On the 2-absorbing Submodules

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Abstract. Let $R$ be a commutative ring and $M$ be an $R$-module. In this paper, we investigate some properties of 2-absorbing submodules of $M$. It is shown that $N$ is a 2-absorbing submodule of $M$ if and only if whenever $IJL \subseteq N$ for some ideals $I, J$ of $R$ and a submodule $L$ of $M$, then $IL \subseteq N$ or $JL \subseteq N$ or $IJ \subseteq N :_R M$. Also, if $N$ is a 2-absorbing submodule of $M$ and $M/N$ is Noetherian, then a chain of 2-absorbing submodules of $M$ is constructed. Furthermore, the annihilation of $E(R/p)$ is studied whenever 0 is a 2-absorbing submodule of $E(R/p)$, where $p$ is a prime ideal of $R$ and $E(R/p)$ is an injective envelope of $R/p$.

Keywords: 2-absorbing ideal, 2-absorbing submodule, A chain of 2-absorbing submodule.


1. Introduction

Throughout this paper $R$ is a commutative ring with non-zero identity and $M$ is an unitary $R$-module. We defined a submodule $N$ of $M$ is 2-absorbing whenever $abm \in N$ for some $a, b \in R$, $m \in M$, then $am \in N$ or $bm \in N$ or $ab \in N :_R M$, see for instance [1, 3, 4, 6, 7, 9, 10]. It is well known that, a submodule $N$ of $M$ is prime if and only if $IL \subseteq N$ for an ideal $I$ of $R$ and

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a submodule $L$ of $M$, then either $L \subseteq N$ or $I \subseteq N :_RM$. This statement persuaded us to prove that, a submodule $N$ of $M$ is 2-absorbing if and only if $IJL \subseteq N$ for some ideals $I, J$ of $R$ and a submodule $L$ of $M$, then $IL \subseteq N$ or $JL \subseteq N$ or $IJ \subseteq N :_RM$. As a corollary of this theorem, it is shown that $L = \{ m \in M : p \subseteq r(N : m) \}$ is a 2-absorbing submodule of $M$, where $N$ is a 2-absorbing submodule of $M$ with $r(N :_RM) = p \cap q$ for some prime ideals $p, q$ of $R$. Also, it is shown that if $M/N$ is Noetherian, then there exists a chain of 2-absorbing submodules of $M$ that begins with $N$. Assume that $E(R/p)$ is an injective envelope of $R/p$, it is shown that if 0 is a 2-absorbing submodule of $E(R/p)$, then $r(0 :_R E(R/p)) = p$ and $0 :_RX$ is determined for all nonzero element $x$ of $E(R/p)$.

Now, we define the concepts that we will use later. For a submodule $L$ of $M$ let $L :_RM$ denote the ideal $\{ r \in R : rM \subseteq L \}$. Similarly, for an element $m \in M$ let $L :_Rm$ denote the ideal $\{ r \in R : rm \in L \}$. If $I$ is an ideal of $R$, then $r(I)$ denotes the radical of $I$. We say that $p \in \text{Spec}(R)$ is an associated prime ideal of $M$ if there exists $m \in M$ with $0 :_RM = p$. The set of associated prime ideals of $M$ is denoted by $\text{Ass}_R(M)$, the set of integers is denoted by $\mathbb{Z}$.

2. 2-absorbing Submodules

Let $N$ be a proper submodule of $M$. We say that $N$ is a 2-absorbing submodule of $M$ if whenever $a, b \in R$, $m \in M$ and $abm \in N$, then $am \in N$ or $bm \in N$ or $ab \in N :_RM$.

**Lemma 2.1.** Let $I$ be an ideal of $R$ and $N$ be a 2-absorbing submodule of $M$. If $a \in R$, $m \in M$ and $iam \subseteq N$, then $am \in N$ or $Im \subseteq N$ or $Ia \subseteq N :_RM$.

**Proof.** Let $am \notin N$ and $Ia \notin N :_RM$. Then there exists $b \in I$ such that $ba \notin N :_RM$. Now, $bam \in N$ implies that $bm \in N$, since $N$ is a 2-absorbing submodule of $M$. We have to show that $Im \subseteq N$. Let $c$ be an arbitrary element of $I$. Thus $(b + c)am \in N$. Hence, either $(b + c)m \in N$ or $(b + c)a \in N :_RM$. If $(b + c)m \in N$, then by $bm \in N$ it follows that $cm \in N$. If $(b + c)a \in N :_RM$, then $ca \notin N :_RM$, but $cam \in N$. Thus $cm \in N$. Hence, we conclude that $Im \subseteq N$. \hfill $\square$

**Lemma 2.2.** Let $I, J$ be ideals of $R$ and $N$ be a 2-absorbing submodule of $M$. If $m \in M$ and $IJm \subseteq N$, then $Im \subseteq N$ or $Jm \subseteq N$ or $IJ \subseteq N :_RM$.

