On the 2-absorbing Submodules

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Abstract. Let \( R \) be a commutative ring and \( M \) be an \( R \)-module. In this paper, we investigate some properties of 2-absorbing submodules of \( M \). It is shown that \( N \) is a 2-absorbing submodule of \( M \) if and only if whenever \( IJL \subseteq N \) for some ideals \( I, J \) of \( R \) and a submodule \( L \) of \( M \), then \( IL \subseteq N \) or \( JL \subseteq N \) or \( IJ \subseteq N \); \( R \). Also, if \( N \) is a 2-absorbing submodule of \( M \) and \( M/N \) is Noetherian, then a chain of 2-absorbing submodules of \( M \) is constructed. Furthermore, the annihilation of \( E(R/p) \) is studied whenever \( 0 \) is a 2-absorbing submodule of \( E(R/p) \), where \( p \) is a prime ideal of \( R \) and \( E(R/p) \) is an injective envelope of \( R/p \).

Keywords: 2-absorbing ideal, 2-absorbing submodule, A chain of 2-absorbing submodule.


1. Introduction

Throughout this paper \( R \) is a commutative ring with non-zero identity and \( M \) is an unitary \( R \)-module. We defined a submodule \( N \) of \( M \) is 2-absorbing whenever \( abm \in N \) for some \( a, b \in R, m \in M \), then \( am \in N \) or \( bm \in N \) or \( ab \in N \); \( R \). It is well known that, a submodule \( N \) of \( M \) is prime if and only if \( IL \subseteq N \) for an ideal \( I \) of \( R \) and

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Received 30 October 2013; Accepted 09 September 2014

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a submodule $L$ of $M$, then either $L \subseteq N$ or $I \subseteq N :_R M$. This statement persuaded us to prove that, a submodule $N$ of $M$ is 2-absorbing if and only if $IJL \subseteq N$ for some ideals $I,J$ of $R$ and a submodule $L$ of $M$, then $IL \subseteq N$ or $JL \subseteq N$ or $IJ \subseteq N :_R M$. As a corollary of this theorem, it is shown that $L = \{ m \in M : p \subseteq r(N : m) \}$ is a 2-absorbing submodule of $M$, where $N$ is a 2-absorbing submodule of $M$ with $r(N :_R M) = p \cap q$ for some prime ideals $p,q$ of $R$. Also, it is shown that if $M/N$ is Noetherian, then there exists a chain of 2-absorbing submodules of $M$ that begins with $N$. Assume that $E(R/p)$ is an injective envelope of $R/p$, it is shown that if 0 is a 2-absorbing submodule of $E(R/p)$, then $r(0 :_R E(R/p)) = p$ and $0 :_R x$ is determined for all nonzero element $x$ of $E(R/p)$.

Now, we define the concepts that we will use later. For a submodule $L$ of $M$ let $L :_R M$ denote the ideal \{ $r \in R : rM \subseteq L$ \}. Similarly, for an element $m \in M$ let $L :_R m$ denote the ideal \{ $r \in R : rm \subseteq L$ \}. If $I$ is an ideal of $R$, then $r(I)$ denotes the radical of $I$. We say that $p \in \text{Spec}(R)$ is an associated prime ideal of $M$ if there exists $m \in M$ with $0 :_R m = p$. The set of associated prime ideals of $M$ is denoted by $\text{Ass}_R(M)$, the set of integers is denoted by $\mathbb{Z}$.

2. 2-absorbing Submodules

Let $N$ be a proper submodule of $M$. We say that $N$ is a 2-absorbing submodule of $M$ if whenever $a,b \in R$, $m \in M$ and $abm \in N$, then $am \in N$ or $bm \in N$ or $ab \in N :_R M$.

Lemma 2.1. Let $I$ be an ideal of $R$ and $N$ be a 2-absorbing submodule of $M$. If $a \in R$, $m \in M$ and $IAM \subseteq N$, then $am \in N$ or $Im \subseteq N$ or $Ia \subseteq N :_R M$.

Proof. Let $am \notin N$ and $Ia \notin N :_R M$. Then there exists $b \in I$ such that $ba \notin N :_R M$. Now, $bam \in N$ implies that $bm \in N$, since $N$ is a 2-absorbing submodule of $M$. We have to show that $Im \subseteq N$. Let $c$ be an arbitrary element of $I$. Thus $(b+c)am \in N$. Hence, either $(b+c)m \in N$ or $(b+c)a \in N :_R M$. If $(b+c)m \in N$, then by $bm \in N$ it follows that $cm \in N$. If $(b+c)a \in N :_R M$, then $ca \notin N :_R M$, but $cam \in N$. Thus $cm \in N$. Hence, we conclude that $Im \subseteq N$. ∎

Lemma 2.2. Let $I,J$ be ideals of $R$ and $N$ be a 2-absorbing submodule of $M$. If $m \in M$ and $IJm \subseteq N$, then $Im \subseteq N$ or $Jm \subseteq N$ or $IJ \subseteq N :_R M$.

