On the 2-absorbing Submodules

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Abstract. Let \(R\) be a commutative ring and \(M\) be an \(R\)-module. In this paper, we investigate some properties of 2-absorbing submodules of \(M\). It is shown that \(N\) is a 2-absorbing submodule of \(M\) if and only if whenever \(IJL \subseteq N\) for some ideals \(I, J\) of \(R\) and a submodule \(L\) of \(M\), then \(IL \subseteq N\) or \(JL \subseteq N\) or \(IJ \subseteq N\).

Keywords: 2-absorbing ideal, 2-absorbing submodule, A chain of 2-absorbing submodule.


1. Introduction

Throughout this paper \(R\) is a commutative ring with non-zero identity and \(M\) is an unitary \(R\)-module. We defined a submodule \(N\) of \(M\) is 2-absorbing whenever \(abm \in N\) for some \(a, b \in R, m \in M\), then \(am \in N\) or \(bm \in N\) or \(ab \in N :_RM\), see for instance [1, 3, 4, 6, 7, 9, 10]. It is well known that, a submodule \(N\) of \(M\) is prime if and only if \(IL \subseteq N\) for an ideal \(I\) of \(R\) and

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a submodule \( L \) of \( M \), then either \( L \subseteq N \) or \( I \subseteq N :_R M \). This statement preserved us to prove that, a submodule \( N \) of \( M \) is 2-absorbing if and only if \( IJL \subseteq N \) for some ideals \( I, J \) of \( R \) and a submodule \( L \) of \( M \), then \( IL \subseteq N \) or \( JL \subseteq N :_R M \). As a corollary of this theorem, it is shown that \( L = \{ m \in M : p \subseteq r(N : m) \} \) is a 2-absorbing submodule of \( M \), where \( N \) is a 2-absorbing submodule of \( M \) with \( r(N :_R M) = p \cap q \) for some prime ideals \( p, q \) of \( R \). Also, it is shown that if \( M/N \) is Noetherian, then there exists a chain of 2-absorbing submodules of \( M \) that begins with \( N \). Assume that \( E(R/p) \) is an injective envelope of \( R/p \), it is shown that if 0 is a 2-absorbing submodule of \( E(R/p) \), then \( r(0 :_R E(R/p)) = p \) and \( 0 :_R x \) is determined for all nonzero element \( x \) of \( E(R/p) \).

Now, we define the concepts that we will use later. For a submodule \( L \) of \( M \) let \( L :_R M \) denote the ideal \( \{ r \in R : rM \subseteq L \} \). Similarly, for an element \( m \in M \) let \( L :_R m \) denote the ideal \( \{ r \in R : rm \in L \} \). If \( I \) is an ideal of \( R \), then \( r(I) \) denotes the radical of \( I \). We say that \( p \in \text{Spec}(R) \) is an associated prime ideal of \( M \) if there exists \( m \in M \) with \( 0 :_R m = p \). The set of associated prime ideals of \( M \) is denoted by \( \text{Ass}_R(M) \), the set of integers is denoted by \( \mathbb{Z} \).

2. 2-absorbing Submodules

Let \( N \) be a proper submodule of \( M \). We say that \( N \) is a 2-absorbing submodule of \( M \) if whenever \( a, b \in R \), \( m \in M \) and \( abm \in N \), then \( am \in N \) or \( bm \in N \) or \( ab \in N :_R M \).

**Lemma 2.1.** Let \( I \) be an ideal of \( R \) and \( N \) be a 2-absorbing submodule of \( M \). If \( a \in R \), \( m \in M \) and \( Iam \subseteq N \), then \( am \in N \) or \( Im \subseteq N \) or \( Ia \subseteq N :_R M \).

**Proof.** Let \( am \notin N \) and \( Ia \nsubseteq N :_R M \). Then there exists \( b \in I \) such that \( ba \notin N :_R M \). Now, \( bam \in N \) implies that \( bm \in N \), since \( N \) is a 2-absorbing submodule of \( M \). We have to show that \( Im \subseteq N \). Let \( c \) be an arbitrary element of \( I \). Thus \( (b+c)am \in N \). Hence, either \((b+c)m \in N \) or \((b+c)a \in N :_R M \). If \((b+c)m \in N \), then by \( bm \in N \) it follows that \( cm \in N \). If \((b+c)a \in N :_R M \), then \( ca \notin N :_R M \), but \( cam \in N \). Thus \( cm \in N \). Hence, we conclude that \( Im \subseteq N \). \( \square \)

**Lemma 2.2.** Let \( I, J \) be ideals of \( R \) and \( N \) be a 2-absorbing submodule of \( M \). If \( m \in M \) and \( IJm \subseteq N \), then \( Im \subseteq N \) or \( Jm \subseteq N \) or \( IJ \subseteq N :_R M \).

