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Representations of Double Coset Lie Hypergroups

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Abstract. We study the double cosets of a Lie group by a compact Lie subgroup. We show that a Weil formula holds for double coset Lie hypergroups and show that certain representations of the Lie group lift to representations of the double coset Lie hypergroup. We characterize smooth (analytic) vectors of these lifted representations.

Keywords: Hypergroup, Lie group, Lie hypergroup, Double coset Lie hypergroup, Representations, Smooth (analytic) vectors.

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1. Introduction

Many classical examples of topological groups are indeed smooth manifolds. This makes many aspects of harmonic analysis (like the notion of Haar measure) much simpler and more natural. Hypergroups are generalizations of (topological) groups coming originally from hypercomplex systems. Roughly speaking, a hypergroup is a (locally compact, Hausdorff) space K with a convolution and

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involution on the measure space $M(K)$ making it a Banach $*$-algebra. In contrast to the group case, in hypergroups the convolution of point masses could have a large (but compact) support [6], [12], [19]. A typical example is the double coset space of a locally compact group by a compact subgroup. We refer the reader to the monograph [5] for more details (we should warn the reader that our approach is topological and is different from the algebraic approach, for instance in [4] or [16]).

A natural question is that if the ambient space of a topological hypergroup is a manifold, could one make sense of the smoothness of convolution in a way that the theory of Lie groups naturally extend to Lie hypergroups? As the multiplication map in a hypergroup sends a pair of elements into a probability measure, the notion of smooth multiplication is more involved here and should be handled with care.

The authors introduced and studied the notion of Lie hypergroups in [1] and showed that many classical hypergroups, including the double coset space of a Lie group by a compact Lie subgroup, are indeed Lie hypergroups. In this paper we prove a Weil formula for the double coset Lie hypergroup and use it to study double homogeneous spaces and smooth (analytic) vectors of smooth representations of double coset spaces.

2. Lie Hypergroups

A (locally compact) *hypergroup* is a locally compact, Hausdorff space K with an involution $^-$ and a binary operation $*$, called a *convolution*, on the Banach space $M(K)$ of all (complex) bounded Radon measures on K such that $(M(K), *)$ is an algebra and for each $x, y \in K$,

(i) $\delta_x * \delta_y$ is a probability measure on K with compact support,
(ii) the maps $(x, y) \in K^2 \mapsto \delta_x * \delta_y \in M(K)$ and $(x, y) \in K^2 \mapsto \operatorname{supp}(\delta_x * \delta_y) \in \mathfrak{C}(K)$ are continuous,
(iii) K admits an *identity* element e satisfying $\delta_e * \delta_x = \delta_x * \delta_e = \delta_x$ such that $e \in \operatorname{supp}(\delta_x * \delta_y)$ if and only if $x = \bar{y}$.
(iv) $(\delta_x * \delta_y)^- = \delta_{\bar{y}} * \delta_{\bar{x}}$,

where $\mathfrak{C}(K)$ is the space of nonvoid compact subsets of K with the Michael topology [5, 17].

If K is a locally compact hypergroup, $M(K)$ is a Banach space in norm topology. However in the next section K is also a real C^∞ (or complex analytic) manifold and we need to consider $M(K)$ as a flat (infinite dimensional) manifold and give it the structure of a topological vector space. The appropriate topology on $M(K)$ is the w^*_∞-topology induced by the space $C^\infty_c(K)$ of C^∞-functions of compact support on K. A net (μ_α) converges to μ in the w^*_∞-topology if and only if $\int_K f d\mu_\alpha \to \int_K f d\mu$ as $\alpha \to \infty$, for each $f \in C^\infty_c(K)$.

We use the notation $M_\infty(K)$ to denote the locally convex topological vector space $M(K)$ with w_∞^*-topology and save the notation $M(K)$ for the Banach space with the norm topology.

