Generalized Douglas-Weyl Finsler Metrics

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ABSTRACT. In this paper, we study generalized Douglas-Weyl Finsler metrics. We find some conditions under which the class of generalized Douglas-Weyl \((\alpha, \beta)\)-metric with vanishing S-curvature reduce to the class of Berwald metrics.

Keywords: Generalized Douglas-Weyl metrics, S-curvature.


1. INTRODUCTION

Let \((M, F)\) be a Finsler manifold. In local coordinates, a curve \(c(t)\) is a geodesic if and only if its coordinates \((c^i(t))\) satisfy \(\ddot{c}^i + 2G^i(\dot{c}) = 0\), where the local functions \(G^i = G^i(x, y)\) are called the spray coefficients [10]. \(F\) is called a Berwald metric, if \(G^i\) are quadratic in \(y \in T_xM\) for any \(x \in M\) or equivalently \(G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^jy^k\). As a generalization of Berwald curvature, Bácsó-Matsumoto introduced the notion of Douglas metrics which are projective invariants in Finsler geometry [2]. \(F\) is called a Douglas metric if \(G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^jy^k + P(x, y)y^i\).

A Finsler metric \(F\) is called generalized Douglas-Weyl metric (briefly, GDW-metric) if \(D^i_{jkl||m}y^m = T_{jkl}y^i\) holds for some tensor \(T_{jkl}\), where \(D^i_{jkl||m}\) denotes the horizontal covariant derivatives of \(D^i_{jkl}\) with respect to the Berwald...
connection of $F$ \[8\] \[18\]. For a manifold $M$, let $GDW(M)$ denotes the class of all Finsler metrics satisfying in above relation for some tensor $T_{jkl}$. In [3], Bácsó-Papp showed that $GDW(M)$ is closed under projective changes. Then, Najafi-Shen-Tayebi characterized generalized Douglas-Weyl Randers metrics [8]. In [18], it is proved that all generalized Douglas-Weyl spaces with vanishing Landsberg curvature have vanishing the quantity $H$. For other works, see [12] and [13].

The notion of S-curvature is originally introduced by Shen for the volume comparison theorem [9]. The Finsler metric $F$ vanishing \[4\], Cheng-Shen show that every $(\alpha, \beta)$-metric with constant Killing 1-form has vanishing S-curvature. In \[9\], the Finsler metric $F = \sqrt{g_{ij}(x)y^{i}y^{j}}$ is a Riemannian metric and $\beta(y) = b(x)y^i$ is a 1-form on $M$ [6]. In this paper, we are going to study generalized Douglas-Weyl $(\alpha, \beta)$-metrics with vanishing S-curvature.

\textbf{Theorem 1.1.} Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an $(\alpha, \beta)$-metric on a manifold $M$ of dimension $n \geq 3$. Suppose that

$$F \neq c_3\alpha \left( \frac{\beta}{\alpha} \right)^{\frac{n+2}{n+4}} \left( c_1 \frac{\beta}{\alpha} + c_2 + 1 \right)^{\frac{1}{n+4}}$$

and

$$F \neq d_1\sqrt{\alpha^2 + d_2\beta^2} + d_3\beta.$$ 

where $c_1, c_2, c_3, d_1, d_2$ and $d_3$ are real constants. Let $F$ has vanishing $S$-curvature. Then $F$ is a GDW-metric if and only if it is a Berwald metric.

2. Preliminary

Given a Finsler manifold $(M,F)$, then a global vector field $G$ is induced by $F$ on $TM_0$, which in a standard coordinate $(x^i, y^i)$ for $TM_0$ is given by $G = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i}$, where

$$G^i := \frac{1}{4} g^{ij} \left( [F^2]_{x^i y^j} - [F^2]_{x^j y^i} \right), \quad y \in T_x M.$$ 

The $G$ is called the spray associated to $F$.