Proof. Let $I \not\subseteq N :_R M$ and $J \not\subseteq N :_R M$. We have to show that $IJ \not\subseteq N :_R M$. Assume that $c \in I$ and $d \in J$. By assumption there exists $a \in I$ such that $am \notin N$ but $aJm \subseteq N$. Now, Lemma 2.1 shows that $aJ \subseteq N :_R M$ and so $(I \setminus N :_R M)J \subseteq N :_R M$, similarly there exists $b \in J \setminus N :_R m$ such that $Ib \subseteq N :_R M$ and also $(I \setminus N :_R m)J \subseteq N :_R M$. Thus we have $ab \in N :_R M$, $ad \in N :_R M$ and $cb \in N :_R M$. By $a+c \in I$ and $b+d \in J$ it follows that $(a+c)(b+d)m \in N$. Therefore, $(a+c)m \in N$ or $(b+d)m \in N$ or
(a + c)(b + d) ∈ N : R M. If (a + c)m ∈ N, then cm ̸∈ N hence, c ∈ I \ N : R m
which implies that cd ∈ N : R M. Similarly by (b + d)m ∈ N, we can deduce
that cd ∈ N : R M. If (a + c)(b + d) ∈ N : R M, then ab + ad + cb + cd ∈ N : R M
and so cd ∈ N : R M. Therefore, IJ ⊆ N : R M. □

Theorem 2.3. Let N be a proper submodule of M. The following statement
are equivalent:

(i) N is a 2-absorbing submodule of M;
(ii) If IJL ⊆ N for some ideals I, J of R and a submodule L of M, then
IL ⊆ N or JL ⊆ N or IJ ⊆ N : R M.

Proof. (ii) ⇒ (i) is obvious. To prove (i) ⇒ (ii), assume that IJL ⊆ N for
some ideals I, J of R and a submodule L of M and IJ ⊄ N : R M. Then
by Lemma 2.2 for all m ∈ L either Im ⊆ N or Jm ⊆ N. If Im ⊆ N, for
all m ∈ L we are done. Similarly if Jm ⊆ N, for all m ∈ L we are done.
Suppose that m, m′ ∈ L are such that Im ⊄ N and Jm′ ⊄ N. Thus Jm ⊆ N
and Im′ ⊆ N. Since IJ(m + m′) ⊆ N we have either I(m + m′) ⊆ N or
J(m + m′) ⊆ N. By I(m + m′) ⊆ N, it follows that Im ⊆ N which is a
contradiction, similarly by J(m + m′) ⊆ N we get a contradiction. Therefore
either IL ⊆ N or JL ⊆ N. □

A submodule N of M is called strongly 2-absorbing if it satisfies in condition
(ii), see [5]. Therefore, Theorem 2.3 shows that N is a 2-absorbing submodule
of M if and only if N is a strongly 2-absorbing submodule of M.

Corollary 2.4. Let M be an R-module and N be a 2-absorbing submodule of
M. Then N : M I = {m ∈ M : Im ⊆ N} is a 2-absorbing submodules of M for
all ideal I of R. Furthermore N : M I^n = N : M I^{n+1}, for all n ≥ 2.

Proof. Let I be an ideal of R, a,b ∈ R, m ∈ M and abm ∈ N : M I. Thus
Iabm ⊆ N. Hence, Im ⊆ N or Iab ⊆ N : R M or abm ∈ N, by Lemma 2.2. If
Im ⊆ N we are done. If Iab ⊆ N : R M, then ab ∈ (N : R M) : R I = (N : M
I) : R M. If abm ∈ N, then am ∈ N or bm ∈ N or ab ∈ N : R M. Thus
Iam ⊆ N or Ibm ⊆ N or Iab ⊆ N : R M which complete the proof.