A *Lie hypergroup* is a hypergroup which is also a C^∞ (or real analytic) manifold K (possibly with boundary) such that the convolution map $m : K \times K \longrightarrow M_\infty(K);$, $(x, y) \mapsto \delta_x * \delta_y$, and involution map $i : K \longrightarrow K$; $x \mapsto \bar{x}$ are C^∞ (real analytic) [1]. The smoothness of the convolution means that the map $(x, y) \mapsto \int_K f d(\delta_x * \delta_y)$ is C^∞ on $K \times K$, for each $f \in C_c^\infty(K)$. Note that, unlike Lie groups, a Lie hypergroup K may be a manifold with boundary [1], and in such a case, the C^∞ maps on K are defined on local charts using half space $H^n := \{(x_1, \ldots, x_n) : x_n \geq 0\}$ (in the real case) where the boundary points are those mapped to $x_n = 0$ by a chart.

3. Double Coset Lie Hypergroups

A typical example of a locally compact hypergroup is the double coset space $H\backslash G/H = \{HxH : x \in G\}$, where G is a locally compact group and H is a compact subgroup [5, 1.1.9]. The space $H\backslash G/H$ (also denoted by $G//H$) is considered with the quotient topology. The identity element is $H = HeH$ and the involution and convolution are given by $(HxH)^{-} = Hx^{-1}H$ and $\delta_{HxH} * \delta_{HyH} = \int_H \delta_{HxtyH} dt$, where the integral is taken against the (normalized) Haar measure of H. The authors showed in [1] that if G is a Lie group and H a compact Lie subgroup of G such that the double coset space $H\backslash G/H$ is a manifold (possibly with boundary), then $H\backslash G/H$ is a Lie hypergroup. Same holds for the orbit space G^H, if H is a compact Lie group acting smoothly on G and G^H is a manifold.

As a concrete example, Let S^2 be the unit sphere in $\mathbb{R}^3$ and $G = SO(3)$, let $n = (0, 0, 1)$ be the north pole in S^2 and $H = \{g \in G : gn = n\}$, then G is a Lie group and H is a compact Lie subgroup, topologically isomorphic to the compact group $\mathbb{T}$, and two elements of G are H-conjugate if and only if they rotate S^2 through the same angle. Hence G^H is a compact commutative Lie hypergroup, homeomorphic to $[0, \pi]$ and $H\backslash G/H$ is a compact commutative Lie hypergroup, homeomorphic to $[-1, 1]$ [5, 1.1.17]. Note that in G^H we have $supp(\delta_\pi * \delta_\pi) = [0, \pi]$ where as in $H\backslash G/H$, $\delta_{-1} * \delta_{-1} = \delta_1$, hence the two hypergroups are not topologically isomorphic. Similarly, the one dimensional hypergroup $\mathbb{R}_+$ (of non compact type) with convolution $\delta_r * \delta_s = \frac{1}{2}(\delta_{|r-s|} + \delta_{r+s})$ is a Lie hypergroup as the double coset hypergroup with $G = \mathbb{R} \rtimes \mathbb{Z}_2$ and $H = \mathbb{Z}_2$ [5, 3.4.5]. This should be handled with care: if we identify the orbit space with $\mathbb{R}_+$ via the quotient map $t \mapsto |t|$, a function $f \in C^\infty(\mathbb{R}_+)$ is identified with an even function $\tilde{f} \in C^\infty(\mathbb{R})$. Now the smoothness of the map

$$(s, t) \in (\mathbb{R}_+ \times \mathbb{R}_+) \mapsto \int f d(\delta_s * \delta_t) = \frac{1}{2}[f(s + t) + f(|s - t|)]$$

follows from that of

$$(s,t) \in \mathbb{R}^2 \mapsto \frac{1}{2}[\tilde{f}(s+t) + \tilde{f}(s-t)].$$

The same holds for the one dimensional hypergroup $[0,1]$ under $\delta_r * \delta_s = \frac{1}{2}(\delta_{|r-s|} + \delta_{|1-|1-r-s||})$ with $G = \mathbb{T} \rtimes \mathbb{Z}_2$ and $H = \mathbb{Z}_2$ [5, 3.4.6]. The Chebychev hypergroup of the second kind identifies with $\mathbb{Z}_+$ with convolution