Define $B_y : T_x M \otimes T_x M \otimes T_x M \rightarrow T_x M$ and $E_y : T_x M \otimes T_x M \rightarrow \mathbb{R}$ by $B_y(u, v, w) := B^i_{jkl}(y) u^j v^k w^l \frac{\partial}{\partial x^i}|_x$ and $E_y(u, v) := E_{jkl}(y) u^j v^k$ where

$$B^i_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}, \quad E_{jkl} := \frac{1}{2} B^m_{jkm}.$$
\[ D^f_{j k l} := \frac{\partial^3}{\partial y^j \partial y^k \partial y^l} (G^i - \frac{1}{n+1} \frac{\partial G^m}{\partial y^m} y^i). \]

It is easy to verify that \( D := D^f_{j k l} dx^j \otimes \partial_i \otimes dx^k \otimes dx^l \) is a well-defined tensor on slit tangent bundle \( T\mathbb{M}_0 \). We call \( D \) the Douglas tensor. A Finsler metric with \( D = 0 \) is called a Douglas metric. The notion of Douglas metrics was proposed by Bácó-Matsumoto as a generalization of Berwald metrics [2].

The Douglas tensor \( D \) is a non-Riemannian projective invariant, namely, if two Finsler metrics \( F \) and \( \bar{F} \) are projectively equivalent, \( G^i = \bar{G}^i + Py^i \), where \( P = P(x, y) \) is positively \( y \)-homogeneous of degree one, then the Douglas tensor of \( F \) is same as that of \( \bar{F} \). Finsler metrics with vanishing Douglas tensor are called Douglas metrics [11].

For a Finsler metric \( F \) on an \( n \)-dimensional manifold \( M \), the Busemann-Hausdorff volume form \( dV_F = \sigma_F (x) dx^1 \cdots dx^n \) is defined by

\[
\sigma_F (x) := \frac{\text{Vol}(\mathbb{B}^n(1))}{\text{Vol}\left(y^i \in \mathbb{R}^n, F\left(y^i \frac{\partial}{\partial x^i}|_x \right) < 1\right)}.
\]

Let \( G^i \) denote the geodesic coefficients of \( F \) in the same local coordinate system. The S-curvature is defined by

\[ S(y) := \frac{\partial G^i}{\partial y^i}(x, y) - y^i \frac{\partial}{\partial x^i} \left( \ln \sigma_F (x) \right), \]

where \( y = y^i \frac{\partial}{\partial x^i}|_x \in T_x M \). \( S \) is said to be isotropic if there is a scalar function \( c = c(x) \) on \( M \) such that \( S = (n+1)cF \).

For an \((\alpha, \beta)\)-metric \( F = \alpha \phi(s) \), \( s = \beta/\alpha \), put

\[ \Phi := -(q - sq')[n\Delta + 1 + sq] - (b^2 - s^2)(1 + sq)q'', \]

where

\[ q := \frac{\phi'}{\phi - sq'}, \quad \Delta := 1 + sq + (b^2 - s^2)q'. \]

In [4], Cheng-Shen characterize \((\alpha, \beta)\)-metrics with isotropic S-curvature.

**Lemma 2.1.** ([4]) Let \( F = \alpha \phi(s) \), \( s = \beta/\alpha \), be an non-Riemannian \((\alpha, \beta)\)-metric on a manifold \( M \) of dimension \( n \geq 3 \). Suppose that \( \phi \neq c_1 \sqrt{1 + c_2 s^2} + c_3 s \) for any constant \( c_1 > 0, c_2 \) and \( c_3 \). Then \( F \) is of isotropic S-curvature \( S = (n+1)cF \) if and only if one of the following holds

(a) \( \beta \) satisfies

\[ r_{ij} = \varepsilon(b^2a_{ij} - b_i b_j), \quad s_j = 0, \quad (2.1) \]
where \( \varepsilon = \varepsilon(x) \) is a scalar function, \( b := \|\beta x\|^{\alpha} \) and \( \phi = \phi(s) \) satisfies
\[
\Phi = -2(n + 1)k \phi \Delta^2 \frac{\phi}{b^2} + C_3 b_j y_{i00} + C_4 y_{jsi0} + C_5 s_j s_{i0} + C_6 b_j t_{0i} + C_7 (y_j t_{0i} + s_{j0} s_{i0}) + C_8 b_j t_{0i},
\]
(3.1)

In this case, \( S = (n + 1)cF \) with \( c = k\varepsilon \).

(b) \( \beta \) satisfies
\[
r_{ij} = 0, \quad s_j = 0
\]
(2.3)

In this case, \( S = 0 \).

The characterization of Finsler metrics with isotropic S-curvature in Cheng-Shen's paper is not complete [4]. Their result is correct for dimension \( n \geq 3 \). For the case dimension \( (M) = 2 \), see [16].