**Proof.** Let $I \notin N :_RM$ and $J \notin N :_RM$. We have to show that $IJ \subseteq N :_RM$. Assume that $c \in I$ and $d \in J$. By assumption there exists $a \in I$ such that $am \notin N$ but $aJm \subseteq N$. Now, Lemma 2.1 shows that $aJ \subseteq N :_RM$ and so $(I \setminus N :_RM)J \subseteq N :_RM$, similarly there exists $b \in J \setminus N :_RM$ such that $Ib \subseteq N :_RM$ and also $(I \setminus N :_RM) \subseteq N :_RM$. Thus we have $ab \in N :_RM$, $ad \in N :_RM$ and $cb \in N :_RM$. By $a + c \in I$ and $b + d \in J$ it follows that $(a + c)(b + d)m \in N$. Therefore, $(a + c)m \in N$ or $(b + d)m \in N$ or
(a + c)(b + d) ∈ N : R M. If (a + c)m ∈ N, then cm ̸∈ N hence, c ∈ I \ N : R m
which implies that cd ∈ N : R M. Similarly by (b + d)m ∈ N, we can deduce
that cd ∈ N : R M. If (a + c)(b + d) ∈ N : R M, then ab + ad + cb + cd ∈ N : R M
and so cd ∈ N : R M. Therefore, IJ ⊆ N : R M.

Theorem 2.3. Let N be a proper submodule of M. The following statement
are equivalent:

(i) N is a 2-absorbing submodule of M;
(ii) If IJL ⊆ N for some ideals I, J of R and a submodule L of M, then
IL ⊆ N or JL ⊆ N or IJ ⊆ N : R M.

Proof. (ii) ⇒ (i) is obvious. To prove (i) ⇒ (ii), assume that IJL ⊆ N
for some ideals I, J of R and a submodule L of M and IJ ̸⊆ N : R M. Then
by Lemma 2.2 for all m ∈ L either Im ⊆ N or Jm ⊆ N. If Im ⊆ N, for
all m ∈ L we are done. Similarly if Jm ⊆ N, for all m ∈ L we are done.
Suppose that m, m′ ∈ L are such that Im ⊆ N and Jm′ ⊆ N. Thus Jm ⊆ N
and Im′ ⊆ N. Since IJ(m + m′) ⊆ N we have either I(m + m′) ⊆ N or
J(m + m′) ⊆ N. By I(m + m′) ⊆ N, it follows that Im ⊆ N which is a
contradiction, similarly by J(m + m′) ⊆ N we get a contradiction. Therefore
either IL ⊆ N or JL ⊆ N.

A submodule N of M is called strongly 2-absorbing if it satisfies in condition
(ii), see [5]. Therefore, Theorem 2.3 shows that N is a 2-absorbing submodule
of M if and only if N is a strongly 2-absorbing submodule of M.

Corollary 2.4. Let M be an R-module and N be a 2-absorbing submodule of
M. Then N : M I = {m ∈ M : Im ⊆ N} is a 2-absorbing submodules of M
for all ideal I of R. Furthermore N : M I^n = N : M I^{n+1}, for all n ≥ 2.

Proof. Let I be an ideal of R, a, b ∈ R, m ∈ M and abm ∈ N : M I. Thus
Iabm ⊆ N. Hence, Im ⊆ N or Iab ⊆ N : R M or abm ∈ N, by Lemma 2.2. If
Im ⊆ N we are done. If Iab ⊆ N : R M, then ab ∈ (N : R M) : R I = (N : M
I) : R M. If abm ∈ N, then am ∈ N or bm ∈ N or ab ∈ N : R M. Thus
Iam ⊆ N or Ibm ⊆ N or Iab ⊆ N : R M which complete the proof.