$$\delta_m * \delta_n = \sum_{k=0}^{m\wedge n} \frac{|m-n| + 2k + 1}{(m+1)(n+1)} \delta_{|m-n|+2k}$$

whose dual object is the Lie hypergroup $SO(3)\backslash SO(4)/SO(3)$. More sophisticated examples [1] come from polynomial hypergroups [5]. The disc polynomial hypergroup is $J_\alpha = (\mathbb{Z}_+^2, *(Q_{m,n}^\alpha))$ with $\alpha > -1$ [13] and its diagonal $H \simeq (\mathbb{Z}_+, *(Q_n^{\alpha,0}))$ is a subhypergroup [3]. The dual hypergroup $\hat{J}_\alpha$ is equal to the closure $\bar{D}$ of the open disc $D = \{(z, \bar{z}) \in \mathbb{C}^2 : |z| < 1\}$. For the compact Lie group $G = U(d)$ and closed subgroup $H = U(d-1)$ the double coset Lie hypergroup $K = H\backslash G/H$ is identified with $\bar{D}$ as the dual of $J_{d-2} = (\mathbb{Z}_+^2, *(Q_{m,n}^{d-2}))$ [3], [5, 3.1.14]. When $\alpha = 0$, $\bar{D}$ is the hypergroup with convolution $\int_{\bar{D}} f d(\delta_{z_1} * \delta_{z_2})$ defined as

$$\frac{1}{2\pi} \int_0^{2\pi} f\big(z_1\bar{z}_2 + e^{it}(1-|z_1|^2)^{\frac{1}{2}}(1-|z_2|^2)^{\frac{1}{2}}\big)dt$$

for $f \in C(\bar{D})$, which is smooth on $\bar{D}$ when $f \in C^\infty(\bar{D})$, turning $\bar{D}$ into a Lie hypergroup.

3.1. **Invariant Measures.** Let G be locally compact group with a closed subgroup H and a compact subgroup K. Let dx, dh, dk, Δ_G and Δ_H denote the corresponding left Haar measures (normalized for K) and modular functions. The *double quotient space of* G by H and K, denoted by $K\backslash G/H$, consists of double cosets KxH for $x \in G$. This is a locally compact space under the quotient topology and the quotient map $q : G \to K\backslash G/H$ is open and continuous [15] on which the normalizer N of K in G acts by translation,

$$n.(KxH) = KnxH \quad (n \in N, x \in G).$$

For $f \in C_c(G)$ define

$$Qf(KxH) = \int_K \int_H f(kxh)dkdh.$$

This is well defined by [15, Lem. 2.3] and $Q : C_c(G) \to C_c(K\backslash G/H)$ and $supp(Qf) \subseteq q(supp(f))$ for $f \in C_c(G)$.

The next two results are proved as in the classical case [7]. We give a sketch of the proof for the sake of completeness.

Lemma 3.1. *For each $f \in C_c(G)$ and $\varphi \in C_c(K\backslash G/H)$, $Q((\varphi \circ q).f) = \varphi.Qf$. If $E \subseteq K\backslash G/H$ is compact, there is a compact subset $F \subseteq G$ and a function $f \in C_c(G)_+$ such that $q(F) = E$ and $Qf = 1$ on E.*

Proof. The first statement is trivial. Let V be a relatively compact open neighbourhood of identity in G and cover E by the sets $q(xV)$, $x \in G$. Choose a subcover $\{q(x_jV)\}_1^n$ and put $F = q^{-1}(E) \cap (\cup_{j=1}^n x_j\bar{V})$. Let E_0 be a compact neighbourhood of E in $K\backslash G/H$ and choose a compact set F_0 in G with $q(F_0) = E_0$. Choose $g \in C_c(G)_+$ such that $g > 0$ on F_0 and $\varphi \in C_c(K\backslash G/H)$ with $supp(\varphi) \subseteq E_0$ and $\varphi = 1$ on E. Put $f = \frac{\varphi\circ q}{Qg\circ q}.g$, then since $Qg > 0$ on $supp(\varphi)$, we have $f \in C_c(G)$, $supp(f) \subseteq supp(g)$ and $Qf = Q((\varphi \circ q).\frac{g}{Qg\circ q}) = \varphi.Q(\frac{g}{Qg\circ q}) = \varphi$. □

