

Frames in 2-inner Product Spaces

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ABSTRACT. In this paper, we introduce the notion of a frame in a 2-inner product space and give some characterizations. These frames can be considered as a usual frame in a Hilbert space, so they share many useful properties with frames.

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1. INTRODUCTION AND PRELIMINARIES

The concept of frames in Hilbert spaces has been introduced by Duffin and Schaeffer [12] in 1952 to study some deep problems in nonharmonic Fourier series. Various generalizations of frames have been proposed; frame of subspaces [2, 6], pseudo-frames [18], oblique frames [10], continuous frames [1, 4, 14] and so on. The concept of frames in Banach spaces have been introduced by Grochenig [16], Casazza, Han and Larson [5] and Christensen and Stoeva [11].

The concept of linear 2-normed spaces has been investigated by S. Gahler in 1965 [15] and has been developed extensively in different subjects by many authors [3, 7, 8, 13, 14, 17]. A concept which is related to a 2-normed space is 2-inner product space which have been intensively studied by many mathematicians in the last three decades. A systematic presentation of the recent results related to the theory of 2-inner product spaces as well as an extensive

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list of the related references can be found in the book [7]. Here we give the basic definitions and the elementary properties of 2-inner product spaces.

Let \mathcal{X} be a linear space of dimension greater than 1 over the field \mathbb{K} , where \mathbb{K} is the real or complex numbers field. Suppose that $(\cdot, \cdot | \cdot)$ is a \mathbb{K} -valued function defined on $\mathcal{X} \times \mathcal{X} \times \mathcal{X}$ satisfying the following conditions:

(I1) $(x, x | z) \geq 0$ and $(x, x | z) = 0$ if and only if x and z are linearly dependent,

(I2) $(x, x | z) = \overline{(z, z | x)}$,

(I3) $(y, x | z) = \overline{(x, y | z)}$,

(I4) $(\alpha x, y | z) = \alpha(x, y | z)$ for all $\alpha \in \mathbb{K}$,

(I5) $(x_1 + x_2, y | z) = (x_1, y | z) + (x_2, y | z)$.

$(\cdot, \cdot | \cdot)$ is called a *2-inner product* on \mathcal{X} and $(\mathcal{X}, (\cdot, \cdot | \cdot))$ is called a *2-inner product space* (or *2-pre Hilbert space*). Some basic properties of 2-inner product $(\cdot, \cdot | \cdot)$ can be immediately obtained as follows ([8, 13]):

- $(0, y | z) = (x, 0 | z) = (x, y | 0) = 0$,
- $(x, \alpha y | z) = \overline{\alpha}(x, y | z)$,
- $(x, y | \alpha z) = |\alpha|^2(x, y | z)$, for all $x, y, z \in \mathcal{X}$ and $\alpha \in \mathbb{K}$.

Using the above properties, we can prove the Cauchy-Schwarz inequality

$$(1.1) \quad |(x, y | z)|^2 \leq (x, x | z)(y, y | z).$$

Example 1.1. If $(\mathcal{X}, \langle \cdot, \cdot \rangle)$ is an inner product space, then the standard 2-inner product $(\cdot, \cdot | \cdot)$ is defined on \mathcal{X} by

$$(1.2) \quad (x, y | z) = \begin{vmatrix} \langle x, y \rangle & \langle x, z \rangle \\ \langle z, y \rangle & \langle z, z \rangle \end{vmatrix} = \langle x, y \rangle \langle z, z \rangle - \langle x, z \rangle \langle z, y \rangle,$$

for all $x, y, z \in \mathcal{X}$.

In any given 2-inner product space $(\mathcal{X}, (\cdot, \cdot | \cdot))$, we can define a function $\|\cdot, \cdot\|$ on $\mathcal{X} \times \mathcal{X}$ by

$$(1.3) \quad \|x, z\| = (x, x | z)^{\frac{1}{2}},$$

for all $x, z \in \mathcal{X}$.

It is easy to see that, this function satisfies the following conditions:

(N1) $\|x, z\| \geq 0$ and $\|x, z\| = 0$ if and only if x and z are linearly dependent,

(N2) $\|x, z\| = \|z, x\|$,

(N3) $\|\alpha x, z\| = |\alpha| \|x, z\|$ for all $\alpha \in \mathbb{K}$,

(N4) $\|x_1 + x_2, z\| \leq \|x_1, z\| + \|x_2, z\|$.