For the second statement, it is enough to show that N : M I^2 = N : M I^3. It
is clear that N : M I^2 ⊆ N : M I^3. Let m ∈ N : M I^3. Then I^3m ⊆ N. Now, by
Lemma 2.2, we have I^2m ⊆ N or Im ⊆ N or I^3 ⊆ N : R M. If I^2m ⊆ N or
Im ⊆ N, we are done. If I^3 ⊆ N : R M, then I^2 ⊆ N : R M since N : R M is a
2-absorbing ideal of R by [9, Theorem 2.3].

□

It is clear that, nZ is a 2-absorbing ideal of Z if and only if n = 0, p, p^2, pq,
where p, q are distinct prime integers. It is easy to see that 4Z : Z 6Z = 2Z
but 4Z : Z 36Z = Z. Hence, the equality mentioned in the Corollary 2.4, is not
necessarily true for n = 1.
Theorem 2.5. Let $N$ be a 2-absorbing submodule of $M$ such that $r(N :_R M) = p \cap q$ where $p$ and $q$ are the only distinct prime ideals of $R$ that are minimal over $N :_R M$. Then $L = \{m \in M : p \subseteq r(N :_R m)\}$ is a 2-absorbing submodule of $M$ containing $N$. Also, either $r(L :_R M) = q$ or $r(L :_R M) = p' \cap q$, where $p'$ is a prime ideal of $R$ containing $p$.

Proof. It is clear that $L$ is a submodule of $M$ containing $N$. Assume that $a, b \in R$, $m \in M$ and $abm \in L$. We have to show that $am \in L$ or $bm \in L$ or $ab \in L :_R M$. Since $p \subseteq r(N :_R abm)$, thus $p^2abm \subseteq N$, by [9, Theorem 2.4] and [2, Theorem 2.4]. Therefore, by Lemma 2.1, we have $abm \in N$ or $p^2m \subseteq N$ or $p^2ab \subseteq N :_R M$. If $abm \in N$, then $am \in N$ or $bm \in N$ or $ab \in N :_R M$ which implies that $am \in L$ or $bm \in L$ or $ab \in L :_R M$. If $p^2m \subseteq N$, then $p^2 \subseteq N :_R m$ and so $p \subseteq r(N :_R m)$ thus $m \in L$ and we are done. If $p^2ab \subseteq N :_R M$, then by [2, Theorem 2.13], we have $p^2a \subseteq N :_R M$ or $p^2b \subseteq N :_R M$ or $ab \in N :_R M$.

In the first case we conclude that $p^2 \subseteq N :_R am$ and so $am \in L$. By a similar argument in the second case we can deduced that $bm \in L$. If $ab \in N :_R M$, then $ab \in L :_R p$. Therefore, the result follows.

For the second statement, first we show that $r(N :_R M) = r(L :_R M) \cap p$. It is clear $r(N :_R M) \subseteq r(L :_R M) \cap p$. Assume that $a \in (L :_R M) \cap p$. Thus $am \subseteq L$ and so, by definition of $L$, $p \subseteq r(N :_R am)$, for all $m \in M$. Hence, [2, Theorem 2.4] shows that $p^3 \subseteq N :_R am$, for all $m \in M$. Therefore, $a^3 \in N :_R m$, for all $m \in M$. So that $a^3 \in N :_R M$ and then $a \in r(N :_R M)$. Thus $r(L :_R M) \cap p \subseteq r(N :_R M)$. Now, $L :_R M$ is a 2-absorbing ideal of $R$, therefore either $r(L :_R M) = p'$ or $r(L :_R M) = p' \cap q'$, for some prime ideals $p', q'$ of $R$. In the first case we have $r(N :_R M) = p \cap p'$ which implies that $p' = q$ and in the second case we have $r(N :_R M) = p \cap p' \cap q'$ which implies that either $p' = q$ or $q' = q$. □

Corollary 2.6. Let $N$ be a 2-absorbing submodule of $M$ such that $r(N :_R M) = p \cap q$ where $p$ and $q$ are the only distinct prime ideals of $R$ that are minimal over $N :_R M$. If $M/N$ is a Noetherian $R$-module, then

(i) there exists a chain $N = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_{n-1} \subseteq L_n = M$ of 2-absorbing submodules of $M$. Furthermore, $\text{Ass}(M) \subseteq \text{Ass}(M/L_{n-1}) \cup \text{Ass}(L_{n-1}/L_{n-2}) \cup \text{Ass}(L_{n-2}/L_{n-3}) \cup \cdots \cup \text{Ass}(L_1/N)$, where $\text{Ass}(L_i/N)$ is the union of at most two totally ordered set, for all $i$.