Proposition 3.2. *If $\varphi \in C_c(K\backslash G/H)$ then there is a function $f \in C_c(G)$ such that $Qf = \varphi$ and $q(supp(f)) \subseteq supp(\varphi)$. Moreover if $\varphi \geq 0$ we may choose $f \geq 0$. When G is a Lie group, the restriction Q^∞ of Q to $C_c^\infty(G)$ is a surjective linear map onto $C_c^\infty(K\backslash G/H)$.*

Proof. There is $g \in C_c(G)_+$ such that $Qg = 1$ on $supp(\varphi)$. Put $f = (\varphi \circ q).g$. For the case of Lie groups, use a similar argument with a smooth version of Urysohn's lemma. □

Now we are ready to prove a version of Weil's formula for double coset spaces (see [15, Thm. 3.2]).

Theorem 3.3. *There is a (unique up to constant factors) N-invariant Radon measure μ on $K\backslash G/H$ if and only if $\Delta_G\big|_H = \Delta_H$. In this case, μ could be suitably to be chosen such that*

$$\int_G f(x)dx = \int_{K\backslash G/H}\int_K\int_H f(kxh)dkdhd\mu(\dot{x}),$$

for $f \in C_c(G)$.

Proof. If μ exists then the uniqueness and the above relation follow from the fact that the (left) Haar measure on G is unique (up to constant factors) and that $f \mapsto \int_{K\backslash G/H} Qfd\mu$ is clearly a left invariant positive linear functional on $C_c(G)$. If μ exists, by the above lemma, an argument similar to [7, 2.49] shows that $\Delta_G = \Delta_H$ on H. Conversely let this equality hold. Take any $f \in C_c(G)$ with $Qf = 0$. Use Lemma 3.1.1 to get an element $\varphi \in C_c(G)$ with $Q\varphi = 1$ on $q(supp(f)) \subseteq K\backslash G/H$. Since

$$Qf(\dot{x}) = \int_K\int_H \Delta_G(h^{-1}k^{-1})\Delta_K(k)f(kxh^{-1})dkdh,$$

vanishes, so does

$$\begin{aligned}&\int_G\int_K\int_H \Delta_G(h^{-1}k^{-1})\Delta_K(k)\varphi(x)f(kxh^{-1})dkdhdx\\ &\quad= \int_G f(x)\int_H\int_K \varphi(kxh)dkdhdx = \int_G f(x)dx.\end{aligned}$$

Hence the map $Qf \mapsto \int_G f$ is a well-defined N-invariant positive linear functional on $C_c(K\backslash G/H)$, and the corresponding Radon measure μ satisfies the desired properties. $\square$

When $K = 1$, the above theorem gives the well-known Weil's formula. When H is compact and $K = H$, we get a Weil's formula for $H\backslash G/H$.

Theorem 3.4. *Let G be a Lie group and H be a Lie subgroup with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, and let K be a compact Lie subgroup. Then there is an N-invariant Radon measure μ on $K\backslash G/H$ iff*

$$\Big|\frac{det(Ad_{\mathfrak{g}}h)}{det(Ad_{\mathfrak{h}}h)}\Big| = 1 \quad (h \in H).$$

The above condition holds if H is a connected nilpotent, or connected semisimple, or compact Lie subgroup.