Any function $\|\cdot, \cdot\|$ defined on $\mathcal{X} \times \mathcal{X}$ and satisfying the conditions (N1)-(N4) is called a *2-norm* on \mathcal{X} and $(\mathcal{X}, \|\cdot, \cdot\|)$ is called a *linear 2-normed space*. Whenever a 2-inner product space $(\mathcal{X}, (\cdot, \cdot | \cdot))$ is given, we consider it as a linear 2-normed space $(\mathcal{X}, \|\cdot, \cdot\|)$ with the 2-norm defined by (1.3).

In the present paper, we shall introduce 2-frames for a 2-inner product space and describe some fundamental properties of them. This implies that each element in the underlying 2-inner product space can be written as an unconditionally convergent infinite linear combination of the frame elements.

2. FRAMES IN THE STANDARD 2-INNER PRODUCT SPACES

Throughout this paper, we assume that \mathcal{H} is a separable Hilbert space, with the inner product $\langle \cdot, \cdot \rangle$ chosen to be linear in the first entry. We first review some basic facts about frames in \mathcal{H} , then try to define them in a standard 2-inner product space.

Definition 2.1. A sequence $\{f_i\}_{i=1}^\infty \subseteq \mathcal{H}$ is called a *frame* for \mathcal{H} if there exist $A, B > 0$ such that

$$(2.1) \quad A\|f\|^2 \leq \sum_{i=1}^\infty |\langle f, f_i \rangle|^2 \leq B\|f\|^2, \quad (f \in \mathcal{H}).$$

The numbers A, B are called *frame bounds*. The frame is called *tight* if $A = B$. Given a frame $\{f_i\}_{i=1}^\infty$, the frame operator is defined by

$$Sf = \sum_{i=1}^\infty \langle f, f_i \rangle f_i.$$

The series defining Sf converges unconditionally for all $f \in \mathcal{H}$ and S is a bounded, invertible, and self-adjoint operator. This leads to the frame decomposition:

$$f = S^{-1}Sf = \sum_{i=1}^\infty \langle f, S^{-1}f_i \rangle f_i, \quad (f \in \mathcal{H}).$$

The possibility of representing every $f \in \mathcal{H}$ in this way is the main feature of a frame. The coefficients $\{\langle f, S^{-1}f_i \rangle\}_{i=1}^\infty$ are called frame coefficients. A sequence satisfying the upper frame condition is called a *Bessel sequence*. A sequence $\{f_i\}_{i=1}^\infty$ is Bessel sequence if and only if the operator $T : \{c_i\} \mapsto \sum_{i=1}^\infty c_i f_i$ is a well-defined operator from l^2 into \mathcal{H} . In that case T , which is called the *pre-frame operator*, is automatically bounded. When $\{f_i\}_{i=1}^\infty$ is a frame, the pre-frame operator T is well-defined and $S = TT^*$. For more details see [9, Section 5.1]. Also see [19] for a class of finite frames.

Let \mathcal{X} be a 2-inner product space. A sequence $\{a_n\}_{n=1}^\infty$ of \mathcal{X} is said to be *convergent* if there exists an element $a \in \mathcal{X}$ such that $\lim_{n \rightarrow \infty} \|a_n - a, x\| = 0$, for all $x \in \mathcal{X}$. Similarly, we can define a Cauchy sequence in \mathcal{X} . A 2-inner product space \mathcal{X} is called a *2-Hilbert space* if it is complete. That is, every Cauchy sequence in \mathcal{X} is convergent in this space [17]. Clearly, the limit of any convergent sequence is unique and if $(\mathcal{X}, (\cdot, \cdot | \cdot))$ is the standard 2-inner product, then this topology is the original topology on \mathcal{X} .

Now we are ready to define 2-frames on a 2-Hilbert space.

Definition 2.2. Let $(\mathcal{X}, (\cdot, \cdot))$ be a 2-Hilbert space and $\xi \in \mathcal{X}$. A sequence $\{x_i\}_{i=1}^{\infty}$ of elements in \mathcal{X} is called a 2-frame (associated to ξ) if there exist $A, B > 0$ such that

$$(2.2) \quad A\|x, \xi\|^2 \leq \sum_{i=1}^{\infty} |(x, x_i|\xi)|^2 \leq B\|x, \xi\|^2, \quad (x \in \mathcal{X}).$$

A sequence satisfying the upper 2-frame condition is called a 2-Bessel sequence. In (2.2) we may assume that every x_i is linearly independent to ξ , by (1.1) and the property (II).

