On the 2-absorbing Submodules

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Abstract. Let $R$ be a commutative ring and $M$ be an $R$-module. In this paper, we investigate some properties of 2-absorbing submodules of $M$. It is shown that $N$ is a 2-absorbing submodule of $M$ if and only if whenever $IJL \subseteq N$ for some ideals $I, J$ of $R$ and a submodule $L$ of $M$, then $IL \subseteq N$ or $JL \subseteq N$ or $IJ \subseteq N$.

Let $N$ be a 2-absorbing submodule of $M$, and $M/N$ is Noetherian, then a chain of 2-absorbing submodules of $M$ is constructed. Furthermore, the annihilation of $E(R/p)$ is studied whenever 0 is a 2-absorbing submodule of $E(R/p)$, where $p$ is a prime ideal of $R$ and $E(R/p)$ is an injective envelope of $R/p$.

Keywords: 2-absorbing ideal, 2-absorbing submodule, A chain of 2-absorbing submodule.


1. Introduction

Throughout this paper $R$ is a commutative ring with non-zero identity and $M$ is an unitary $R$-module. We defined a submodule $N$ of $M$ is 2-absorbing whenever $abm \in N$ for some $a, b \in R$, $m \in M$, then $am \in N$ or $bm \in N$ or $ab \in N :_RM$, see for instance [1, 3, 4, 6, 7, 9, 10]. It is well known that, a submodule $N$ of $M$ is prime if and only if $IL \subseteq N$ for an ideal $I$ of $R$ and

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a submodule \( L \) of \( M \), then either \( L \subseteq N \) or \( I \subseteq N :_RM \). This statement persuaded us to prove that, a submodule \( N \) of \( M \) is 2-absorbing if and only if \( IJL \subseteq N \) for some ideals \( I, J \) of \( R \) and a submodule \( L \) of \( M \), then \( IL \subseteq N \) or \( JL \subseteq N \) or \( IJ \subseteq N :_RM \). As a corollary of this theorem, it is shown that \( L = \{m \in M : p \subseteq r(N : m)\} \) is a 2-absorbing submodule of \( M \), where \( N \) is a 2-absorbing submodule of \( M \) with \( r(N :_RM) = p \cap q \) for some prime ideals \( p, q \) of \( R \). Also, it is shown that if \( M/N \) is Noetherian, then there exists a chain of 2-absorbing submodules of \( M \) that begins with \( N \). Assume that \( E(R/p) \) is an injective envelope of \( R/p \), it is shown that if \( 0 \) is a 2-absorbing submodule of \( E(R/p) \), then \( r(0 :_R E(R/p)) = p \) and \( 0 :_R x \) is determined for all nonzero element \( x \) of \( E(R/p) \).

Now, we define the concepts that we will use later. For a submodule \( L \) of \( M \) let \( L :_RM \) denote the ideal \( \{r \in R : rM \subseteq L\} \). Similarly, for an element \( m \in M \) let \( L :_R m \) denote the ideal \( \{r \in R : rm \in L\} \). If \( I \) is an ideal of \( R \), then \( r(I) \) denotes the radical of \( I \). We say that \( p \in \text{Spec}(R) \) is an associated prime ideal of \( M \) if there exists \( m \in M \) with \( 0 :_R m = p \). The set of associated prime ideals of \( M \) is denoted by \( \text{Ass}_R(M) \), the set of integers is denoted by \( \mathbb{Z} \).

2. 2-absorbing Submodules

Let \( N \) be a proper submodule of \( M \). We say that \( N \) is a \textit{2-absorbing} submodule of \( M \) if whenever \( a, b \in R, m \in M \) and \( abm \in N \), then \( am \in N \) or \( bm \in N \) or \( ab \in N :_R M \).

**Lemma 2.1.** Let \( I \) be an ideal of \( R \) and \( N \) be a 2-absorbing submodule of \( M \). If \( a \in R, m \in M \) and \( Iam \subseteq N \), then \( am \in N \) or \( Im \subseteq N \) or \( Ia \subseteq N :_R M \).

**Proof.** Let \( am \notin N \) and \( Ia \notin N :_R M \). Then there exists \( b \in I \) such that \( ba \notin N :_R M \). Now, \( bam \in N \) implies that \( bm \in N \), since \( N \) is a 2-absorbing submodule of \( M \). We have to show that \( Im \subseteq N \). Let \( c \) be an arbitrary element of \( I \). Thus \( (b+c)am \in N \). Hence, either \( (b+c)m \in N \) or \( (b+c)a \in N :_R M \). If \( (b+c)m \in N \), then by \( bm \in N \) it follows that \( cm \in N \). If \( (b+c)a \in N :_R M \), then \( ca \notin N :_R M \), but \( cam \in N \). Thus \( cm \in N \). Hence, we conclude that \( Im \subseteq N \). \( \square \)

**Lemma 2.2.** Let \( I, J \) be ideals of \( R \) and \( N \) be a 2-absorbing submodule of \( M \). If \( m \in M \) and \( IJm \subseteq N \), then \( Im \subseteq N \) or \( Jm \subseteq N \) or \( IJ \subseteq N :_R M \).