Proof. It is a classical result [7, 2.30] that for a Lie group G with Lie algebra $\mathfrak{g}$, the modular function of G is calculated via

$$\Delta_G(x) = |det(Ad_{\mathfrak{g}}x^{-1})| \quad (x \in G).$$

Hence the restriction of Δ_G to H is equal to Δ_H iff the equality in the statement of the theorem holds. In particular, H is connected and nilpotent, then $\Delta_H = 1$ on H. Also if H is connected and semisimple, then $\mathfrak{h}$ is a direct sum of simple algebras, hence $[\mathfrak{h}, \mathfrak{h}] = \mathfrak{h}$. Now $[\mathfrak{h}, \mathfrak{h}]$ is the Lie algebra of $[H, H]$ hence $[H, H] = H$, since H is connected. Now clearly $\Delta_H = 1$ on $[H, H]$, hence $\Delta_H = 1$ on H. Finally if H is compact, then $\Delta_H(H)$ is a compact subgroup of the multiplicative group $\mathbb{R}^+$, and hence is the trivial subgroup, and again $\Delta_H = 1$ on H. It is easy to see that in these cases, $\Delta_G = 1$ on H [7] and the result follows. $\square$

3.2. **Smooth vectors.** In this section we study smooth (analytic) vectors for representations of double coset Lie hypergroups. To motivate our problem, let us consider the Fourier transform in the complex domain as the analytic continuation of a representation of the additive group $\mathbb{R}$ to its complexification $\mathbb{C}$. Let U be the regular representation of R on $L^2(\mathbb{R})$ defined by $U(t)f(x) = f(x + t)$. Suppose that the Fourier-Plancherel transform of $f \in L^2(\mathbb{R})$ satisfies Paley-Wiener condition

$$\int_{-\infty}^{+\infty} e^{2r|\xi|}|\hat{f}(\xi)|^2 d\xi < \infty,$$

for some $r > 0$ then f has an analytic continuation to the function $f(z) = \int e^{iz\xi}\hat{f}(\xi)d\xi$ on the strip $|Imz| < r$ and $U(z)f(x) = f(x + z)$, $|Imz| < r$ defines a local representation of the additive group $\mathbb{C}$ on a subspace of $L^2(\mathbb{R})$.

To get the right idea for a generalization to representations of Lie groups, the space of functions satisfying Paley-Wiener condition should be replaced by the space of analytic vectors for the representation U, namely those functions f for which $U(t)f$ is an analytic function from $\mathbb{R}$ to $L^2(\mathbb{R})$. These are shown to be the functions f holomorphic in some strip $|Imz| < r$ satisfying

$$\sup_{|y|<r} \int_{-\infty}^{+\infty} |f(x+iy)|^2 dx < \infty.$$

The first Paley-Wiener condition means that f is an analytic vector for the operator $B = [1 - (d/dx)^2]^{\frac{1}{2}}$, defined by $(Bf)\check{}(\xi) = (l + \xi^2)^{\frac{1}{2}} \hat{f}(\xi)$. To define such an operator for general Lie groups, let π be a strongly continuous unitary representation of a Lie group G on a Hilbert space $\mathcal{H}_\pi$ and $\mathfrak{g}$ be the Lie algebra of G, lift π to the representation $\partial\pi$ of the universal enveloping algebra of $\mathfrak{g}$ on the space $\mathcal{H}_\pi^\infty$ of the C^∞ vectors of π. Choose a basis $X_1, ..., X_d$ for $\mathfrak{g}$ and put $\Delta = X_1^2 + ... + X_d^2$ and $A = \partial\pi(1 - \Delta)^-$, where the bar stands for the operator closure. The operator A is selfadjoint and positive, as shown by Nelson and Stinespring [18], and for $B = A^{\frac{1}{2}}$, the space $\mathcal{H}_\pi^\omega$ of analytic vectors for π is precisely the space of analytic vectors for the operator B [8].

For a vector $v \in \mathcal{H}_\pi$, the map $x \mapsto \pi(x)v$ is C^∞ on G iff the complex valued maps $x \mapsto \langle \pi(x)v, w \rangle$ are C^∞ on G for each $w \in \mathcal{H}_\pi$. In this case we call v a C^∞-vector of π and write $v \in \mathcal{H}_\pi^\infty$. The analytic vectors $\mathcal{H}_\pi^\omega$ are defined similarly [9].