3. Proof of Main Results

Let \( F := \alpha \phi(s) \), \( s = \beta/\alpha \), be an \((\alpha, \beta)\)-metric on a manifold \( M \), where
\[
\alpha = \sqrt{a_{ij}(x)}y^i y^j \quad \text{and} \quad \beta(y) = b_i(x)y^i.
\]
Define \( b_{ij} \) by \( b_{ij} \theta^i := db_i - b_j \theta^j \), where \( \theta^i := dx^i \) and \( \theta^i_j := \Gamma^i_{jk}dx^k \) denote the Levi-Civita connection forms of \( \alpha \). Let
\[
\begin{align*}
r_{ij} &:= \frac{1}{2} \left[ b_{ij} + b_{ji} \right], \\
s_{ij} &:= \frac{1}{2} \left[ b_{ij} - b_{ji} \right], \\
r_{00} &:= r_{ij} y^i y^j, \\
r_{0i} &:= r_{ij} y^i y^j, \\
sh_{i0} &:= s_{ij} y^j, \\
s_j &:= b^i s_{ij}, \\
r_0 &:= r_{ij} y^j, \\
s_0 &:= s_j y^j.
\end{align*}
\]
Then \( \beta = b_i(x)y^i \) is a constant Killing one-form on \( M \) if \( r_{ij} = s_j = 0 \) hold. By definition, we have
\[
b_{ij} = s_{ij} + r_{ij}.
\]
Since \( y^i|_0 = 0 \), then for a constant Killing 1-form \( \beta \) we have
\[
r_{00} = 0, \quad r_i + s_i = 0.
\]

For an \((\alpha, \beta)\)-metric \( F = \alpha \phi(s) \), \( s = \beta/\alpha \), the following hold.

**Proposition 3.1.** Let \( F = \alpha \phi(s) \), \( s = \beta/\alpha \), be an \((\alpha, \beta)\)-metric on an \( n \)-dimensional manifold \( M \) of dimension \( n \geq 3 \), where \( \alpha = \sqrt{a_{ij}(x)}y^i y^j \) is a Riemannian metric and \( \beta = b_i(x)y^i \) is a one-form on \( M \). Suppose that \( F \) is of vanishing S-curvature. Then \( F \) is a GDW-metric if and only if the following holds
\[
C_1 s_{j0|0} y^i - (C_2 y_j + C_3 b_j) y^i t_{00} = C_4 y_j s^i_{0j0} + C_5 (b_j s^i_{0j0} + s_{j0} s^i_{0}) + C_6 s^i_{j0|0} + C_7 (y_j t^i_{0} + s_{j0} s^i_{0}) + C_8 b_j t^i_{0},
\]
(3.1)
where
\[ C_1 := -\left[(n+1)Q_\alpha + 2\beta Q_{\alpha\beta}\right]^{\alpha-3} - \left[Q_{\alpha\alpha} + b^2 Q_{\beta\beta}\right]^{\alpha-2}, \]
\[ C_2 := (n+1)\left[Q_\alpha^2 + Q_{\alpha\alpha} - \alpha^{-1} Q_\alpha\right]^{\alpha-4} - 2\left[Q_\alpha Q_\beta + Q_{\alpha\alpha\beta}\right]^{\beta\alpha^{-5}} \]