**Proof.** Let \( I \not\subseteq N :_R m \) and \( J \not\subseteq N :_R m \). We have to show that \( IJ \not\subseteq N :_R M \). Assume that \( c \in I \) and \( d \in J \). By assumption there exists \( a \in I \) such that \( am \notin N \) but \( aJm \subseteq N \). Now, Lemma 2.1 shows that \( aJ \subseteq N :_R M \) and so \( (I \setminus N :_R m)J \subseteq N :_R M \), similarly there exists \( b \in J \setminus N :_R m \) such that \( Ib \subseteq N :_R M \) and also \( I(J \setminus N :_R m) \subseteq N :_R M \). Thus we have \( ab \in N :_R M \), \( ad \in N :_R M \) and \( cb \in N :_R M \). By \( a+c \in I \) and \( b+d \in J \) it follows that \( (a+c)(b+d)m \in N \). Therefore, \( (a+c)m \in N \) or \( (b+d)m \in N \) or
(a + c)(b + d) ∈ N :R M. If (a + c)m ∈ N, then cm ̸∈ N hence, c ∈ I \ N :R m
which implies that cd ∈ N :R M. Similarly by (b + d)m ∈ N, we can deduce
that cd ∈ N :R M. If (a + c)(b + d) ∈ N :R M, then ab + ad + cb + cd ∈ N :R M
and so cd ∈ N :R M. Therefore, IJ ⊆ N :R M. □

Theorem 2.3. Let N be a proper submodule of M. The following statement
are equivalent:

(i) N is a 2-absorbing submodule of M;
(ii) If I J L ⊆ N for some ideals I, J of R and a submodule L of M, then
IL ⊆ N or JL ⊆ N or IJ ⊆ N :R M.

Proof. (ii) ⇒ (i) is obvious. To prove (i) ⇒ (ii), assume that I J L ⊆ N for
some ideals I, J of R and a submodule L of M and I J ̸⊆ N :R M. Then
by Lemma 2.2 for all m ∈ L either Im ⊆ N or Jm ⊆ N. If Im ⊆ N, for
all m ∈ L we are done. Similarly if Jm ⊆ N, for all m ∈ L we are done.
Suppose that m, m′ ∈ L are such that Im ̸⊆ N and Jm′ ̸⊆ N. Thus Jm ⊆ N
and Im′ ⊆ N. Since IJ(m + m′) ⊆ N we have either I(m + m′) ⊆ N or
J(m + m′) ⊆ N. By I(m + m′) ⊆ N, it follows that Im ⊆ N which is a
contradiction, similarly by J(m + m′) ⊆ N we get a contradiction. Therefore
either IL ⊆ N or JL ⊆ N.

A submodule N of M is called strongly 2-absorbing if it satisfies in condition
(ii), see [5]. Therefore, Theorem 2.3 shows that N is a 2-absorbing submodule
of M if and only if N is a strongly 2-absorbing submodule of M.

Corollary 2.4. Let M be an R-module and N be a 2-absorbing submodule of
M. Then N :M I = {m ∈ M : Im ⊆ N} is a 2-absorbing submodules of M for
all ideal I of R. Furthermore N :M I^n = N :M I^{n+1}, for all n ≥ 2.

Proof. Let I be an ideal of R, a, b ∈ R, m ∈ M and abm ∈ N :M I. Thus
Iabm ⊆ N. Hence, Im ⊆ N or Iab ⊆ N :R M or abm ∈ N, by Lemma 2.2. If
Im ⊆ N we are done. If Iab ⊆ N :R M, then ab ∈ (N :R M) :R I = (N :M
I) :R M. If abm ∈ N, then am ∈ N or bm ∈ N or ab ∈ N :R M. Thus
Iam ⊆ N or Ibm ⊆ N or Iab ⊆ N :R M which complete the proof.