For a locally compact group G and compact subgroup H with Haar measure σ such that $\sigma(H) = 1$, let $\pi \in Rep(G)$ be a unitary representation of G such that $\pi(\sigma) \neq 0$. Then $\dot{\pi}(\dot{x}) = \pi(\sigma * \delta_x * \sigma)$ defines a representation of $H\backslash G/H$ [12, section 14]. When π is irreducible, the condition $\pi(\sigma) \neq 0$ holds and the restriction of $\dot{\pi}$ to $\pi(\sigma)\mathcal{H}_\pi$ is irreducible, and the set of all such liftings exhaust all irreducible representations of $H\backslash G/H$ [10, Lemma 4.1]. If each positive definite function on $H\backslash G/H$ lifts to a positive definite function on G, $Rep(H\backslash G/H)$ is homeomorphic to the open subset of $Rep(G)$ consisting of those unitary representations π for which $\pi(\sigma) \neq 0$ [11, 12]. If the quotient map $q : G \to H\backslash G/H$ is open, the above lifting property of positive definite functions holds [2].

Let G be a Lie group and H be a compact subgroup with Haar measure σ such that $\sigma(H) = 1$.

Lemma 3.5. *If $\pi \in Rep(G)$ and $\pi(\sigma) \neq 0$, then for each $v \in \mathcal{H}_\pi^\infty$ we have $\dot{v} := \pi(\sigma)v \in \mathcal{H}_{\dot{\pi}}^\infty$. A similar statement is valid for analytic vectors.*

Proof. Since H is compact, σ is self-adjoint idempotent in the Banach $*$-algebra $M(G)$, that is $\sigma^* = \sigma$ and $\sigma * \sigma = \sigma$, and

$$\begin{aligned} \dot{\pi}(\dot{x})\dot{v} &= \pi(\sigma * \delta_x * \sigma * \sigma)v \\ &= \pi(\sigma * \delta_x * \sigma)v, \end{aligned}$$

similarly for each $w \in \mathcal{H}_\pi$,

$$\begin{aligned}\langle \dot{\pi}(\dot{x})\dot{v}, \dot{w}\rangle &= \langle \pi(\sigma^* * \sigma * \delta_x * \sigma)v, w\rangle \\ &= \langle \pi(\sigma * \delta_x * \sigma)v, w\rangle.\end{aligned}$$

Consider $f(x) := \langle \pi(x)v, w\rangle$ then $f \in C^\infty(G)$ and by Proposition 3.1.2, $Q^\infty f \in C^\infty(H\backslash G/H)$ and

$$\begin{aligned}Q^\infty f(\dot{x}) &= \int_G \langle \pi(t)v, w\rangle d(\sigma * \delta_x * \sigma)(t) \\ &= \langle \pi(\sigma * \delta_x * \sigma)v, w\rangle \\ &= \langle \dot{\pi}(\dot{x})\dot{v}, \dot{w}\rangle,\end{aligned}$$

hence $\dot{v} \in \mathcal{H}^\infty_{\dot{\pi}}$. A similar argument works for analytic vectors. □

Next let $X_1, \ldots, X_r$ be a basis for the Lie algebra $\mathfrak{g}$ of G and put $\dot{X}_k := X_k + \mathfrak{h} \in \mathfrak{g}/\mathfrak{h}$. We may suppose that $\dot{X}_1, \ldots, \dot{X}_d$ is a basis for the linear space $\mathfrak{g}/\mathfrak{h}$, where $d = dim\mathfrak{g} - dim\mathfrak{h}$. Let $\pi \in Rep(G)$ be a unitary representation of G and let

$$\partial\pi(X)v = \frac{d}{dt}\big(\pi(e^{tX})v\big)\big|_{t=0} \quad (X \in \mathfrak{g}, v \in \mathcal{H}^\infty_\pi)$$

be the corresponding representation of $\mathfrak{g}$. Put

$$\partial\dot{\pi}(\dot{X})\dot{v} = \int_H \int_H \pi(k)\partial\pi(X)\pi(h)v dk dh,$$

then if $X, X^{'} \in \mathfrak{g}$ and $X = X^{'} + Y$ for some $Y \in \mathfrak{h}$,

$$\begin{aligned}\partial\dot{\pi}(\dot{X})\dot{v} &= \int_H \int_H \frac{d}{dt}\big(\pi(ke^{tX^{'}+tY}h)\big)\big|_{t=0} v dk dh \\ &= \int_H \int_H \frac{d}{dt}\big(\pi(ke^{tX^{'}})\pi(h)\big)\big|_{t=0} v dk dh \\ &= \partial\dot{\pi}(\dot{X}^{'})\dot{v},\end{aligned}$$

hence $\partial\dot{\pi}$ is well-defined.