\[ + 2\left[2Q_\alpha Q_{\alpha\beta} + Q_{\alpha\alpha} Q_\beta + Q_{\alpha\alpha\beta}\right]^{\beta\alpha^{-4}} - b^2\left[2Q_\alpha Q_\beta + Q_\alpha Q_{\beta\beta}\right]^{\beta\alpha^{-3}} \]
\[ + b^2Q_{\alpha\beta\beta} + 3Q_\alpha Q_{\alpha\alpha} + Q_{\alpha\alpha\alpha\alpha}\right]^{\alpha^{-3}}, \]
\[ C_3 := (n+3)\left[Q_\alpha Q_\beta + Q_{\alpha\alpha\beta}\right]^{\alpha^{-3}} + 2\left[Q_\alpha Q_\beta + Q_{\alpha\alpha\beta}\right]^{\beta\alpha^{-3}} \]
\[ + b^2\left[3Q_\beta Q_{\beta\beta} + Q_{\beta\beta\beta}\right]^{\beta\alpha^{-2}}, \]
\[ C_4 := -\left[(n+1)Q_\alpha + 2\beta Q_{\alpha\beta}\right]^{\alpha-3} + 2\left[Q_{\alpha\alpha\beta} + Q_{\alpha\alpha}\right]^{\beta\alpha^{-2}} \]
\[ + b^2\left[3Q_\alpha Q_{\alpha\beta} + Q_{\alpha\alpha\alpha}\right]^{\alpha^{-1}}, \]
\[ C_5 := (n+3)\left[Q_{\alpha\alpha\beta} + Q_{\alpha\alpha\alpha} + 2\beta\alpha^{-1} Q_{\alpha\beta\beta} + b^2 Q_{\beta\beta\beta}, \right. \]
\[ C_6 := (n+1)\alpha^{-1}Q_\alpha + Q_{\alpha\alpha} + 2\beta\alpha^{-1} Q_{\alpha\beta\beta} + b^2 Q_{\beta\beta\beta}, \]
\[ C_7 := (n+1)\alpha^{-3}Q_\alpha - (n+1)\alpha^{-2}\left(Q_\alpha^2 + Q_{\alpha\alpha}\right) - 2\beta\alpha^{-2} Q_{\alpha\alpha\beta} \]
\[ + 2\left[Q_\alpha Q_\beta + Q_{\alpha\alpha}\right]^{\beta\alpha^{-3}} - b^2\left[2Q_\alpha Q_\beta + 2Q_\alpha Q_{\beta\beta}\right]^{\beta\alpha^{-1}} \]
\[ - 2\left[2Q_\alpha Q_{\beta\beta} + Q_{\beta\beta\beta}\right]^{\beta\alpha^{-2}} \]
\[ - b^2\alpha^{-1}Q_\alpha Q_{\beta\beta} - 3\alpha^{-1} Q_{\alpha\alpha\alpha} - 2\alpha^{-1} Q_{\alpha\alpha\alpha\alpha}, \]
\[ C_8 := -(n+3)\left[Q_\alpha Q_{\alpha\beta} + Q_{\alpha\alpha\beta}\right]^{\alpha^{-1}} - 2\left[2Q_\beta Q_{\alpha\beta} + Q_\alpha Q_{\beta\beta}\right]^{\beta\alpha^{-1}} \]
\[ - b^2\left[Q_{\beta\beta\beta} + 3Q_\beta Q_{\beta\beta}\right]^{\beta\alpha^{-1}} - Q_\beta Q_{\alpha\alpha} - Q_{\alpha\alpha\beta} - 2Q_\alpha Q_{\alpha\beta}. \]

