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Median and Center of Zero-Divisor Graph of Commutative Semigroups

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Abstract. For a commutative semigroup S with 0, the zero-divisor graph of S denoted by $\Gamma(S)$ is the graph whose vertices are nonzero zero-divisor of S, and two vertices x, y are adjacent in case $xy = 0$ in S. In this paper we study median and center of this graph. Also we show that if $Ass(S)$ has more than two elements, then the girth of $\Gamma(S)$ is three.

Keywords: Commutative semigroup; Zero-divisor graph; Center of a graph; Median of a graph.

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1. Introduction

In [8] Beck introduced the concept of a zero-divisor graph $\mathrm{G}(R)$ of a commutative ring R. However, he lets all elements of R be vertices of the graph and his work was mostly concerned with coloring of rings. Later, D. F. Anderson and Livingston in [6] studied the subgraph $\Gamma(R)$ of $\mathrm{G}(R)$ whose vertices are the nonzero zero-divisors of R. The zero-divisor graph of a commutative ring has been studied extensively by several authors, e.g. [1], [5], [7], [9], [11], [17]–[20], [23], and etc.

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This notion has also been extended to (commutative) semigroups with zero, e.g. [13], [14], [24], and [25]. Throughout S denotes a commutative semigroup with 0. According to [14], the zero-divisor graph, $\Gamma(S)$, is an undirected graph with vertices $\mathrm{Z}(S)^* = \mathrm{Z}(S)\backslash\{0\}$, the set of nonzero zero-divisors of S, where for distinct $x, y \in \mathrm{Z}(S)^*$, the vertices x and y are adjacent if and only if $xy = 0$. In this paper we study commutative semigroups and compare the algebraic structure of commutative semigroup S with the combinatorial structure of $\Gamma(S)$.

For the sake of completeness, we state some definitions and notions used throughout to keep this paper as self contained as possible.

For a graph G, the set of vertices of G is denoted by $\mathrm{V}(G)$. The *degree* of a vertex v in G is the number of edges of G incident with v. For a nontrivial connected graph G and a pair u, v of vertices of G, the distance $\mathrm{d}\,(u, v)$ between u and v is the length of shortest path from u to v in G. If $\mathrm{d}\,(u, v) < k$ for an integer k and for any $u, v \in \mathrm{V}(G)$, then the *eccentricity* $\mathrm{e}(v)$ of a vertex v in graph G is the distance from v to a vertex farthest from v, that is,

$$\mathrm{e}(v) = \max\{\mathrm{d}\,(x, v) | x \in \mathrm{V}(G)\}.$$

The *radius* $\mathrm{rad}(G)$ of a connected graph is defined as

$$\mathrm{rad}(G) = \min\{\mathrm{e}(v) | v \in \mathrm{V}(G)\},$$

and the *diameter* $\mathrm{diam}\,(G)$ of a connected graph G is defined as

$$\mathrm{diam}\,(G) = \max\{\mathrm{e}(v) | v \in \mathrm{V}(G)\}.$$

It is known that (e.g. [10, Theorem 4.3])

$$\mathrm{rad}(G) \leq \mathrm{diam}\,(G) \leq 2\,\mathrm{rad}(G).$$

A graph in which each pair of distinct vertices is joined by an edge is called a *complete* graph. We use K_n for the complete graph with n vertices. An *r-partite* graph is a graph whose vertex set can be partitioned into r subsets so that no edge has both ends in any one subset. A *complete r-partite* graph is one in which each vertex is joined to every vertex that is not in the same subset as the given vertex. The *complete bipartite* (i.e., complete 2-partite) graph is denoted by $K_{m,n}$ where the set of partition has sizes m and n. The *girth* of a graph G is the length of a shortest cycle in G and is denoted by $\mathrm{girth}\,(G)$. We define a *coloring* of a graph G to be an assignment of colors (elements of some set) to the vertices of G, one color to each vertex, so that adjacent vertices are assigned distinct colors. If n colors are used, then the coloring is referred to as an *n-coloring*. If there exists an n-coloring of a graph G, then G is called n-colorable. The minimum n for which a graph G is n-colorable is called the *chromatic number* of G, and is denoted by $\chi(G)$. A *clique* of a graph is a maximal complete subgraph and the number of vertices in the largest clique of graph G, denoted by $\omega(G)$, is called the *clique number* of G. Obviously $\chi(G) \geq \omega(G)$ for general graph G (see [10, page 289]).