**Proof.** Let \( I \nsubseteq N :_R m \) and \( J \nsubseteq N :_R m \). We have to show that \( IJ \subseteq N :_R M \). Assume that \( c \in I \) and \( d \in J \). By assumption there exists \( a \in I \) such that \( am \notin N \) but \( aJm \subseteq N \). Now, Lemma 2.1 shows that \( aJ \subseteq N :_R M \) and so \((I \setminus N :_R m)J \subseteq N :_R M \), similarly there exists \( b \in J \setminus N :_R m \) such that \( Ib \subseteq N :_R M \) and also \( I(J \setminus N :_R m) \subseteq N :_R M \). Thus we have \( ab \in N :_R M \), \( ad \in N :_R M \) and \( cb \in N :_R M \). By \( a + c \in I \) and \( b + d \in J \) it follows that \((a + c)(b + d)m \in N \). Therefore, \((a + c)m \in N \) or \((b + d)m \in N \) or
(a + c)(b + d) ∈ N : R M. If (a + c)m ∈ N, then cm ̸∈ N hence, c ∈ I \ N : R m
which implies that cd ∈ N : R M. Similarly by (b + d)m ∈ N, we can deduce
that cd ∈ N : R M. If (a + c)(b + d) ∈ N : R M, then ab + ad + cb + cd ∈ N : R M
and so cd ∈ N : R M. Therefore, IJ ⊆ N : R M. □

Theorem 2.3. Let N be a proper submodule of M. The following statement
are equivalent:

(i) N is a 2-absorbing submodule of M;
(ii) If IJL ⊆ N for some ideals I, J of R and a submodule L of M, then
IL ⊆ N or JL ⊆ N or IJ ⊆ N : R M.

Proof. (ii) ⇒ (i) is obvious. To prove (i) ⇒ (ii), assume that IJL ⊆ N for
some ideals I, J of R and a submodule L of M and IJ ̸⊆ N : R M. Then
by Lemma 2.2 for all m ∈ L either Im ⊆ N or Jm ⊆ N. If Im ⊆ N, for
all m ∈ L we are done. Similarly if Jm ⊆ N, for all m ∈ L we are done.
Suppose that m, m′ ∈ L are such that Im ̸⊆ N and Jm′ ̸⊆ N. Thus Jm ⊆ N
and Im′ ⊆ N. Since IJ(m + m′) ⊆ N we have either I(m + m′) ⊆ N or
J(m + m′) ⊆ N. By (m + m′) ⊆ N, it follows that Im ⊆ N which is a
contradiction, similarly by J(m + m′) ⊆ N we get a contradiction. Therefore
either IL ⊆ N or JL ⊆ N.

A submodule N of M is called strongly 2-absorbing if it satisfies in condition
(ii), see [5]. Therefore, Theorem 2.3 shows that N is a 2-absorbing submodule
of M if and only if N is a strongly 2-absorbing submodule of M.

Corollary 2.4. Let M be an R-module and N be a 2-absorbing submodule of
M. Then N : M I = {m ∈ M : Im ⊆ N} is a 2-absorbing submodules of M for
all ideal I of R. Furthermore N : M In = N : M I n+1, for all n ≥ 2.

Proof. Let I be an ideal of R, a, b ∈ R, m ∈ M and abm ∈ N : M I. Thus
Iabm ⊆ N. Hence, Im ⊆ N or Iab ⊆ N : R M or abm ∈ N, by Lemma 2.2. If
Im ⊆ N we are done. If Iab ⊆ N : R M, then ab ∈ (N : R M) : R I = (N : M
I) : R M. If abm ∈ N, then am ∈ N or bm ∈ N or ab ∈ N : R M. Thus
Iam ⊆ N or Ibm ⊆ N or Iab ⊆ N : R M which complete the proof.