Theorem 3.6. *Let $\pi \in Rep(G)$ and $\pi(\sigma) \neq 0$. For each $v \in \mathcal{H}_\pi$, if $\dot{v} := \pi(\sigma)v \in \mathcal{H}_{\dot{\pi}}$ is in the domain of $\partial\dot{\pi}((\dot{X}_k)^n)$ for each $n \geq 1$ and $1 \leq k \leq d$, then $\dot{v} := \pi(\sigma)v \in \mathcal{H}^\infty_{\dot{\pi}}$.*

Proof. Let $v_k(t) = \pi(e^{tX_k})v$ for $1 \leq k \leq r$ then $v_k : \mathbb{R} \to \mathcal{H}_\pi$ is C^∞ and $\left(\frac{d}{dt}\right)^n v_k\big|_{t=0} = \partial\pi((X_k)^n)v$ [9, Theorem 1.1]. Let $\phi \in C^\infty_c(G)$ and $w \in \mathcal{H}_\pi$, then by the proof of [9, Theorem 1.1],

$$\begin{aligned}&\int_G \phi(x)\langle \pi(x)\partial\pi((X_k)^n)v, w\rangle dx \\ &\qquad = (-1)^n \int_G \tilde{X}^n_k \phi(x)\langle \pi(x)v, w\rangle dx,\end{aligned}$$

where $\tilde{X}\phi(x) = \frac{d}{dt}\phi(xe^{tX})\big|_{t=0}$. Note that each $\dot{X}_j$, for $d+1 \le j \le r$ is a finite linear combination of $\dot{X}_k$'s for $1 \le k \le d$ and hence the right hand side of the above equality makes sense for each $1 \le k \le r$. Now by Theorem 3.1.4 and Proposition 3.1.2, the left hand side is equal to

$$\int_{H\backslash G/H} Q^{\infty}\phi(\dot{x})\langle\dot{\pi}(\dot{x})\partial\pi((\dot{X}_k)^n)\dot{v},\dot{w}\rangle d\mu(\dot{x}),$$

and the right hand side is equal to

$$(-1)^n \int_{H\backslash G/H} \dot{\tilde{X}}_k^n\dot{\phi}(\dot{x})\langle\dot{\pi}(\dot{x})\dot{v},\dot{w}\rangle d\mu(\dot{x}),$$

where $\dot{\tilde{X}}\dot{\phi}(\dot{x}) = \frac{d}{dt}\dot{\phi}(\dot{x}_t)\big|_{t=0}$ with $x_t = xe^{tX}$.

Consider the elliptic operator $\Delta_m = \tilde{X}_1^{2m} + \dots \tilde{X}_r^{2m}$ on G and let $\dot{\Delta}_m = \dot{\tilde{X}}_1^{2m} + \dots \dot{\tilde{X}}_r^{2m}$ be the corresponding elliptic operator on $H\backslash G/H$, then the above observation means that $f_w(\dot{x}) := \langle\dot{\pi}(\dot{x})\dot{v},\dot{w}\rangle$ is a weak solution of the elliptic differential equation $\dot{\Delta}_m f_w = g_m$ where

$$g_m(\dot{x}) = \sum_{k=1}^{r}\langle\dot{\pi}(\dot{x})\partial\pi((\dot{X}_k)^{2m})\dot{v},\dot{w}\rangle.$$

Now f_w is continuous and $\dot{\Delta}_m$ is elliptic of order $2m$, hence by local regularity theorem, f_w has locally-L^2-derivatives up to order $2m$, for each m. By Sobolev lemma, f_w is C^∞ and therefore $\dot{v} \in \mathcal{H}_{\dot{\pi}}^{\infty}$. □

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