The following proposition shows that in the standard 2-inner product spaces every frame is a 2-frame.

Proposition 2.3. Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space and $\{x_i\}_{i=1}^{\infty}$ is a frame for \mathcal{H} . Then it is a 2-frame with the standard 2-inner product on \mathcal{H} .

Proof. Suppose that $\{x_i\}_{i=1}^{\infty}$ is a frame with the bounds A, B and $\xi \in \mathcal{H}$ such that $\|\xi\| = 1$. Then by using (2.1) and (1.2) for every $x \in \mathcal{H}$ we have

$$\begin{aligned} \sum_{i=1}^{\infty} |(x, x_i|\xi)|^2 &= \sum_{i=1}^{\infty} |\langle x - \langle x, \xi \rangle \xi, x_i \rangle|^2 \\ &\leq B\|x - \langle x, \xi \rangle \xi\|^2 \\ &\leq B(\|x\|^2 - |\langle x, \xi \rangle|^2) \\ &= B(x, x|\xi). \end{aligned}$$

The argument for lower bound is similar. □

The converse of the above proposition is not true. In fact, by the following proposition, every 2-frame is a frame for a closed subspace of \mathcal{H} with codimension 1. For each $\xi \in \mathcal{H}$ we denote by L_{ξ} the subspace generated with ξ .

Proposition 2.4. Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space and $\xi \in \mathcal{H}$. Every 2-frame associated to ξ is a frame for L_{ξ}^{\perp} .

Proof. If $\{x_i\}_{i=1}^{\infty}$ is a 2-frame with the bounds A, B then (2.2) implies that there exist $A, B > 0$ such that

$$A(\|x\|^2 - |\langle x, \xi \rangle|^2) \leq \sum_{i=1}^{\infty} |\langle x - \langle x, \xi \rangle \xi, x_i \rangle|^2 \leq B(\|x\|^2 - |\langle x, \xi \rangle|^2), \quad (x \in \mathcal{H}).$$

Therefore, $\{x_i\}_{i=1}^{\infty}$ is a frame for the Hilbert space L_{ξ}^{\perp} . □

Remark 2.5. Let \mathcal{H} be a Hilbert space and $\{x_i\}_{i=1}^{\infty}$ is a frame for \mathcal{H} with the frame operator S . If $\langle x_j, S^{-1}x_j \rangle = 1$ for some $j \in \mathbb{N}$, then $\{x_i\}_{i \neq j}$ is incomplete

and therefore it is not a frame for \mathcal{H} [9, Theorem 5.3.9]. Assume that $\|x_j\| = 1$ and consider the standard 2-inner product on \mathcal{H} . It is not difficult to see that

$$\sum_{i=1}^{\infty} |(x, x_i | x_j)|^2 = \sum_{i=1, i \neq j}^{\infty} |(x, x_i | x_j)|^2.$$

Now the proof of Proposition 2.3 shows that $\{x_i\}_{i \neq j}$ is a 2-frame for \mathcal{H} associated to x_j .

3. SOME PROPERTIES OF 2-FRAMES

This section is devoted to establishing pre-frame and frame operator for a 2-frame. To extend a well-known result in Hilbert spaces to 2-inner product spaces.

Lemma 3.1. *Let $(\mathcal{X}, (., . | .))$ be a 2-inner product space and $x, z \in \mathcal{X}$. Then*

$$(3.1) \quad \|x, z\| = \sup\{|(x, y | z)|; \quad y \in \mathcal{X}, \|y, z\| = 1\}.$$

Proof. By the Cauchy-Schwarz inequality (1.1) we observe that

$$(x, y | z) \leq \|x, z\| \|y, z\| = \|x, z\|$$

for every $y \in \mathcal{X}$ such that $\|y, z\| = 1$. Moreover, if $y = \frac{1}{\|x, z\|}x$, then $\|y, z\| = 1$ and therefore $(x, y | z) = \|x, z\|$. \square

For the remainder, we assume $(\mathcal{X}, (., . | .))$ is a 2-Hilbert space and L_ξ the subspace generated with ξ for a fix element ξ in \mathcal{X} . Denote by \mathcal{M}_ξ the algebraic complement of L_ξ in \mathcal{X} . So $L_\xi \oplus \mathcal{M}_\xi = \mathcal{X}$.