For the second statement, it is enough to show that N :M I^2 = N :M I^3. It
is clear that N :M I^2 ⊆ N :M I^3. Let m ∈ N :M I^3. Then I^3m ⊆ N. Now, by
Lemma 2.2, we have I^2m ⊆ N or Im ⊆ N or I^3 ⊆ N :R M. If I^2m ⊆ N or
Im ⊆ N, we are done. If I^3 ⊆ N :R M, then I^2 ⊆ N :R M since N :R M is a
2-absorbing ideal of R by [9, Theorem 2.3]. □

It is clear that, nZ is a 2-absorbing ideal of Z if and only if n = 0, p, p^2, pq,
where p, q are distinct prime integers. It is easy to see that 4Z :Z 6Z = 2Z
but 4Z :Z 36Z = Z. Hence, the equality mentioned in the Corollary 2.4, is not
necessarily true for n = 1.
Theorem 2.5. Let \( N \) be a 2-absorbing submodule of \( M \) such that \( r(N :_R M) = p \cap q \) where \( p \) and \( q \) are the only distinct prime ideals of \( R \) that are minimal over \( N :_R M \). Then \( L = \{ m \in M : p \subseteq r(N :_R m) \} \) is a 2-absorbing submodule of \( M \) containing \( N \). Also, either \( r(L :_R M) = q \) or \( r(L :_R M) = p' \cap q \), where \( p' \) is a prime ideal of \( R \) containing \( p \).

Proof. It is clear that \( L \) is a submodule of \( M \) containing \( N \). Assume that \( a, b \in R \), \( m \in M \) and \( abm \in L \). We have to show that \( am \in L \) or \( bm \in L \) or \( ab \in L :_R M \). Since \( p \subseteq r(N :_R abm) \), thus \( p^2abm \subseteq N \), by [9, Theorem 2.4] and [2, Theorem 2.4]. Therefore, by Lemma 2.1, we have \( abm \subseteq N \) or \( p^2m \subseteq N \) or \( p^2ab \subseteq N :_R M \). If \( abm \in N \), then \( am \in L \) or \( bm \in L \) or \( ab \in L :_R M \). Thus \( p \subseteq r(N :_R m) \) which implies that \( a \) \( m \in L \) or \( bm \in L \) or \( ab \in L :_R M \). If \( p^2m \subseteq N \), then \( p^2 \subseteq N :_R m \) and so \( p \subseteq r(N :_R m) \) thus \( m \in L \) and we are done. If \( p^2ab \subseteq N :_R M \), then by [2, Theorem 2.13], we have \( p^2a \subseteq N :_R M \) or \( p^2b \subseteq N :_R M \) or \( ab \subseteq N :_R M \). In the first case we conclude that \( p^2 \subseteq N :_R am \) and so \( am \in L \). By a similar argument in the second case we can deduced that \( bm \in L \). If \( ab \in N :_R M \), then \( ab \subseteq L :_R M \). Therefore, the result follows.

For the second statement, first we show that \( r(N :_R M) = r(L :_R M \cap p) \). It is clear \( r(N :_R M) \subseteq r(L :_R M \cap p) \). Assume that \( a \in (L :_R M) \cap p \). Thus \( aM \subseteq L \) and so, by definition of \( L \), \( p \subseteq r(N :_R am) \), for all \( m \in M \). Hence, [2, Theorem 2.4] shows that \( p^2 \subseteq N :_R am \), for all \( m \in M \). Therefore, \( a^3 \in N :_R m \), for all \( m \in M \). So that \( a^3 \in N :_R M \) and then \( a \in r(N :_R M) \). Thus \( r(L :_R M) \cap p \subseteq r(N :_R M) \). Now, \( L :_R M \) is a 2-absorbing ideal of \( R \), therefore either \( r(L :_R M) = p' \) or \( r(L :_R M) = p' \cap q' \), for some prime ideals \( p', q' \) of \( R \). In the first case we have \( r(N :_R M) = p \cap p' \) which implies that \( p' = q \) and in the second case we have \( r(N :_R M) = p \cap p' \cap q' \) which implies that either \( p' = q \) or \( q' = q \).

\( \square \)

Corollary 2.6. Let \( N \) be a 2-absorbing submodule of \( M \) such that \( r(N :_R M) = p \cap q \) where \( p \) and \( q \) are the only distinct prime ideals of \( R \) that are minimal over \( N :_R M \). If \( M/N \) is a Noetherian \( R \)-module, then

(i) there exists a chain \( N = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_{n-1} \subseteq L_n = M \) of 2-absorbing submodules of \( M \). Furthermore, \( \text{Ass}(M) \subseteq \text{Ass}(M/L_{n-1}) \cup \text{Ass}(L_{n-1}/L_{n-2}) \cup \text{Ass}(L_{n-2}/L_{n-3}) \cup \cdots \cup \text{Ass}(L_1/N) \), where \( \text{Ass}(L_i/N) \) is the union of at most two totally ordered set, for all \( i \).