For the second statement, it is enough to show that N : M I 2 = N : M I 3. It
is clear that N : M I 2 ⊆ N : M I 3. Let m ∈ N : M I 3. Then I 3m ⊆ N. Now, by
Lemma 2.2, we have I 2m ⊆ N or Im ⊆ N or I 3 ⊆ N : R M. If I 2m ⊆ N or
Im ⊆ N, we are done. If I 3 ⊆ N : R M, then I 2 ⊆ N : R M since N : R M is a
2-absorbing ideal of R by [9, Theorem 2.3]. □

It is clear that, nZ is a 2-absorbing ideal of Z if and only if n = 0, p, p 2, pq,
where p, q are distinct prime integers. It is easy to see that 4Z : Z 6Z = 2Z
but 4Z : Z 36Z = Z. Hence, the equality mentioned in the Corollary 2.4, is not
necessarily true for n = 1.
Theorem 2.5. Let $N$ be a 2-absorbing submodule of $M$ such that $r(N :_RM) = p \cap q$ where $p$ and $q$ are the only distinct prime ideals of $R$ that are minimal over $N :_RM$. Then $L = \{ m \in M : p \subseteq r(N :_RM) \}$ is a 2-absorbing submodule of $M$ containing $N$. Also, either $r(L :_RM) = q$ or $r(L :_RM) = p' \cap q$, where $p'$ is a prime ideal of $R$ containing $p$.

Proof. It is clear that $L$ is a submodule of $M$ containing $N$. Assume that $a, b \in R$, $m \in M$ and $abm \in L$. We have to show that $am \in L$ or $bm \in L$ or $ab \in L :_RM$. Since $p \subseteq r(N :_Rabm)$, thus $p^2abm \subseteq N$, by [9, Theorem 2.4] and [2, Theorem 2.4]. Therefore, by Lemma 2.1, we have $abm \in N$ or $p^2m \subseteq N$ or $p^2ab \subseteq N :_RM$. If $abm \in N$, then $am \in N$ or $bm \in N$ or $ab \in N :_RM$ which implies that $am \in L$ or $bm \in L$ or $ab \in L :_RM$. If $p^2m \subseteq N$, then $p^2 \subseteq N :_RM$ and so $p \subseteq r(N :_RM)$ thus $m \in L$ and we are done. If $p^2ab \subseteq N :_RM$, then by [2, Theorem 2.13], we have $p^2a \subseteq N :_RM$ or $p^2b \subseteq N :_RM$ or $ab \in N :_RM$.

In the first case we conclude that $p^2 \subseteq N :_R am$ and so $am \in L$. By a similar argument in the second case we can deduced that $bm \in L$. If $ab \in N :_RM$, then $ab \in L :_R M$. Therefore, the result follows.

For the second statement, first we show that $r(N :_RM) = r(L :_RM) \cap p$. It is clear $r(N :_RM) \subseteq r(L :_RM) \cap p$. Assume that $a \in (L :_RM) \cap p$. Thus $aM \subseteq L$ and so, by definition of $L$, $p \subseteq r(N :_R am)$, for all $m \in M$. Hence, [2, Theorem 2.4] shows that $p^2 \subseteq N :_R am$, for all $m \in M$. Therefore, $a^3 \in N :_RM$, for all $m \in M$. So that $a^3 \in N :_RM$ and then $a \in r(N :_RM)$. Thus $r(L :_RM) \cap p \subseteq r(N :_RM)$. Now, $L :_RM$ is a 2-absorbing ideal of $R$, therefore either $r(L :_RM) = p'$ or $r(L :_RM) = p' \cap q'$, for some prime ideals $p', q'$ of $R$. In the first case we have $r(N :_RM) = p \cap p'$ which implies that $p' = q$ and in the second case we have $r(N :_RM) = p \cap p' \cap q'$ which implies that either $p' = q$ or $q' = q$.

\[ \square \]

Corollary 2.6. Let $N$ be a 2-absorbing submodule of $M$ such that $r(N :_RM) = p \cap q$ where $p$ and $q$ are the only distinct prime ideals of $R$ that are minimal over $N :_RM$. If $M/N$ is a Noetherian $R$-module, then

(i) there exists a chain $N = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_{n-1} \subseteq L_n = M$ of 2-absorbing submodules of $M$. Furthermore, $\text{Ass}(M) \subseteq \text{Ass}(M/L_{n-1}) \cup \text{Ass}(L_{n-1}/L_{n-2}) \cup \text{Ass}(L_{n-2}/L_{n-3}) \cup \cdots \cup \text{Ass}(L_1/N)$, where $\text{Ass}(L_i/N)$ is the union of at most two totally ordered set, for all $i$.

(ii) there exists a chain $N \subseteq L_n \subseteq L_{n-1} \subseteq \cdots \subseteq L_1 \subseteq L_0 = M$ of submodules of $M$ such that $L_i$ is a 2-absorbing submodule of $L_{i+1}$, for all $0 \leq i \leq n-1$.