For the second statement, it is enough to show that N : M I^2 = N : M I^3. It
is clear that N : M I^2 ⊆ N : M I^3. Let m ∈ N : M I^3. Then I^3m ⊆ N. Now, by
Lemma 2.2, we have I^2m ⊆ N or Im ⊆ N or I^3 ⊆ N : R M. If I^2m ⊆ N or
Im ⊆ N, we are done. If I^3 ⊆ N : R M, then I^2 ⊆ N : R M since N : R M is a
2-absorbing ideal of R by [9, Theorem 2.3]. □

It is clear that, nZ is a 2-absorbing ideal of Z if and only if n = 0, p, p^2, pq,
where p, q are distinct prime integers. It is easy to see that 4Z : 2Z = 2Z
but 4Z : 2Z = 36Z = Z. Hence, the equality mentioned in the Corollary 2.4, is not
necessarily true for n = 1.
Theorem 2.5. Let \( N \) be a 2-absorbing submodule of \( M \) such that \( r(N : R M) = p \cap q \) where \( p \) and \( q \) are the only distinct prime ideals of \( R \) that are minimal over \( N : R M \). Then \( L = \{ m \in M : p \subseteq r(N : R m) \} \) is a 2-absorbing submodule of \( M \) containing \( N \). Also, either \( r(L : R M) = q \) or \( r(L : R M) = p' \cap q \), where \( p' \) is a prime ideal of \( R \) containing \( p \).

Proof. It is clear that \( L \) is a submodule of \( M \) containing \( N \). Assume that \( a, b \in R, m \in M \) and \( abm \in L \). We have to show that \( am \in L \) or \( bm \in L \) or \( ab \in L : R M \). Since \( p \subseteq r(N : R abm) \), thus \( p^2 abm \subseteq N \), by [9, Theorem 2.4] and [2, Theorem 2.4]. Therefore, by Lemma 2.1, we have \( abm \in N \) or \( p^2 m \subseteq N \) or \( p^2 ab \subseteq N : R M \). If \( abm \in N \), then \( am \in N \) or \( bm \in N \) or \( ab \in N : R M \) which implies that \( am \in L \) or \( bm \in L \) or \( ab \in L : R M \). If \( p^2 m \subseteq N \), then \( p^2 \subseteq N : R m \) and so \( p \subseteq r(N : R m) \) thus \( m \in L \) and we are done. If \( p^2 ab \subseteq N : R M \), then by [2, Theorem 2.13], we have \( p^2 a \subseteq N : R M \) or \( p^2 b \subseteq N : R M \) or \( ab \in N : R M \).

In the first case we conclude that \( p^2 \subseteq N : R am \) and so \( am \in L \). By a similar argument in the second case we can deduced that \( bm \in L \). If \( ab \in N : R M \), then \( ab \in L : R M \). Therefore, the result follows.

For the second statement, first we show that \( r(N : R M) = r(L : R M) \cap p \). It is clear \( r(N : R M) \subseteq r(L : R M) \cap p \). Assume that \( a \in (L : R M) \cap p \). Thus \( aM \subseteq L \) and so, by definition of \( L \), \( p \subseteq r(N : R am) \), for all \( m \in M \). Hence, [2, Theorem 2.4] shows that \( p^2 \subseteq N : R am \), for all \( m \in M \). Therefore, \( a^3 \in N : R m \), for all \( m \in M \). So that \( a^3 \in N : R M \) and then \( a \in r(N : R M) \). Thus \( r(L : R M) \cap p \subseteq r(N : R M) \). Now, \( L : R M \) is a 2-absorbing ideal of \( R \), therefore either \( r(L : R M) = p' \) or \( r(L : R M) = p' \cap q' \), for some prime ideals \( p', q' \) of \( R \). In the first case we have \( r(N : R M) = p \cap p' \) which implies that \( p' = q \) and in the second case we have \( r(N : R M) = p \cap p' \cap q' \) which implies that either \( p' = q \) or \( q' = q \).

\[ \square \]

Corollary 2.6. Let \( N \) be a 2-absorbing submodule of \( M \) such that \( r(N : R M) = p \cap q \) where \( p \) and \( q \) are the only distinct prime ideals of \( R \) that are minimal over \( N : R M \). If \( M/N \) is a Noetherian \( R \)-module, then

(i) there exists a chain \( N = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_{n-1} \subseteq L_n = M \) of 2-absorbing submodules of \( M \). Furthermore, \( \text{Ass}(M) \subseteq \text{Ass}(M/L_{n-1}) \cup \text{Ass}(L_{n-1}/L_{n-2}) \cup \cdots \cup \text{Ass}(L_1/N) \) where \( \text{Ass}(L_i/N) \) is the union of at most two totally ordered set, for all \( i \).