Suppose that S is a commutative semigroup with zero. For ideal theory in commutative semigroup we refer to the survey of D.D. Anderson and Johnson [3] (also see [2]). Here we just recall some of the notions. A non-empty subset I of S is called *ideal* if $xS \subseteq I$ for any $x \in I$. An ideal $\mathfrak{p}$ of a commutative semigroup is called a *prime ideal* of S if for any two element $x, y \in S$, $xy \in \mathfrak{p}$ implies $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$. Let $\mathrm{Z}(S)$ be its set of zero-divisors of S. In order that $\Gamma(S)$ be non empty, we usually assume S always contains at least one nonzero zero divisor. In [14] DeMeyer, McKenzie, and Schneider observe that $\Gamma(S)$ (as in the ring case) is always connected, and the diameter of $\Gamma(S) \leq 3$. If $\Gamma(S)$ has a cycle then girth$\,(\Gamma(S)) \leq 4$. They also show that the number of minimal ideals of S gives a lower bound to the clique number of S. In [26] Zue and Wu studied a graph $\overline{\Gamma}(S)$ where the vertex set of this graph is $\mathrm{Z}(S)^*$ and for distinct elements $x, y \in \mathrm{Z}(S)^*$, if $xSy = 0$, then there is an edge connecting x and y. Note that $\Gamma(S)$ is a subgraph of $\overline{\Gamma}(S)$. Recently, F. DeMeyer and L. DeMeyer studied further the graph $\Gamma(S)$ and its extension to a simplicial complex, cf. [13]. Clearly for any prime ideal $\mathfrak{p}$ if x and y are adjacent in $\Gamma(S)$, then $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$. So for every prime ideal $\mathfrak{p}$ and every edge e, one of the end points of e belongs to $\mathfrak{p}$.

One may address three major problems in this area: characterization of the resulting graphs, characterization of the commutative semigroups with isomorphic graphs and realization of the connections between the structures of a commutative semigroup and the corresponding graph. In this paper we focus on the third problem.

The organization of this paper is as follows:

In Section 2, it is shown that if the set of associated primes of S, $\mathrm{Ass}\,(S)$, has more than two elements then the girth of $\Gamma(S)$ (i.e. the length of the shortest cycle in $\Gamma(S)$) is three.

2. Some special ideals and girth of $\Gamma(S)$

Let S be a commutative semigroup with 0. It is known that the following hold:

(a) $Z(S)$ is an ideal of S;
(b) $S' = S \setminus Z(S)$ and $S' \cup 0$ are subsemigroup of S with no nonzero zero-divisors.

Let T be a non-empty set of vertices of the graph G. The subgraph induced by T is the greatest subgraph of G with vertex set T, and is denoted by $G[T]$, that is, $G[T]$ contains precisely those edges of G joining two vertices of T.

The following result is an elementary statement about algebraic semigroup but expressed in graph-theoretical term.

Proposition 2.1. *Let N be the set of nilpotent elements of S. If $N^* = N \setminus \{0\}$ is a non-empty set, then $\Gamma(S)[N^*]$ is a connected subgraph of $\Gamma(S)$ of diameter at most 2.*

Proof. Since N is a commutative semigroup we have that $\Gamma(N) = \Gamma(S)[N^*]$ is connected, see [14, Theorem 1.2]. In addition, N is nilpotent commutative semigroup and so $\operatorname{diam}\Gamma(N) \leq 2$, see [13, Theorem 5].

□

The *distance* $\mathrm{d}\,(v)$ of a vertex v in a connected finite graph G is the sum of the distances v to each vertex of G. The *median* $M(G)$ of a graph G is the subgraph induced by the set of vertices having minimum distance.

Let G be a connected graph, and $T \subseteq \mathrm{V}(G)$. We say T is a *cut vertex set* if $G \setminus T$ is disconnected. Also the cut vertex set T is called a minimal cut vertex set for G if no proper subset of T is a cut vertex set. In addition, if $T = \{x\}$, then x is called a *cut vertex*.

Theorem 2.2. *The set of vertices of $M(\Gamma(S)) \bigcup \{0\}$ is an ideal of S. In addition, if T is a minimal cut vertex set of $\Gamma(S)$, then $T \cup \{0\}$ is an ideal of S.*

Proof. Let x be a vertex of $M(\Gamma(S))$ and $y \in S$. Suppose that $xy \neq 0$. Let z be a vertex of $\Gamma(S)$ and $\mathrm{d}\,(x, z) = t$. Then there is a shortest path from x to z of length t,

$$x — x_1 — x_2 — \cdots — x_{t-1} — z$$

and so

$$xy — x_1 — x_2 — \cdots — x_{t-1} — z,$$

is a walk of length t from xy to z. Thus $\mathrm{d}\,(xy, z) \leq \mathrm{d}\,(x, z)$. Since $\mathrm{d}\,(r, r) = 0$, we have the following (in)equalities:

$$\mathrm{d}\,(xy) = \sum_{z \in \mathrm{V}(\Gamma(S))} \mathrm{d}\,(xy, z) \leq \sum_{z \in \mathrm{V}(\Gamma(S))} \mathrm{d}\,(x, z) = \mathrm{d}\,(x).$$

Since $x \in M(\Gamma(S))$, we have $\mathrm{d}\,(xy) = \mathrm{d}\,(x)$, and hence xy belongs to the vertex set of $M(\Gamma(S))$.