Proof. (i) Let $L_1 = \{ m \in M : p \subseteq r(N :_RM) \}$. Then by Corollary 2.4, $L_1$ is a 2-absorbing submodule of $M$ and so either $r(L_1 :_RM) = q$ or $r(L_1 :_RM) = p_1 \cap q$, where $p_1$ is a prime ideal of $R$ containing $p$. If $r(L_1 :_RM) = q$, then choose $L_2 = \{ m \in M : q \subseteq r(L_1 :_RM) \} = M$. Hence, $N \subseteq L_1 \subseteq L_2 = M$ is requested chain. If $r(L_1 :_RM) = p_1 \cap q$, set $L_2 = \{ m \in M : p_1 \subseteq r(L_1 :_RM) \}$
and so either \( r(L_2 :_R M) = q \) or \( r(L_2 :_R M) = p_2 \cap q \), where \( p_2 \) is a prime ideal of \( R \) containing \( p_1 \). Proceeding in this way, we can achieve \( N \subseteq L_0 \subseteq L_1 \subseteq \cdots \subseteq L_{n-1} \subseteq L_n = M \) of 2-absorbing submodules of \( M \). The last statement is obvious, by [9, Theorem 2.6].

(ii) Let \( L_1 = \{ m \in M : p \subseteq r(N :_R M) \} \). Then \( N \) is a 2-absorbing submodule of \( L_1 \). So that either \( r(N :_R L_1) = p_1 \) or \( r(N :_R L_1) = p_1 \cap q_1 \), for some prime ideals \( p_1, q_1 \) of \( R \). If \( r(N :_R L_1) = p_1 \), then choose \( L_2 = \{ x \in L_1 : p_1 \subseteq r(N :_R x) \} = N \). Hence, in this case \( N \subseteq L_1 \subseteq L_0 = M \) is the requested chain. If \( r(N :_R L_1) = p_1 \cap q_1 \), then set \( L_2 = \{ x \in L_1 : p_1 \subseteq r(N :_R x) \} \) and continue the same way to achieve the chain \( N \subseteq L_n \subseteq L_{n-1} \subseteq \cdots \subseteq L_1 \subseteq L_0 = M \) of 2-absorbing submodules of \( M \).

\[ \square \]

**Theorem 2.7.** Let \( N \) be a 2-absorbing submodule of \( M \). Then \( N :_R M \) is a prime ideal of \( R \) if and only if \( N :_R m \) is a prime ideal of \( R \) for all \( m \in M \setminus N \).

**Proof.** Assume that \( a, b \in R, m \in M \setminus N \) and \( ab \in N :_R M \). Then \( abm \subseteq N \). We have \( am \in N \) or \( bm \in N \) or \( ab \in N :_R M \) since \( N \) is a 2-absorbing submodule of \( M \). If \( am \in N \) or \( bm \in N \) we are done. If \( ab \in N :_R M \), then by assumption either \( a \in N :_R M \) or \( b \in N :_R M \). Thus either \( a \in N :_R m \) or \( b \in N :_R m \). So \( N :_R m \) is a prime ideal.

Conversely, suppose that \( ab \in N :_R M \) for some \( a, b \in R \) and assume that there exist \( m, m' \in M \) such that \( am \notin N \) and \( bm' \notin N \). By \( abm, abm' \in N \) it follows that \( bm \in N \) and \( am' \in N \) since \( N :_R M \) and \( N :_R m' \) are prime ideals of \( R \). If \( m + m' \in N \), then \( am \in N \) which is a contradiction. Thus \( m + m' \notin N \). Now by \( ab(m' + m'') \in N \) we have \( a(m' + m'') \in N \) or \( b(m' + m'') \in N \) which is a contradiction. Thus \( aM \subseteq N \) or \( bM \subseteq N \) which implies that \( N :_R M \) is prime.

\[ \square \]

**Corollary 2.8.** Let \( N \) be a 2-absorbing submodule of \( M \). Then \( N :_R M \) is a prime ideal of \( R \) if and only if \( N :_R K \) is a prime ideal of \( R \) for all submodules \( K \) of \( M \) containing \( N \).