(ii) there exists a chain \( N \subseteq L_n \subseteq L_{n-1} \subseteq \cdots \subseteq L_1 \subseteq L_0 = M \) of submodules of \( M \) such that \( L_i \) is a 2-absorbing submodule of \( L_{i+1} \), for all \( 0 \leq i \leq n-1 \).

Proof. (i) Let \( L_1 = \{ m \in M : p \subseteq r(N : R m) \} \). Then by Corollary 2.4, \( L_1 \) is a 2-absorbing submodule of \( M \) and so either \( r(L_1 : R M) = q \) or \( r(L_1 : R M) = p_1 \cap q \), where \( p_1 \) is a prime ideal of \( R \) containing \( p \). If \( r(L_1 : R M) = q \), then choose \( L_2 = \{ m \in M : q \subseteq r(L_1 : R m) \} = M \). Hence, \( N \subseteq L_1 \subseteq L_2 = M \) is requested chain. If \( r(L_1 : R M) = p_1 \cap q \), set \( L_2 = \{ m \in M : p_1 \subseteq r(L_1 : R m) \} \).
and so either \( r(L_2 :_RM) = q \) or \( r(L_2 :_RM) = p_2 \cap q \), where \( p_2 \) is a prime ideal of \( R \) containing \( p_1 \). Proceeding in this way, we can achieve \( N \subseteq L_0 \subseteq L_1 \subseteq \cdots \subseteq L_{n-1} \subseteq L_n = M \) of 2-absorbing submodules of \( M \). The last statement is obvious, by [9, Theorem 2.6].

(ii) Let \( L_1 = \{ m \in M : p \subseteq r(N :_R m) \} \). Then \( N \) is a 2-absorbing submodule of \( L_1 \). So that either \( r(N :_RL_1) = p_1 \) or \( r(N :_RL_1) = p_1 \cap q_1 \), for some prime ideals \( p_1, q_1 \) of \( R \). If \( r(N :_RL_1) = p_1 \), then choose \( L_2 = \{ x \in L_1 : p_1 \subseteq r(N :_Rx) \} = N \). Hence, in this case \( N \subseteq L_1 \subseteq L_0 = M \) is the requested chain. If \( r(N :_RL_1) = p_1 \cap q_1 \), then set \( L_2 = \{ x \in L_1 : p_1 \subseteq r(N :_Rx) \} \) and continue the same way to achieve the chain \( N \subseteq L_n \subseteq L_{n-1} \subseteq \cdots \subseteq L_1 \subseteq L_0 = M \) of 2-absorbing submodules of \( M \).

**Theorem 2.7.** Let \( N \) be a 2-absorbing submodule of \( M \). Then \( N :_RM \) is a prime ideal of \( R \) if and only if \( N :_Rm \) is a prime ideal of \( R \) for all \( m \in M \setminus N \).

**Proof.** Assume that \( a, b \in R, m \in M \setminus N \) and \( ab \in N :_Rm \). Then \( abm \subseteq N \). We have \( am \in N \) or \( bm \in N \) or \( ab \in N :_R M \) since \( N \) is a 2-absorbing submodule of \( M \). If \( am \in N \) or \( bm \in N \) we are done. If \( ab \in N :_R M \), then by assumption either \( a \in N :_RM \) or \( b \in N :_RM \). Thus either \( a \in N :_RM \) or \( b \in N :_RM \). So \( N :_Rm \) is a prime ideal.

 Conversely, suppose that \( ab \in N :_R M \) for some \( a, b \in R \) and assume that there exist \( m, m' \in M \) such that \( am \notin N \) and \( bm' \notin N \). By \( abm, abm' \in N \) it follows that \( bm \in N \) and \( am' \in N \) since \( N :_R m \) and \( N :_R m' \) are prime ideals of \( R \). If \( m + m' \in N \), then \( am \in N \) which is a contradiction. Thus \( m + m' \notin N \). Now by \( ab(m' + m'') \in N \) we have \( a(m' + m'') \in N \) or \( b(m' + m'') \in N \) which is a contradiction. Thus \( aM \subseteq N \) or \( bM \subseteq N \) which implies that \( N :_R M \) is prime.

**Corollary 2.8.** Let \( N \) be a 2-absorbing submodule of \( M \). Then \( N :_R M \) is a prime ideal of \( R \) if and only if \( N :_RK \) is a prime ideal of \( R \) for all submodules \( K \) of \( M \) containing \( N \).

**Proof.** By Theorem 2.7 and [9, Theorem 2.6] it follows that \( \{ N :_R x : x \in K \setminus N \} \) is a totally ordered set of prime ideals of \( R \). Hence, \( N :_RK = \cap_{x \in K} N :_Rx \) is a prime ideal of \( R \).