**Proof.** Let \( G^i \) and \( G_0^i \) denote the spray coefficients of \( F \) and \( \alpha \), respectively, in the same coordinate system. Then, we have
\[
G^i = G_0^i + Py^i + Q^i, \quad (3.2)
\]
where
\[
Q := \alpha q = \frac{\alpha \phi'}{\phi - s \phi'},
\]
\[
P := \alpha^{-1} \Theta(r_{00} - 2Qs_0), \quad Q^i := Qs_0^i + \Psi(r_{00} - 2Qs_0)b^i,
\]
\[
\Theta = \frac{q - sq'}{2\Delta} = \frac{\phi \phi' - s(\phi \phi'' + \phi' \phi')}{2\phi \left(\phi - s \phi'\right) + (b^2 - s^2) \phi''}.
\]
\[
\Psi = \frac{q'}{2\Delta} = \frac{1}{2} \frac{\phi''}{\left(\phi - s \phi'\right) + (b^2 - s^2) \phi''}.
\]
By Lemma 2.1, we have \( r_{00} = s_0 = 0 \). Then (3.2) reduces to following

\[
G^i = G^i_\alpha + QS^i_0. \tag{3.3}
\]

Let “\(|\)” and “\(|\)” denote the covariant differentiations with respect to \( G^i \) and \( G^i_\alpha \) respectively. Then by (3.3), we have

\[
D^i_{jkl|m}y^m = D^i_{jkl|m}y^m - 2QS^i_0 \frac{\partial D^i_{jkl}}{\partial y^p} + D^p_{jkl} \tilde{N}^i_p - D^i_{jkl} \tilde{N}^p_k - D^i_{jkl} \tilde{N}^p_j, \tag{3.4}
\]

where

\[
D^i_{jkl|m}y^m = \alpha^{-4}Q_{\alpha\alpha} - \alpha^{-1}Q_{\alpha}(A_{jk}y_l + A_{kl}y_j + A_{jl}y_k)s^i_{0|0} + \alpha^{-3}Q_{\alpha}(A_{jk}s^i_{|0} + A_{kl}s^i_{j|0} + A_{jl}s^i_{k|0}) + \alpha^{-3}Q_{\alpha\beta}[(A_{jk}b_l + A_{kl}b_j + A_{jl}b_k)s^i_{0|0} + (A_{jk}s_{i|0} + A_{kl}s_{j|0} + A_{jl}s_{k|0})s^i_0] + \alpha^{-2}Q_{\alpha\beta\gamma}(y_{jk}y_{bl} + y_{lk}y_{bj} + y_{lj}y_{bk})s^i_{0|0} + (y_{jk}y_{b|0} + y_{lk}y_{s|0} + y_{lj}y_{s|0})s^i_0 + \alpha^{-1}Q_{\alpha\beta\gamma}[y_{jk}s^i_{b|0} + y_{lk}s^i_{j|0} + y_{lj}s^i_{k|0} + Q_{\beta\beta\beta}(b_{jk}s_{i|0} + b_{jl}s_{k|0} + b_{lk}s_{j|0})s^i_0 + \alpha^{-3}Q_{\alpha\alpha\alpha}y_{jk}y_{s^i_{|0}} + \alpha^{-1}Q_{\alpha\beta\gamma}[(y_{jk}b_l + y_{lk}b_j + y_{lj}b_k)s^i_{0|0} + (y_{jk}s_{i|0} + y_{lk}s_{j|0} + y_{lj}s_{k|0})s^i_0 + (y_{jk}s_{i|0} + y_{lk}s_{j|0} + y_{lj}s_{k|0})s^i_k] + Q_{\beta\beta\beta}[b_{jk}s^i_{b|0} + b_{lk}s^i_{j|0} + b_{jl}s^i_{k|0} + (s_{j|0}b_k + b_{jk}s_{k|0})s^i_l + (s_{k|0}b_l + b_{kl}s_{j|0})s^i_j + Q_{\beta\beta\beta}b_{jk}s^i_{0|0} + Q_{\beta\beta\beta}b_{lk}s^i_{j|0} + Q_{\beta\beta\beta}b_{jl}s^i_{k|0} + Q_{\beta\beta\beta}s^i_j + Q_{\beta\beta\beta}s^i_k + Q_{\beta\beta\beta}s^i_l]. \tag{3.5}
\]

and

\[
A_{ij} = \alpha^2 a_{ij} - y_{i}y_{j}, \tag{3.6}
\]

\[
\tilde{N}^i_p = Qs^i_{p|0} + \left[ \alpha^{-1}Qs_{p|0} + Q_{\beta}b_{p|0} \right] s^i_0, \tag{3.7}
\]

\[
\frac{\partial D^i_{jkl}}{\partial y^p} = Q_{jklp}s^i_{0|0} + Q_{jkl}|s^i_{p|0} + Q_{jklp}s^i_{j|0} + Q_{jklp}s^i_{k|0} + Q_{jklp}s^i_{l|0}. \tag{3.8}
\]

Let \( F \) is a \textit{GDW}-metric. Then there exists a tensor \( D^i_{jkl} \) such that

\[
D^i_{jkl|m}y^m = D^i_{jkl}y^i. \]
By (3.4), we have

$$D_{jkl} y^i = D^k_{jkl|m} y^m - 2Q \frac{\partial D^p_{jkl}}{\partial y^p} s^p_0 + D^p_{jkl} \tilde{N}^i_p - D^p_{pkl} \tilde{N}^i_j - D^p_{jpl} \tilde{N}^i_k - D^p_{jkm} \tilde{N}^i_l. \quad (3.9)$$