We first define the inner product $\langle ., . \rangle_\xi$ on \mathcal{X} as following:

$$\langle x, z \rangle_\xi = (x, z | \xi).$$

A straightforward calculations shows that $\langle ., . \rangle_\xi$ is a semi-inner product on \mathcal{X} . It is well-known that this semi-inner product induces an inner product on the quotient space \mathcal{X}/L_ξ as

$$\langle x + L_\xi, z + L_\xi \rangle_\xi = \langle x, z \rangle_\xi, \quad (x, z \in \mathcal{X}).$$

By identifying \mathcal{X}/L_ξ with \mathcal{M}_ξ in an obvious way, we obtain an inner product on \mathcal{M}_ξ . Define

$$(3.2) \quad \|x\|_\xi = \sqrt{\langle x, x \rangle_\xi} \quad (x \in \mathcal{M}_\xi).$$

Then $(\mathcal{M}_\xi, \|\cdot\|_\xi)$ is a norm space.

Now if $\{x_i\}_{i=1}^\infty \subseteq \mathcal{X}$ is a 2-frame associated to ξ with bounds A and B , then we can rewrite (2.2) as

$$A\|x\|_\xi^2 \leq \sum_{i=1}^{\infty} |\langle x, x_i \rangle_\xi|^2 \leq B\|x\|_\xi^2, \quad (x \in \mathcal{M}_\xi).$$

That is, $\{x_i\}_{i=1}^\infty$ is a frame for \mathcal{M}_ξ . Let \mathcal{X}_ξ be the completion of the inner product space \mathcal{M}_ξ . Due to Lemma 5.1.2 of [9] the sequence $\{x_i\}_{i=1}^\infty$ is also a frame for \mathcal{X}_ξ with the same bounds. To summarize, we have the following theorem.

Theorem 3.2. *Let $(\mathcal{X}, (\cdot, \cdot | \cdot))$ be a 2-Hilbert space. Then $\{x_i\}_{i=1}^\infty \subseteq \mathcal{X}$ is a 2-frame associated to ξ with bounds A and B if and only if it is a frame for the Hilbert space \mathcal{X}_ξ with bounds A and B .*

By the above theorem, every question about 2-frames in a 2-Hilbert space can be solved as a question about frames in a Hilbert space.

Lemma 3.3. *Let $\{x_i\}_{i=1}^\infty$ be a 2-Bessel sequence in \mathcal{X} . Then the 2-pre frame operator $T_\xi : l^2 \rightarrow \mathcal{X}_\xi$ defined by*

$$(3.3) \quad T_\xi\{c_i\} = \sum_{i=1}^{\infty} c_i x_i$$

is well-defined and bounded.

Proof. Suppose $\{c_i\}_{i=1}^\infty \in l^2$, then by using (3.1) and (3.2) we have

$$\begin{aligned} \left\| \sum_{i=1}^m c_i x_i - \sum_{i=1}^n c_i x_i \right\|_\xi^2 &= \left\| \sum_{i=1}^m c_i x_i - \sum_{i=1}^n c_i x_i, \xi \right\|^2 \\ &= \sup \left\{ \left| \left(\sum_{i=n}^m c_i x_i, y | \xi \right) \right|^2, y \in \mathcal{X}, \|y, \xi\| = 1 \right\} \\ &\leq \sum_{i=n}^m |c_i|^2 \sup \left\{ \left| (x_i, y | \xi) \right|^2, y \in \mathcal{X}, \|y, \xi\| = 1 \right\} \\ &\leq B \sum_{i=n}^m |c_i|^2 \end{aligned}$$

where B is the (upper) bound of $\{x_i\}_{i=1}^\infty$. This implies that $\sum_{i=1}^\infty c_i x_i$ is well-defined as an element of \mathcal{X}_ξ . Moreover, if $\{c_i\}_{i=1}^\infty$ is a sequence in l^2 , then an argument as above shows that $\|T_\xi\{c_i\}\|_\xi \leq \sqrt{B} \|\{c_i\}\|_2$. In particular, $\|T_\xi\| \leq \sqrt{B}$. \square

Next, we can compute T_ξ^* , the adjoint of T_ξ as

$$T_\xi^* : \mathcal{X}_\xi \rightarrow l^2; \quad T_\xi^* x = \{(x, x_i | \xi)\}_{i=1}^\infty.$$

It is easy to check that T_ξ^* is well-defined. Moreover, it follows by (2.2) that $\|T_\xi^*\| \leq \sqrt{B}$.