(ii) there exists a chain \( N \subseteq L_n \subseteq L_{n-1} \subseteq \cdots \subseteq L_1 \subseteq L_0 = M \) of submodules of \( M \) such that \( L_i \) is a 2-absorbing submodule of \( L_{i+1} \), for all \( 0 \leq i \leq n - 1 \).

Proof. (i) Let \( L_1 = \{ m \in M : p \subseteq r(N :_R m) \} \). Then by Corollary 2.4, \( L_1 \) is a 2-absorbing submodule of \( M \) and so either \( r(L_1 :_R M) = q \) or \( r(L_1 :_R M) = p_1 \cap q \), where \( p_1 \) is a prime ideal of \( R \) containing \( p \). If \( r(L_1 :_R M) = q \), then choose \( L_2 = \{ m \in M : q \subseteq r(L_1 :_R m) \} = M \). Hence, \( N \subseteq L_1 \subseteq L_2 = M \) is requested chain. If \( r(L_1 :_R M) = p_1 \cap q \), set \( L_2 = \{ m \in M : p_1 \subseteq r(L_1 :_R m) \} \).
and so either \(r(L_2 :_RM) = q\) or \(r(L_2 :_RM) = p_2 \cap q\), where \(p_2\) is a prime ideal of \(R\) containing \(p_1\). Proceeding in this way, we can achieve \(N \subseteq L_0 \subseteq L_1 \subseteq \cdots \subseteq L_{n-1} \subseteq L_n = M\) of 2-absorbing submodules of \(M\). The last statement is obvious, by [9, Theorem 2.6].

(ii) Let \(L_1 = \{m \in M : p \subseteq r(N :_R m)\}\). Then \(N\) is a 2-absorbing submodule of \(L_1\). So that either \(r(N :_RL_1) = p_1\) or \(r(N :_RL_1) = p_1 \cap q_1\), for some prime ideals \(p_1, q_1\) of \(R\). If \(r(N :_RL_1) = p_1\), then choose \(L_2 = \{x \in L_1 : p_1 \subseteq r(N :_Rx)\}\). Hence, in this case \(N \subseteq L_1 \subseteq L_0 = M\) is the requested chain. If \(r(N :_RL_1) = p_1 \cap q_1\), then set \(L_2 = \{x \in L_1 : p_1 \subseteq r(N :_Rx)\}\) and continue the same way to achieve the chain \(N \subseteq L_n \subseteq L_{n-1} \subseteq \cdots \subseteq L_1 \subseteq L_0 = M\) of 2-absorbing submodules of \(M\).

\[\square\]

**Theorem 2.7.** Let \(N\) be a 2-absorbing submodule of \(M\). Then \(N :_RM\) is a prime ideal of \(R\) if and only if \(N :_R m\) is a prime ideal of \(R\) for all \(m \in M \setminus N\).

**Proof.** Assume that \(a, b \in R, m \in M \setminus N\) and \(ab \in N :_R m\). Then \(abm \subseteq N\). We have \(am \in N\) or \(bm \in N\) or \(ab \in N :_R M\) since \(N\) is a 2-absorbing submodule of \(M\). If \(am \in N\) or \(bm \in N\) we are done. If \(ab \in N :_R M\), then by assumption either \(a \in N :_R M\) or \(b \in N :_R M\). Thus either \(a \in N :_R m\) or \(b \in N :_R m\). So \(N :_R m\) is a prime ideal.

Conversely, suppose that \(ab \in N :_R M\) for some \(a, b \in R\) and assume that there exist \(m, m' \in M\) such that \(am \notin N\) and \(bm' \notin N\). By \(abm, abm' \in N\) it follows that \(bm \in N\) and \(am' \in N\) since \(N :_R m\) and \(N :_R m'\) are prime ideals of \(R\). If \(m + m' \in N\), then \(am \in N\) which is a contradiction. Thus \(m + m' \notin N\). Now by \(ab(m' + m'') \in N\) we have \(a(m' + m'') \in N\) or \(b(m' + m'') \in N\) which is a contradiction. Thus \(aM \subseteq N\) or \(bM \subseteq N\) which implies that \(N :_R M\) is prime.

\[\square\]

**Corollary 2.8.** Let \(N\) be a 2-absorbing submodule of \(M\). Then \(N :_R M\) is a prime ideal of \(R\) if and only if \(N :_R K\) is a prime ideal of \(R\) for all submodules \(K\) of \(M\) containing \(N\).