Now let T be a minimal cut vertex set of $\Gamma(S)$, and $x \in T$, $r \in S$. Since $T \setminus \{x\}$ is not a cut vertex of $\Gamma(S)$, there exist two vertices z, y of the graph $\Gamma(S)$ such that $y — x — z$ is a path in $\Gamma(S)$, and y, z belong to two distinct connected components of $\Gamma(S) \setminus T$. Now if $rx \neq 0$, and $rx \notin T$, then rx is a vertex of $\Gamma(S) \setminus T$. Therefore we have the following path in $\Gamma(S) \setminus T$;

$$y — rx — z,$$

which is a contradiction. Thus $rx \in T \cup \{0\}$ and so $T \cup \{0\}$ is an ideal of S.

□

The techniques of the proof of Theorem 2.2 can be applied to obtain the following result.

Corollary 2.3. *Let x be a cut vertex of $\Gamma(S)$. Then $\{0, x\}$ is an ideal of S. In this case either x is adjacent to every vertex of $\Gamma(S)$ or $x \in Sx$.*

The *center* $\mathrm{C}(G)$ of a connected finite graph G is the subgraph induced by the vertices of G with eccentricity equal the radius of G.

Theorem 2.4. *For a finite commutative semigroup S, the set $V(C(\Gamma(S)))\cup\{0\}$ is an ideal of S.*

Proof. Let $x \in \mathrm{V}(\mathrm{C}(\Gamma(S)))$, and $r \in S$. Suppose that $rx \neq 0$. Then

$$\mathrm{e}(rx) = \max\{\mathrm{d}\,(u, rx)|u \in \mathrm{V}(G)\} \leq \max\{\mathrm{d}\,(u, x)|u \in \mathrm{V}(G)\} = \mathrm{e}(x).$$

Thus $\mathrm{e}(rx) = \mathrm{e}(x)$, and so $rx \in \mathrm{V}(\mathrm{C}(\Gamma(S))) \cup \{0\}$. □

A subgraph H of a graph G is a *spanning* subgraph of G if $\mathrm{V}(H) = \mathrm{V}(G)$. If U is a set of edges of a graph G, then $G \setminus U$ is the spanning subgraph of G obtained by deleting the edges in U from $\mathrm{E}(G)$. A subset U of the edge set of a connected graph G is an *edge cutset* of G if $G \setminus U$ is disconnected. An edge cutset of G is *minimal* if no proper subset of U is edge cutset. If e is an edge of G, such that $G \setminus \{e\}$ is disconnected, then e is called a *bridge*. Note that if U is a minimal edge cutset, then $G \setminus U$ has exactly two connected components.

Theorem 2.5. *Let $e = xy$ be a bridge of $\Gamma(S)$ such that the two connected components G_1, G_2 of $\Gamma(S) \setminus \{e\}$ have at least two vertices. Then $Sx = \{0, x\}$ and $Sy = \{0, y\}$ are two minimal ideals of S. Also if G_1 or G_2 has only one vertex (i.e. $\deg x = 1$ or $\deg y = 1$), then $\{0, x, y\}$ is an ideal.*

Proof. Since G_1 and G_2 have at least two vertices, there exists vertices g_1 and g_2 of $\Gamma(S)$ with $g_1 \in \mathrm{V}(G_1)$, $g_2 \in \mathrm{V}(G_2)$, and x adjacent to g_1 (in G_1) and y adjacent to g_2 (in G_2). Suppose that $r \in S$ and $rx \neq 0$. Then $rx \in \mathrm{Z}(S)$. If $rx \in G_2$, then rx is adjacent to g_1 in $\Gamma(S) \setminus \{e\}$, which is a contradiction. Therefore $rx \in G_1$. We claim that $rx = x$. In the other case rx is adjacent to y in $\Gamma(S) \setminus \{e\}$, which is a contradiction. Since $g_2 x \neq 0$ we have that $g_2 x = x$ and so $Sx = \{0, x\}$ is a minimal ideal of S. Similarly $Sy = \{0, y\}$ is a minimal ideal of S. The last part follows by a similar argument. □

The techniques of the proof of Theorem 2.5 can be applied to obtain the following result.