Definition 3.4. Let $\{x_i\}_{i=1}^{\infty}$ be a 2-frame associated to ξ with bounds A and B in a 2-Hilbert space \mathcal{X} . The operator $S_{\xi} : \mathcal{X}_{\xi} \rightarrow \mathcal{X}_{\xi}$ defined by

$$(3.4) \quad S_{\xi}x = \sum_{i=1}^{\infty} (x, x_i|_{\xi})x_i$$

is called the 2-frame operator for $\{x_i\}_{i=1}^{\infty}$.

Clearly, $S_{\xi} = T_{\xi}T_{\xi}^*$ and therefore $\|S_{\xi}\| \leq B$. We can conclude the boundedness of S_{ξ} directly. Indeed, we see from (I3),(I4),(I5) and (3.1) that

$$\begin{aligned} \|S_{\xi}x\|_{\xi}^2 &= \|S_{\xi}x, \xi\|^2 \\ &= \sup\{|(S_{\xi}x, y|_{\xi})|^2, \quad y \in \mathcal{X}, \|y, \xi\| = 1\} \\ &\leq \sup\{\sum_{i=1}^{\infty} |(x, x_i|_{\xi})|^2 \sum_{i=1}^{\infty} |(y, x_i|_{\xi})|^2, \quad y \in \mathcal{X}, \|y, \xi\| = 1\} \\ &\leq B^2\|x\|_{\xi}^2. \end{aligned}$$

Now we state some of the important properties of S_{ξ} .

Theorem 3.5. Let $\{x_i\}_{i=1}^{\infty}$ be a 2-frame associated to ξ for a 2-Hilbert space $(\mathcal{X}, (\cdot, \cdot|_{\xi}))$ with 2-frame operator S_{ξ} and frame bounds A, B . Then S_{ξ} is invertible, self-adjoint, and positive.

Proof. Obviously, the operator S_{ξ} is self-adjoint. The inequality (2.2) means that

$$A\|x\|_{\xi}^2 \leq \langle S_{\xi}x, x \rangle_{\xi} \leq B\|x\|_{\xi}^2, \quad (x \in \mathcal{X}_{\xi}).$$

Hence, S_{ξ} is a positive element in the set of all bounded operators on the Hilbert space \mathcal{X}_{ξ} . More precisely, with symbols $AI \leq S_{\xi} \leq BI$ where I is the identity operator on \mathcal{X}_{ξ} . Furthermore,

$$\|I - B^{-1}S_{\xi}\| = \sup_{\|x\|_{\xi}=1} |\langle (I - B^{-1}S_{\xi})x, x \rangle_{\xi}| \leq \frac{B - A}{B} < 1.$$

This shows that S_{ξ} is invertible. \square

Corollary 3.6. Let $\{x_i\}_{i=1}^{\infty}$ be a 2-frame in a 2-Hilbert space \mathcal{X} with frame operator S_{ξ} . Then each $x \in \mathcal{X}_{\xi}$ has an expansion of the following

$$x = S_{\xi}S_{\xi}^{-1}x = \sum_{i=1}^{\infty} (S_{\xi}^{-1}x, x_i|_{\xi})x_i.$$

Remark 3.7. If $\{x_i\}_{i=1}^{\infty}$ is a 2-frame associated to ξ , then every $x \in \mathcal{X}$ has a representation as

$$x = \alpha\xi + \sum_{i=1}^{\infty} c_i x_i,$$

for some $\alpha \in \mathbb{C}$ and $\{c_i\}_{i=1}^{\infty} \in l^2$. The coefficients $\{c_i\}_{i=1}^{\infty}$ are not unique, but the frame coefficients $\{(S_{\xi}^{-1}x, x_i|\xi)\}_{i=1}^{\infty}$ introduced in the Corollary 3.6 have minimal l^2 -norm among all sequences representing x , see Lemma 5.3.6 of [9].

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