(ii) there exists a chain $N \subseteq L_n \subseteq L_{n-1} \subseteq \cdots \subseteq L_1 \subseteq L_0 = M$ of submodules of $M$ such that $L_i$ is a 2-absorbing submodule of $L_{i+1}$, for all $0 \leq i \leq n-1$.

Proof. (i) Let $L_1 = \{m \in M : p \subseteq r(N :_R m)\}$. Then by Corollary 2.4, $L_1$ is a 2-absorbing submodule of $M$ and so either $r(L_1 :_R M) = q$ or $r(L_1 :_R M) = p_1 \cap q$, where $p_1$ is a prime ideal of $R$ containing $p$. If $r(L_1 :_R M) = q$, then choose $L_2 = \{m \in M : q \subseteq r(L_1 :_R m)\} = M$. Hence, $N \subseteq L_1 \subseteq L_2 = M$ is requested chain. If $r(L_1 :_R M) = p_1 \cap q$, set $L_2 = \{m \in M : p_1 \subseteq r(L_1 :_R m)\}$.
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and so either \( r(L_2 :_RM) = q \) or \( r(L_2 :_RM) = p_2 \cap q \), where \( p_2 \) is a prime ideal of \( R \) containing \( p_1 \). Proceeding in this way, we can achieve \( N \subseteq L_0 \subseteq L_1 \subseteq \cdots \subseteq L_{n-1} \subseteq L_n = M \) of 2-absorbing submodules of \( M \). The last statement is obvious, by [9, Theorem 2.6].

(ii) Let \( L_1 = \{ m \in M : p \subseteq r(N :_RM) \} \). Then \( N \) is a 2-absorbing submodule of \( L_1 \). So that either \( r(N :_RL_1) = p_1 \) or \( r(N :_RL_1) = p_1 \cap q_1 \), for some prime ideals \( p_1, q_1 \) of \( R \). If \( r(N :_RL_1) = p_1 \), then choose \( L_2 = \{ x \in L_1 : p_1 \subseteq r(N :_Rx) \} = N \). Hence, in this case \( N \subseteq L_1 \subseteq L_0 = M \) is the requested chain. If \( r(N :_RL_1) = p_1 \cap q_1 \), then set \( L_2 = \{ x \in L_1 : p_1 \subseteq r(N :_Rx) \} \) and continue the same way to achieve the chain \( N \subseteq L_n \subseteq L_{n-1} \subseteq \cdots \subseteq L_1 \subseteq L_0 = M \) of 2-absorbing submodules of \( M \). 

Theorem 2.7. Let \( N \) be a 2-absorbing submodule of \( M \). Then \( N :_RM \) is a prime ideal of \( R \) if and only if \( N :_RM \) is a prime ideal of \( R \) for all \( m \in M \setminus N \).

Proof. Assume that \( a, b \in R, m \in M \setminus N \) and \( ab \in N :_RM \). Then \( abm \subseteq N \). We have \( am \in N \) or \( bm \in N \) or \( ab \in N :_RM \) since \( N \) is a 2-absorbing submodule of \( M \). If \( am \in N \) or \( bm \in N \) we are done. If \( ab \in N :_RM \), then by assumption either \( a \in N :_RM \) or \( b \in N :_RM \). Thus either \( a \in N :_RM \) or \( b \in N :_RM \). So \( N :_RM \) is a prime ideal.

Conversely, suppose that \( ab \in N :_RM \) for some \( a, b \in R \) and assume that there exist \( m, m' \in M \) such that \( am \notin N \) and \( bm' \notin N \). By \( abm, abm' \in N \) it follows that \( bm \in N \) and \( am' \in N \) since \( N :_RM \) and \( N :_Rm' \) are prime ideals of \( R \). If \( m + m' \in N \), then \( am \in N \) which is a contradiction. Thus \( m + m' \notin N \).

Now by \( ab(m' + m'') \in N \) we have \( a(m' + m'') \in N \) or \( b(m' + m'') \in N \) which is a contradiction. Thus \( aM \subseteq N \) or \( bM \subseteq N \) which implies that \( N :_RM \) is prime.

Corollary 2.8. Let \( N \) be a 2-absorbing submodule of \( M \). Then \( N :_RM \) is a prime ideal of \( R \) if and only if \( N :_RK \) is a prime ideal of \( R \) for all submodules \( K \) of \( M \) containing \( N \).