**Theorem 2.9.** Let \( p \) be a prime ideal of \( R \) and \( E(R/p) \) be an injective envelope of \( R/p \). If \( 0 \) is a 2-absorbing submodule of \( E(R/p) \), then

(i) \( p^2 \subseteq 0 :_RE(R/p) \subseteq p \) so that \( r(0 :_RE(R/p)) = p \).

(ii) \( p^2 \subseteq 0 :_R x \subseteq 0 :_Rx \) for all non-zero element \( x \) of \( E(R/p) \) and all \( a \in p \setminus 0 :_Rx \).

(iii) \( p^2 \subseteq 0 :_Rx = 0 :_R a^n x \subseteq p \), for all \( a \notin p \).

**Proof.** (i) We have \( r(0 :_Rx) = p \) for all non-zero element \( x \) of \( E(R/p) \), by [8, Theorem 18.4]. Also it is obvious \( 0 :_RE(R/p) \subseteq 0 :_Rx \). Thus \( 0 :_RE(R/p) \subseteq p \).
Now, assume that $a \in p^2$ and $x$ is a non-zero element of $E(R/p)$. Since $0$ is a 2-absorbing submodule of $M$, $0:_R x$ is a 2-absorbing ideal of $R$, by [9, Theorem 2.4]. Therefore we have $p^2$ is a subset of $0:_R x$, by [2, Theorem 2.4]. Hence, $ax = 0$ and therefore $aE(R/p) = 0$ and $p^2 \subseteq 0:_R E(R/p)$.

(ii) Let $x$ be a non-zero element of $E(R/p)$. Then we have $p^2 \subseteq 0:_R x \subseteq p$. Assume that $a \in p \setminus 0 :_R x$. Thus $ax \neq 0$ but $a^2x = 0$ which shows that $0 :_R x \subsetneq 0 :_R ax$. If $b \in p$, then $ab \in p^2$ and $abx = 0$. Thus $b \in 0 :_R ax$ and so $p \subseteq 0 :_R ax$.

(iii) Assume that $a \notin p$. It is obvious that $0 :_R x \subsetneq 0 :_R a^nx$, for all $n \in \mathbb{N}$. Let $b \in \text{Ann}_R(a^n x)$. Thus $ba^n x = 0$. But multiplication by $a^n$ is an automorphism on $E(R/p)$, so that $bx = 0$ and $b \in 0 :_R x$. Therefore, $0 :_R x = 0 :_R a^nx$. □

**Corollary 2.10.** Let $R$ be a principal ideal domain and $p$ is a prime ideal of $R$. If $0$ is a 2-absorbing submodule of $E(R/p)$, then for all non-zero element $x$ of $E(R/p)$ either $0 :_R x = p^2$ or $0 :_R x = p$.

**Proof.** Let $p = (a)$. Then $p^2 = (a^2)$. Let $x$ be a non-zero element of $E(R/p)$. Then $p^2 \subseteq 0 :_R x = (b) \subseteq p$ by Theorem 2.9(ii). Thus $a^2 = bc$ and $b = ae$ for some $c, e \in R$. Hence, $a^2 = ace$. So $a = ec \in p$. Therefore, either $c \in p$ or $e \in p$. If $c \in p$, then $c = ae'$ and so $a = eac'$ which implies that $1 = ec'$ and $a = bc' \therefore 0 :_R x = p$. If $e \in p$, then $e = ae'$ and so $a = ae'c$ which implies that $1 = e'c$ and $b = a^2c'$ thus $0 :_R x = p^2$. □

The following example shows that the condition “0 is a 2-absorbing submodule of $E(R/p)$” is essential. It is well-known that $E(\mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}(p^{\infty}) = \{m/n + \mathbb{Z} : m, n \in \mathbb{Z}, n \neq 0\}$, where $p$ is a prime integer. But neither $p^2\mathbb{Z} = 0 :\mathbb{Z} 1/p^3 + \mathbb{Z}$ nor $0 :\mathbb{Z} 1/p^3 + \mathbb{Z} = p\mathbb{Z}$. Hence, 0 is not a 2-absorbing submodule of $E(\mathbb{Z}/p\mathbb{Z})$. Also, this example shows that if 0 is a 2-absorbing submodule of $M$, then it is not necessarily a 2-absorbing submodule of $E(M)$.

**Acknowledgments**

We would like to thank the referee for a careful reading of our article.

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