**Proof.** By Theorem 2.7 and [9, Theorem 2.6] it follows that \(\{N :_Rx : x \in K \setminus N\}\) is a totally ordered set of prime ideals of \(R\). Hence, \(N :_RK = \cap_{x \in K} N :_Rx\) is a prime ideal of \(R\).

\[\square\]

**Theorem 2.9.** Let \(p\) be a prime ideal of \(R\) and \(E(R/p)\) be an injective envelop of \(R/p\). If \(0\) is a 2-absorbing submodule of \(E(R/p)\), then

(i) \(p^2 \subseteq 0 :_R E(R/p) \subseteq p\) so that \(r(0 :_R E(R/p)) = p\).

(ii) \(p^2 \subseteq 0 :_R x \subseteq 0 :_R ax\), for all non-zero element \(x\) of \(E(R/p)\) and all \(a \in p \setminus 0 :_R x\).

(iii) \(p^2 \subseteq 0 :_R x = 0 :_R ax \subseteq p\), for all \(a \notin p\).

**Proof.** (i) We have \(r(0 :_Rx) = p\) for all non-zero element \(x\) of \(E(R/p)\), by [8, Theorem 18.4]. Also it is obvious \(0 :_R E(R/p) \subseteq 0 :_R x\). Thus \(0 :_R E(R/p) \subseteq p\).
Now, assume that $a \in p^2$ and $x$ is a non-zero element of $E(R/p)$. Since 0 is a 2-absorbing submodule of $M$, 0 :$_R x$ is a 2-absorbing ideal of $R$, by [9, Theorem 2.4]. Therefore we have $p^2$ is a subset of 0 :$_R x$, by [2, Theorem 2.4]. Hence, $ax = 0$ and therefore $aE(R/p) = 0$ and $p^2 \subseteq 0 :_R E(R/p)$.

(ii) Let $x$ be a non-zero element of $E(R/p)$. Then we have $p^2 \subseteq 0 :_R x \subseteq p$. Assume that $a \in p \setminus 0 :_R x$. Thus $ax \neq 0$ but $a^2x = 0$ which shows that 0 :$_R x \subset 0 :_R ax$. If $b \in p$, then $ab \in p^2$ and $abx = 0$. Thus $b \in 0 :_R ax$ and so $p \subseteq 0 :_R ax$.

(iii) Assume that $a \notin p$. It is obvious that 0 :$_R x \subseteq 0 :_R a^n x$, for all $n \in \mathbb{N}$. Let $b \in \text{Ann}_R(a^n x)$. Thus $ba^n x = 0$. But multiplication by $a^n$ is an automorphism on $E(R/p)$, so that $bx = 0$ and $b \in 0 :_R x$. Therefore, 0 :$_R x = 0 :_R a^n x$.

\[\square\]

**Corollary 2.10.** Let $R$ be a principal ideal domain and $p$ is a prime ideal of $R$. If 0 is a 2-absorbing submodule of $E(R/p)$, then for all non-zero element $x$ of $E(R/p)$ either 0 :$_R x = p^2$ or 0 :$_R x = p$.

**Proof.** Let $p = (a)$. Then $p^2 = (a^2)$. Let $x$ be a non-zero element of $E(R/p)$. Then $p^2 \subseteq 0 :_R x = (b) \subseteq p$ by Theorem 2.9(ii). Thus $a^2 = bc$ and $b = ae$ for some $c, e \in R$. Hence, $a^2 = ace$. So $a = ec \in p$. Therefore, either $c \in p$ or $e \in p$. If $c \in p$, then $c = ae$ and so $a = eac$ which implies that $1 = ee'$ and $a = be'$ thus 0 :$_R x = p$. If $e \in p$, then $e = ae'$ and so $a = ae'c$ which implies that $1 = e'c$ and $b = a^2e'$ thus 0 :$_R x = p^2$.

\[\square\]

The following example shows that the condition “0 is a 2-absorbing submodule of $E(R/p)$” is essential. It is well-known that $E(\mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}(p^\infty) = \{m/n + \mathbb{Z} : m, n \in \mathbb{Z}, n \neq 0\}$, where $p$ is a prime integer. But neither $p^2 \mathbb{Z} = 0 : \mathbb{Z} 1/p^3 + \mathbb{Z}$ nor 0 :$_\mathbb{Z} 1/p^3 + \mathbb{Z} = p\mathbb{Z}$. Hence, 0 is not a 2-absorbing submodule of $E(\mathbb{Z}/p\mathbb{Z})$. Also, this example shows that if 0 is a 2-absorbing submodule of $M$, then it is not necessarily a 2-absorbing submodule of $E(M)$.

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