Proof. By Theorem 2.7 and [9, Theorem 2.6] it follows that \( \{ N :_R x : x \in K \setminus N \} \) is a totally ordered set of prime ideals of \( R \). Hence, \( N :_RK = \cap_{x \in K} N :_Rx \) is a prime ideal of \( R \).

Theorem 2.9. Let \( p \) be a prime ideal of \( R \) and \( E(R/p) \) be an injective envelop of \( R/p \). If \( 0 \) is a 2-absorbing submodule of \( E(R/p) \), then

(i) \( p^2 \subseteq 0 :_R E(R/p) \subseteq p \) so that \( r(0 :_R E(R/p)) = p \).

(ii) \( p^2 \subseteq 0 :_R x \subseteq 0 :_R ax \), for all non-zero element \( x \) of \( E(R/p) \) and all \( a \in p \setminus 0 :_R x \).

(iii) \( p^2 \subseteq 0 :_R x = 0 :_R a^nx \subseteq p \), for all \( a \notin p \).

Proof. (i) We have \( r(0 :_R x) = p \) for all non-zero element \( x \) of \( E(R/p) \), by [8, Theorem 18.4]. Also it is obvious \( 0 :_R E(R/p) \subseteq 0 :_R x \). Thus \( 0 :_R E(R/p) \subseteq p \).
Now, assume that $a \in p^2$ and $x$ is a non-zero element of $E(R/p)$. Since $0$ is a 2-absorbing submodule of $M$, $0 : R x$ is a 2-absorbing ideal of $R$, by [9, Theorem 2.4]. Therefore we have $p^2$ is a subset of $0 : R x$, by [2, Theorem 2.4]. Hence, $ax = 0$ and therefore $aE(R/p) = 0$ and $p^2 \subseteq 0 : R E(R/p)$.

(ii) Let $x$ be a non-zero element of $E(R/p)$. Then we have $p^2 \subseteq 0 : R x \subseteq p$. Assume that $a \in p \setminus 0 : R x$. Thus $ax \neq 0$ but $a^2 x = 0$ which shows that $0 : R x \subset 0 : R ax$. If $b \in p$, then $ab \in p^2$ and $abx = 0$. Thus $b \in 0 : R ax$ and so $p \subseteq 0 : R ax$.

(iii) Assume that $a \notin p$. It is obvious that $0 : R x \subseteq 0 : R a^n x$, for all $n \in N$. Let $b \in \text{Ann}_R(a^n x)$. Thus $ba^n x = 0$. But multiplication by $a^n$ is an automorphism on $E(R/p)$, so that $bx = 0$ and $b \in 0 : R x$. Therefore, $0 : R x = 0 : R a^n x$.

\[ \Box \]

**Corollary 2.10.** Let $R$ be a principal ideal domain and $p$ is a prime ideal of $R$. If $0$ is a 2-absorbing submodule of $E(R/p)$, then for all non-zero element $x$ of $E(R/p)$ either $0 : R x = p^2$ or $0 : R x = p$.

**Proof.** Let $p = (a)$. Then $p^2 = (a^2)$. Let $x$ be a non-zero element of $E(R/p)$. Then $p^2 \subseteq 0 : R x = (b) \subseteq p$ by Theorem 2.9(ii). Thus $a^2 = bc$ and $b = ae$ for some $c, e \in R$. Hence, $a^2 = ace$. So $a = ec \in p$. Therefore, either $c \in p$ or $e \in p$. If $c \in p$, then $c = ac'$ and so $a = eac'$ which implies that $1 = ec'$ and $a = bc'$ thus $0 : R x = p$. If $e \in p$, then $e = ae'$ and so $a = ae'c$ which implies that $1 = e'c$ and $b = a^2 e'$ thus $0 : R x = p^2$.

\[ \Box \]

The following example shows that the condition “$0$ is a 2-absorbing submodule of $E(R/p)$” is essential. It is well-known that $E(\mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}(p^\infty) = \{m/n + \mathbb{Z} : m, n \in \mathbb{Z}, n \neq 0\}$, where $p$ is a prime integer. But neither $p^2 \mathbb{Z} = 0 \not\subseteq 1/p^3 + \mathbb{Z}$ nor $0 \not\subseteq 1/p^3 + \mathbb{Z} = p\mathbb{Z}$. Hence, $0$ is not a 2-absorbing submodule of $E(\mathbb{Z}/p\mathbb{Z})$. Also, this example shows that if $0$ is a 2-absorbing submodule of $M$, then it is not necessarily a 2-absorbing submodule of $E(M)$.

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