**Proof.** By Theorem 2.7 and [9, Theorem 2.6] it follows that \( \{ N :_R x : x \in K \setminus N \} \) is a totally ordered set of prime ideals of \( R \). Hence, \( N :_R K = \cap_{x \in K \setminus N} N :_R x \) is a prime ideal of \( R \).

\[ \square \]

**Theorem 2.9.** Let \( p \) be a prime ideal of \( R \) and \( E(R/p) \) be an injective envelop of \( R/p \). If \( 0 \) is a 2-absorbing submodule of \( E(R/p) \), then

i) \( p^2 \subseteq 0 :_R E(R/p) \subseteq p \) so that \( r(0 :_R E(R/p)) = p \).

ii) \( p^2 \subseteq 0 :_R x \subseteq 0 :_R ax = p \), for all non-zero element \( x \) of \( E(R/p) \) and all \( a \in p \setminus 0 :_R x \).

iii) \( p^2 \subseteq 0 :_R x = 0 :_R a^nx \subseteq p \), for all \( a \notin p \).

**Proof.** (i) We have \( r(0 :_R x) = p \) for all non-zero element \( x \) of \( E(R/p) \), by [8, Theorem 18.4]. Also it is obvious \( 0 :_R E(R/p) \subseteq 0 :_R x \). Thus \( 0 :_R E(R/p) \subseteq p \).
Now, assume that \( a \in p^2 \) and \( x \) is a non-zero element of \( E(R/p) \). Since \( 0 \) is a 2-absorbing submodule of \( M \), \( 0 :_R x \) is a 2-absorbing ideal of \( R \), by [9, Theorem 2.4]. Therefore we have \( p^2 \) is a subset of \( 0 :_R x \), by \[2,\text{Theorem 2.4}\]. Hence, \( ax = 0 \) and therefore \( aE(R/p) = 0 \) and \( p^2 \subseteq 0 :_R E(R/p) \).

(ii) Let \( x \) be a non-zero element of \( E(R/p) \). Then we have \( p^2 \subseteq 0 :_R x \subseteq p \).

Assume that \( a \in p \setminus 0 :_R x \). Thus \( ax \neq 0 \) but \( a^2x = 0 \) which shows that \( 0 :_R x \subset 0 :_R ax \). If \( b \in p \), then \( ab \in p^2 \) and \( abx = 0 \). Thus \( b \in 0 :_R ax \) and so \( p \subseteq 0 :_R ax \).

(iii) Assume that \( a \notin p \). It is obvious that \( 0 :_R x \subseteq 0 :_R a^n x \), for all \( n \in \mathbb{N} \). Let \( b \in \text{Ann}_R(a^n x) \). Thus \( ba^n x = 0 \). But multiplication by \( a^n \) is an automorphism on \( E(R/p) \), so that \( bx = 0 \) and \( b \in 0 :_R x \). Therefore, \( 0 :_R x = 0 :_R a^n x \).

\[ \square \]

**Corollary 2.10.** Let \( R \) be a principal ideal domain and \( p \) is a prime ideal of \( R \). If \( 0 \) is a 2-absorbing submodule of \( E(R/p) \), then for all non-zero element \( x \) of \( E(R/p) \) either \( 0 :_R x = p^2 \) or \( 0 :_R x = p \).

**Proof.** Let \( p = (a) \). Then \( p^2 = (a^2) \). Let \( x \) be a non-zero element of \( E(R/p) \). Then \( p^2 \subseteq 0 :_R x = (b) \subseteq p \) by Theorem 2.9(ii). Thus \( a^2 = bc \) and \( b = ae \) for some \( c, e \in R \). Hence, \( a^2 = ace \). So \( a = ce \in p \). Therefore, either \( c \in p \) or \( e \in p \). If \( c \in p \), then \( c = ac' \) and so \( a = cac' \) which implies that \( 1 = ac' \) and \( a = bc' \) thus \( 0 :_R x = p \). If \( e \in p \), then \( e = ae' \) and so \( a = ae'c \) which implies that \( 1 = c'e' \) and \( b = a^2 e' \) thus \( 0 :_R x = p^2 \).

\[ \square \]

The following example shows that the condition “\( 0 \) is a 2-absorbing submodule of \( E(R/p) \)” is essential. It is well-known that \( E(\mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}(p^\infty) = \{m/n + \mathbb{Z} : m, n \in \mathbb{Z}, n \neq 0\} \), where \( p \) is a prime integer. But neither \( p^2 \mathbb{Z} = 0 :_p 1/p^3 + \mathbb{Z} \) nor \( 0 :_p 1/p^3 + \mathbb{Z} = p\mathbb{Z} \). Hence, \( 0 \) is not a 2-absorbing submodule of \( E(\mathbb{Z}/p\mathbb{Z}) \). Also, this example shows that if \( 0 \) is a 2-absorbing submodule of \( M \), then it is not necessarily a 2-absorbing submodule of \( E(M) \).

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**References**