left[ (y_j b_k + y_k b_j)s_{l0} \right] 
+ D_4 \left[ b_j b_k s_{l0} + b_k b_l s_{j0} + b_j b_l s_{k0} \right] 
+ D_5 \left[ A_{jk} y_l + A_{kl} y_j + A_{lj} y_k \right] t_{00} 
+ D_6 \left[ A_{jk} b_l + A_{kl} b_j + A_{lj} b_k \right] t_{00} 
+ D_7 \left[ y_j y_k b_l + y_k y_l b_j + y_j y_l b_k \right] t_{00} 
+ D_8 \left[ y_j b_k b_l + y_k b_l b_j + y_j b_l b_k \right] t_{00} 
+ D_9 \left[ y_j y_k b_l \right] t_{00} + D_{10} \left[ b_j b_k b_l \right] t_{00} 
+ D_{11} \left[ y_j s_{k0} s_{l0} + y_j s_{k0} s_{l0} \right] 
+ D_{12} \left[ b_j s_{k0} s_{l0} + b_j s_{k0} s_{l0} + b_k s_{j0} s_{l0} \right], \]

where

\[ D_1 := -\alpha^{-5}Q_\alpha, \]
\[ D_2 := -\alpha^{-4}Q_{\alpha\alpha}, \]
\[ D_3 := -\alpha^{-3}Q_{\alpha\beta}, \]
\[ D_4 := -\alpha^{-2}Q_{\beta\beta}, \]
\[ D_5 := -\alpha^{-6}Q_\alpha - \alpha^{-6}Q_{\alpha\alpha} + \alpha^{-7}Q_\alpha, \]
\[ D_6 := -\alpha^{-5}Q_\alpha Q_\beta - \alpha^{-5}Q_{\alpha\alpha}, \]
\[ D_7 := -\alpha^{-4}Q_{\alpha\alpha} Q_\beta - 2\alpha^{-4}Q_{\alpha\beta} Q_\alpha - \alpha^{-4}Q_{\alpha\alpha\beta}, \]
\[ D_8 := -\alpha^{-3}Q_{\beta\beta} Q_\alpha - 2\alpha^{-3}Q_{\alpha\beta} Q_\beta - \alpha^{-3}Q_{\alpha\beta\beta}, \]
\[ D_9 := -3\alpha^{-5}Q_{\alpha\alpha} Q_\alpha - \alpha^{-5}Q_{\alpha\alpha\alpha}, \]
\[ D_{10} := -3\alpha^{-2}Q_{\beta\beta} Q_\beta - 2\alpha^{-2}Q_{\beta\beta\beta}, \]
\[ D_{11} := -3\alpha^{-3}Q_{\alpha\beta} + 2\alpha^{-3}Q_\alpha + 2\alpha^{-4}Q_{\alpha\alpha} - 2\alpha^{-5}Q_\alpha, \]
\[ D_{12} := -3\alpha^{-2}Q_{\beta\beta} + 2\alpha^{-3}Q_{\alpha\beta} + 2\alpha^{-3}Q_\alpha Q_\beta. \]
Now, by plugging (3.10) into (3.9), and contracting the obtained result with $a^{kl}$, we get (3.1).

□

Proof of Theorem 1.1: Let $F = \alpha \phi(s)$, $s = \beta/\alpha$, be an $(\alpha, \beta)$-metric on an $n$-dimensional manifold $M$. By multiplying (3.1) with $y_i$ and $y^j$, we get

$$-\alpha QQ_{\alpha \alpha} t_{00} = 0. \tag{3.11}$$

If $Q_{\alpha \alpha} = 0$ then

$$Q = c_1 \alpha + c_2 \frac{\alpha^2}{\beta},$$

where $c_1$ and $c_2$ are real constants. Thus, we get

$$F = c_3 \alpha \left( \frac{\beta}{\alpha} \right)^{\frac{c_2}{1+c_2}} \left( c_1 \frac{\beta}{\alpha} + c_2 + 1 \right)^{\frac{1}{1+c_2}},$$

where $c_3$ is a real constant. This is a contradiction with our assumption. Then by (3.11), we get $t_{00} = 0$ which results that $s_{i0} = 0$. This means that $\beta$ is a closed one-form. By assumption, $\beta$ is parallel one-form and then $F$ reduces to a Berwald metric. □

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