

Application of He's homotopy perturbation method for Schrodinger equation

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ABSTRACT. In this paper, He's homotopy perturbation method is applied to solve linear Schrodinger equation. The method yields solutions in convergent series forms with easily computable terms. The result show that these method is very convenient and can be applied to large class of problems. Some numerical examples are given to effectiveness of the method.

Keywords: He's homotopy perturbation method, Linear Schrodinger equations.

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1. INTRODUCTION

The Schrodinger equation has been widely used in various application areas, e. g, quantum mechanics, optics, seismology and plasma physics. Since analytic approaches to the Schrodinger equation have limited applicability science and engineering problems, there is a growing interest in exploring new methods to solve the equation more accurately and efficiently. The homotopy

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perturbation method is a new approach which searches for an analytical approximate solution of the linear and nonlinear problems. This method was proposed first by J. Huan He in 1998 and was further developed and improved by him[1,2,3]. The homotopy perturbation method is, in fact, a coupling of the perturbation method and homotopy in topology[4]. This method are based on an assumption that a small parameter must exist. The determination of small parameters seems to be a special art requiring special techniques. An appropriate choice of small parameters leads to ideal results. Here parameter takes the values from 0 to 1. If parameter is 0 , the problem usually reduce to a sufficiently simplified form, which normally admits a rather simple solution. As parameter gradually increases to 1 , the problem goes through a sequence of deformation. At 1 , the problem takes the original form and the final stage of deformation gives the desired solution. The homotopy perturbation method has been applied to Volterra's integro differential equation[5], nonlinear oscillators[6], bifurcation of nonlinear problems[7], bifurcation of delay differential equations[8], nonlinear wave equations[9], boundary value problems[10] and to other fields[11,12]. In this paper, we will apply the He's homotopy perturbation method to linear Schrodinger equation to show the simplicity and straightforwardness of the method. The outline of this paper is as follows: in Section 2 details of He's homotopy perturbation method are explained and we show how it can be applied to Schrodinger equation. Section 3 presents our numerical results for the various cases of the problem. The last section is conclusion.

2. HOMOTOPY PERTURBATION METHOD

In this section, we illustrate the basic ideas of the method, then we apply it for linear Schrodinger equation. Consider the following nonlinear differential equation:

$$(1) \quad L(u) + N(u) = f(\mathbf{r}), \quad \mathbf{r} \in \Omega,$$

with boundary conditions:

$$(2) \quad B(u, \frac{\partial u}{\partial n}) = 0, \quad \mathbf{r} \in \Gamma,$$

where L is a linear operator, N is a non-linear operator, $f(\mathbf{r})$ is a known analytical function, B is a boundary operator and Γ is the boundary of the domain Ω .

By using homotopy technique, one can construct a homotopy $U(\mathbf{r}, p) : \Omega \times [0, 1] \rightarrow \mathbf{R}$ which satisfies:

$$(3) \quad H(U, p) = (1-p)[L(U) - L(u_0)] + p[L(U) + N(U) - f(\mathbf{r})] = 0, \quad p \in [0, 1],$$

or:

$$(4) \quad H(U, p) = L(U) - L(u_0) + pL(u_0) + p[N(U) - f(\mathbf{r})] = 0,$$

where $p \in [0, 1]$ is an embedding parameter, and u_0 is the initial approximation of Eq.(1) which satisfies the boundary conditions. Clearly, we have:

$$(5) \quad H(U, 0) = L(U) - L(u_0) = 0,$$

$$(6) \quad H(U, 1) = L(U) + N(U) - f(\mathbf{r}) = 0.$$

The changing process of p from zero to unity is just that of $U(\mathbf{r}, p)$ changing from $u_0(\mathbf{r})$ to $u(\mathbf{r})$. This is called deformation, and also $L(U) - L(u_0)$ and $L(U) + N(U) - f(\mathbf{r})$ are called homotopy in topology. If the embedding parameter $p(0 \leq p \leq 1)$ is considered as a "small parameter", applying the classical perturbation technique, we can assume that the solution of Eq.(3) or (4) can be given as a power series in p , i.e.,

$$(7) \quad U = u_0 + pu_1 + p^2u_2 + \dots,$$

and setting $p = 1$ result in the approximate solution of Eq(1) as:

$$(8) \quad u = \lim U = u_0 + u_1 + u_2 + \dots$$

It is worth to note that the major advantage of He's homotopy perturbation method is that the perturbation equation can be freely constructed in many ways (therefor is problem dependent) by homotopy in topology and the initial approximation can also be freely selected.

In this paper, the linear Schrodinger equation is considered as follows:

$$(9) \quad \frac{\partial u}{\partial t} + i \frac{\partial^2 u}{\partial x^2} = 0, \quad u(x, 0) = f(x), \quad i^2 = -1,$$

according to the He's homotopy perturbation method, we have from Eq.(9):

$$(1-p) \left[\frac{\partial u}{\partial t} - \frac{\partial u_0}{\partial t} \right] + p \left[\frac{\partial u}{\partial t} + i \frac{\partial^2 u}{\partial x^2} \right] = 0, \quad p \in [0, 1],$$

or:

$$(10) \quad \frac{\partial u}{\partial t} - \frac{\partial u_0}{\partial t} + p \left[\frac{\partial u_0}{\partial t} + i \frac{\partial^2 u}{\partial x^2} \right] = 0,$$