Corollary 2.6. *Let T be the minimal edge cutset of $\Gamma(S)$, and G_1, G_2 are two parts of $G \setminus T$. Then the following hold.*

(a) *For any $i = 1, 2$, $(V(G_i) \cap V(T)) \cup \{0\}$ is ideal of S provided G_i has at least two vertices.*

(b) *$V(T) \cup \{0\}$ is an ideal if G_1 or G_2 has only one vertex.*

A commutative semigroup is called *reduced* if for any $x \in S$, $x^n = 0$ implies $x = 0$. The annihilator of $x \in S$ is denoted by $\mathrm{Ann}\,(x)$ and it is defined as

$$\mathrm{Ann}\,(x) = \{a \in S | ax = 0\}.$$

In [22] Satyanarayana gave some characterization of semigroups satisfying the a.c.c. for right ideals possesses zero divisors. In the following we bring a necessarily condition for a commutative and reduced semigroup to satisfying the a.c.c on annihilators.

Proposition 2.7. *Let S be a commutative and reduced semigroup in which $\Gamma(S)$ does not contain an infinite clique. Then S satisfies the a.c.c on annihilators.*

Proof. Suppose that $\mathrm{Ann}\, x_1 < \mathrm{Ann}\, x_2 < \cdots$ be an increasing chain of ideals. For each $i \geq 2$, choose $a_i \in \mathrm{Ann}\, x_i \setminus \mathrm{Ann}\, x_{i-1}$. Then each $y_n = x_{n-1}a_n$ is nonzero, for $n = 2, 3, \cdots$. Also $y_i y_j = 0$ for any $i \neq j$. Since S is a commutative and reduced semigroup, we have $y_i \neq y_j$ when $i \neq j$. Therefore we have an infinite clique in S. This is a contradiction and so the assertion holds. □

Lemma 2.8. *Let S be a commutative semigroup and let $Ann\, a$ be a maximal element of $\{Ann\, x : 0 \neq x \in S\}$. Then $Ann\, a$ is a prime ideal.*

Proof. Let $xSy \subseteq \mathrm{Ann}\, a$, and $x, y \notin \mathrm{Ann}\, a$. Then $xxy \in \mathrm{Ann}\, a$, and so $x^2ya = 0$. Since $ya \neq 0$ and $\mathrm{Ann}\, a \subset \mathrm{Ann}\, ya$, we have $\mathrm{Ann}\, a = \mathrm{Ann}\, ya$. Thus $x^2 \in \mathrm{Ann}\, a$ and hence $x \in \mathrm{Ann}\, xa = \mathrm{Ann}\, a$. This is a contradiction. □

Recall that the set of associated primes of a commutative semigroup S is denoted by $\mathrm{Ass}\,(S)$ and it is the set of prime ideals $\mathfrak{p}$ of S such that there exists $x \in S$ with $\mathfrak{p} = \mathrm{Ann}\,(x)$. The next result gives some information of $\Gamma(S)$.

Theorem 2.9. *Let S be a commutative semigroup. Then the following hold:*

(a) *If $|Ass\,(S)| \geq 2$ and $\mathfrak{p} = Ann\,(x)$, $\mathfrak{q} = Ann\,(y)$ are two distinct elements of $Ass\,(S)$, then $xy = 0$.*

(b) *If $|Ass\,(S)| \geq 3$, then $girth\,(\Gamma(S)) = 3$.*

(c) *If $|Ass\,(S)| \geq 5$, then $\Gamma(S)$ is not planar (A graph G is planar if it can be drawn in the plane in such a way that no two edges meet except at vertex with which they are both incident).*

Proof. (a). We can assume that there exists $r \in \mathfrak{p} \setminus \mathfrak{q}$. Then $rx = 0$ and so $rSx = 0 \in \mathfrak{q}$. Since $\mathfrak{q}$ is a prime ideal, $x \in \mathfrak{q}$ and hence $xy = 0$.

(b). Let $\mathfrak{p}_1 = \mathrm{Ann}\,(x_1)$, $\mathfrak{p}_2 = \mathrm{Ann}\,(x_2)$, and $\mathfrak{p}_3 = \mathrm{Ann}\,(x_3)$ belong to $\mathrm{Ass}\,(S)$. Then x_1—x_2—x_3—x_1 is a cycle of length 3.

(c). Since $|\mathrm{Ass}\,(S)| \geq 5$, K_5 is a subgraph of $\Gamma(S)$, and hence by Kuratowski's Theorem $\Gamma(S)$ is not planar.

□

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