By contracting (3.9) with $y_i$ and using (3.5), (3.7) and (3.8) we get the following

$$D_{jkl} = D_1 [A_{jkl} s_{00} + A_{kl} s_{j0} + A_{jl} s_{k0}]$$

$$+ D_2 [y_j y_k s_{00} + y_j y_l s_{j0} + y_l y_j s_{k0}]$$

$$+ D_3 [(y_j b_k + y_k b_j) s_{00} + (y_j b_l + y_l b_j) s_{j0} + (y_j b_l + y_l b_j) s_{k0}]$$

$$+ D_4 [b_j b_k s_{00} + b_k b_j s_{j0} + b_j b_k s_{k0}]$$

$$+ D_5 [A_{jkl} y_j + A_{kl} y_j + A_{jl} y_k] t_{00}$$

$$+ D_6 [A_{jkl} b_l + A_{kl} b_j + A_{jl} b_k] t_{00}$$

$$+ D_7 [y_j y_k b_l + y_j y_l b_j + y_l y_j b_k] t_{00}$$

$$+ D_8 [y_j b_k b_l + y_l b_j b_k + y_k b_j b_l] t_{00}$$

$$+ D_9 [y_j y_k t_{00} + D_{10} b_j b_k t_{00}$$

$$+ D_{11} [y_j s_{j0} s_{k0} + y_j s_{k0} s_{j0} + y_k s_{j0} s_{00}]$$

$$+ D_{12} [b_j s_{j0} s_{k0} + b_j s_{k0} s_{j0} + b_k s_{j0} s_{00}], \quad (3.10)$$

where

$$D_1 := -\alpha^{-5} Q_\alpha,$$

$$D_2 := -\alpha^{-4} Q_{\alpha\alpha},$$

$$D_3 := -\alpha^{-3} Q_\alpha \beta,$$

$$D_4 := -\alpha^{-2} Q_\beta \beta,$$

$$D_5 := -\alpha^{-6} Q^\alpha_\alpha - \alpha^{-6} Q Q^\alpha_\alpha + \alpha^{-7} Q Q_\alpha,$$

$$D_6 := -\alpha^{-5} Q_\alpha Q_\beta - \alpha^{-5} Q Q_\alpha \beta,$$

$$D_7 := -\alpha^{-4} Q_{\alpha\alpha} Q_\beta - 2\alpha^{-4} Q_{\alpha\beta} Q_\alpha - \alpha^{-4} Q Q_{\alpha\beta},$$

$$D_8 := -\alpha^{-3} Q_\beta \beta \beta \alpha - 2\alpha^{-3} Q_{\alpha\beta} Q_\beta - \alpha^{-3} Q Q_{\alpha\beta \beta},$$

$$D_9 := -3\alpha^{-3} Q_{\alpha\alpha} Q_\alpha - \alpha^{-5} Q Q_{\alpha\alpha\alpha},$$

$$D_{10} := -3\alpha^{-2} Q_\beta \beta Q_\alpha - \alpha^{-2} Q Q_{\beta\beta\alpha},$$

$$D_{11} := -2\alpha^{-3} Q_{\alpha\alpha} + 2\alpha^{-2} Q^2_\alpha + 2\alpha^{-4} Q Q_{\alpha\alpha} - 2\alpha^{-5} Q Q_\alpha,$$

$$D_{12} := -2\alpha^{-2} Q_\beta \beta + 2\alpha^{-3} Q Q_{\alpha\beta} + 2\alpha^{-3} Q_{\alpha} Q_\beta.$$
Now, by plugging (3.10) into (3.9), and contracting the obtained result with $a^{kl}$, we get (3.1).

□

Proof of Theorem 1.1: Let $F = \alpha \phi(s), s = \beta/\alpha$, be an $(\alpha, \beta)$-metric on an $n$-dimensional manifold $M$. By multiplying (3.1) with $y_i$ and $y^j$, we get

$$-\alpha Q_{\alpha\alpha \alpha} t_{00} = 0. \quad (3.11)$$

If $Q_{\alpha\alpha \alpha} = 0$ then

$$Q = c_1 \alpha + c_2 \frac{\alpha^2}{\beta},$$

where $c_1$ and $c_2$ are real constants. Thus, we get

$$F = c_3 \alpha \left( \frac{\beta}{\alpha} \right)^{\frac{c_2}{c_1 + c_2}} \left( c_1 \frac{\beta}{\alpha} + c_2 + 1 \right)^{\frac{1}{c_2}},$$

where $c_3$ is a real constant. This is a contradiction with our assumption. Then by (3.11), we get $t_{00} = 0$ which results that $s_{i0} = 0$. This means that $\beta$ is a closed one-form. By assumption, $\beta$ is parallel one-form and then $F$ reduces to a Berwald metric.

□

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References