Suppose the solution of Eq.(10) to be as (7). Substituting(7) into Eq.(10), and equating the terms with identical powers of p , we can obtain a series of linear

equations of the form as follows:

$$\begin{aligned}
 (11) \quad p^0 & : \frac{\partial u_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0, \\
 p^1 & : \frac{\partial u_1}{\partial t} + \frac{\partial u_0}{\partial t} + i \frac{\partial^2 u_0}{\partial x^2} = 0, \quad u_1(x, 0) = 0, \\
 p^2 & : \frac{\partial u_2}{\partial t} + i \frac{\partial^2 u_1}{\partial x^2} = 0, \quad u_2(x, 0) = 0, \\
 & \cdot \\
 & \cdot \\
 & \cdot \\
 p^k & : \frac{\partial u_k}{\partial t} + i \frac{\partial^2 u_{k-1}}{\partial x^2} = 0, \quad u_k(x, 0) = 0, \\
 & \cdot \\
 & \cdot \\
 & \cdot
 \end{aligned}$$

The approximate solution of Eq(9) can be obtained by setting $p = 1$:

$$(12) \quad u = \lim U = u_0 + u_1 + u_2 + \dots$$

3. EXAMPLES

This section is devoted to computational results. We applied the method presented in this paper and solved some examples.

Example 3.1 Consider the following linear Schrodinger equation :

$$(13) \quad \frac{\partial u}{\partial t}(x, t) + i \frac{\partial^2 u}{\partial x^2}(x, t) = 0,$$

$$u_0(x) = 1 + \cosh 2x, x \in R.$$

According to He's homotopy perturbation method, we construct the following homotopy:

$$(14) \quad \frac{\partial u}{\partial t} - \frac{\partial u_0}{\partial t} + p \left[\frac{\partial u_0}{\partial t} + i \frac{\partial^2 u}{\partial x^2} \right] = 0,$$

substituting (7) into (14), we get following set of differential equations:

$$(15) \quad \begin{aligned} p^0 & : \frac{\partial u_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0, & u_0(x) &= 1 + \cosh 2x, \\ p^1 & : \frac{\partial u_1}{\partial t} + \frac{\partial u_0}{\partial t} + i \frac{\partial^2 u_0}{\partial x^2} = 0, & u_1(x, 0) &= 0, \\ p^2 & : \frac{\partial u_2}{\partial t} + i \frac{\partial^2 u_1}{\partial x^2} = 0, & u_2(x, 0) &= 0, \\ & \cdot \\ & \cdot \\ & \cdot \end{aligned}$$

solving the systems accordingly, thus we obtain:

$$\begin{aligned} u_0(x) &= 1 + \cosh 2x, \\ u_1(x) &= (-4it) \cosh 2x, \\ u_2(x) &= \frac{(-4it)^2}{2!} \cosh 2x, \\ u_3(x) &= \frac{(-4it)^3}{3!} \cosh 2x, \\ &\cdot \\ &\cdot \\ &\cdot \end{aligned}$$

By setting $p = 1$ in Eq.(7), the solution of (13) can be obtained as:

$$(16) \quad U = u_0 + u_1 + u_2 + \dots$$

Thus we have:

$$U(x, t) = 1 + \cosh 2x \left(1 + (-4it) + \frac{(-4it)^2}{2!} + \frac{(-4it)^3}{3!} + \dots \right).$$

In the limit of infinitely many terms, yields the closed-form solution:

$$u(x, t) = 1 + e^{-4it} \cosh 2x,$$

which is the exact solution.

Example 3.2 Consider the following linear Schrodinger equation :

$$(17) \quad \begin{aligned} \frac{\partial u}{\partial t}(x, t) + i \frac{\partial^2 u}{\partial x^2}(x, t) &= 0, \\ u_0(x) &= e^{3ix}. \end{aligned}$$

According to He's homotopy perturbation method, we construct the following homotopy:

$$(18) \quad \frac{\partial u}{\partial t} - \frac{\partial u_0}{\partial t} + p \left[\frac{\partial u_0}{\partial t} + i \frac{\partial^2 u}{\partial x^2} \right] = 0,$$

substituting (7) into (18), we get following set of differential equations:

$$\begin{aligned}
 p^0 & : \frac{\partial u_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0, \quad u_0(x) = e^{3ix}, \\
 p^1 & : \frac{\partial u_1}{\partial t} + \frac{\partial u_0}{\partial t} + i \frac{\partial^2 u_0}{\partial x^2} = 0, \quad u_1(x, 0) = 0, \\
 p^2 & : \frac{\partial u_2}{\partial t} + i \frac{\partial^2 u_1}{\partial x^2} = 0, \quad u_2(x, 0) = 0, \\
 & \cdot \\
 & \cdot \\
 & \cdot
 \end{aligned}
 \tag{19}$$

solving the systems accordingly, thus we obtain:

$$\begin{aligned}
 u_0(x) & = e^{3ix}, \\
 u_1(x) & = 9ite^{3ix}, \\
 u_2(x) & = -\frac{81}{2}t^2e^{3ix}, \\
 u_3(x) & = -\frac{243}{2}it^3e^{3ix}, \\
 & \cdot \\
 & \cdot \\
 & \cdot
 \end{aligned}$$

By setting $p = 1$ in Eq.(7), the solution of (17) can be obtained as:

$$U = u_0 + u_1 + u_2 + \dots
 \tag{20}$$

Thus we have:

$$U(x, t) = e^{3ix} \left(1 + (9it) + \frac{(9it)^2}{2!} + \frac{(9it)^3}{3!} + \dots \right).$$

In the limit of infinitely many terms, yields the closed-form solution,

$$u(x, t) = e^{3i(x+3t)},$$

which is the exact solution.

CONCLUSIONS

In this paper, He's homotopy perturbation method has been successfully applied to find the solution of the linear Schrodinger equation. The method is reliable and easy to use. The main advantage of the method is the fact that it provides its user with an analytical approximation, in many cases an exact solution, in a rapidly convergent sequence with elegantly computed term.

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