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Sum Formula for Maximal Abstract Monotonicity and Abstract Rockafellar's Surjectivity Theorem

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Abstract. In this paper, we present an example in which the sum of two maximal abstract monotone operators is maximal. Also, we shall show that the necessary condition for Rockafellar's surjectivity which was obtained in ([19], Theorem 4.3) can be sufficient.

Keywords: Monotone operator, Abstract monotonicity, Abstract convex function, Abstract convexity, Rockafellar's surjectivity theorem.

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1. Introduction

Abstract convexity has found many applications in the study of problems of mathematical analysis and optimization. Abstract convexity has mainly been used for the study of point-to-point functions. Examples of its use in the analysis of multifunctions can be found in [3, 13, 14, 24].
Several approaches to the theory of monotone multifunctions have established links between maximal monotone multifunctions and convex functions (see

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[4, 5, 9, 12, 15, 16, 23, 33, 34]). The richness of the theory of monotone operators has given rise to a great number of works and the simplification of proofs and theory that has resulted from the use of convex analysis techniques justifies an interest in these links.

The theory of monotone multifunctions have found many applications in optimization and variational analysis [1, 11]. Let X be a real Banach space and X^* be the dual space of X. Denote by $\langle .,. \rangle$ the duality product between X and X^*.

Rockafellar in [26] proved that subdifferentials of proper lower semi-continuous convex functions on X are maximal monotone. In general, maximal monotone operators are not subdifferentials of convex functions. Krauss in [12] managed to represent maximal monotone operators by subdifferentials of saddle functions on $X \times X$. After that, Fitzpatrick [9] proved that the family

$$\mathcal{H}(A) := \{h \in \Gamma(X \times X^*);\ h(x,x^*) \geq \langle x, x^* \rangle\ \forall\ (x,x^*),\ h(x,x^*) = \langle x, x^* \rangle\ \forall (x,x^*) \in G(A)\}$$

is non-empty, where $A : X \longrightarrow 2^{X^*}$ is an arbitrary maximal monotone operator and $\Gamma(X \times X^*)$ is the set of all lower semi continuous convex functions $h : X \times X^* \longrightarrow (-\infty, +\infty]$. He defined the function $\varphi_A : X \times X^* \longrightarrow (-\infty, +\infty]$ by

$$\varphi_A(x,x^*) := \sup_{(y,y^*) \in G(A)} \langle x - y, y^* - x^* \rangle + \langle x, x^* \rangle, \quad \forall\ (x,x^*) \in X \times X^*,$$

and showed that $\varphi_A \in \mathcal{H}(A)$. It is worth noting that φ_A is called the Fitzpatrick function and moreover φ_A represents A, that is, $\varphi_A \in \mathcal{H}(A)$.

In a recent paper, Martínez-Legaz and Théra [16] rediscovered the Fitzpatrick function associated to maximal monotone operators and characterized the family

$$\{\varphi_A : A : X \longrightarrow 2^{X^*} \textit{ is a maximal monotone operator}\}.$$

In [4] Burachik and Svaiter also rediscovered Fitzpatrick functions and studied the whole family of lower semi-continuous convex functions associated with a given maximal monotone operator A, that is, those functions $h \in \mathcal{H}(A)$. They proved that this family is invariant under a suitable generalized conjugation operator and has a biggest element. Recently, Martínez-Legaz and Svaiter [15] extended the representation of maximal monotone operators by lower semi-continuous convex functions to a larger class of monotone operators. They showed that, in the finite-dimensional case, the class of representable operators is the one consisting of the intersections of maximal monotone operators.

In the sequel, we present a citation of the available literature in the topic of abstract convexity.

The theory of Fenchel conjugation and subdifferentials plays a central role in convex analysis. Fenchel's theorem on the second conjugate and duality for the sum of two convex functions, and the Fenchel-Rockafellar's theorem on the sum of the subdifferentails have substantially influenced the development of convex analysis and its applications in various ways. For instance, Fenchel's duality theorem, which states an equality between the minimization of a sum of two convex functions and the maximization of the sum of concave functions, using conjugates, is fundamental to the study of convex optimization.

In 1970, Moreau [21] observed that Fenchel conjugation and the second conjugation theorem can be established in a very general setting, using two arbitrary sets and arbitrary coupling functions. The second conjugation theorem in this setting, known as Fenchel-Moreau theorem, has given rise to the rich theory of abstract convexity (see [22, 27, 32]). Moreover, extensions of Fenchel's duality theorem and Fenchel-Rockafellar's theorem, which have played key roles in the application of convex analysis, have been presented for abstract convex functions in [10].

Abstract convexity has found many applications in mathematical analysis and optimization. Also, it has found interesting applications to the theory of inequalities (see [27]). Abstract convexity opens the way for extending some main ideas and results from classical convex analysis to much more general classes of functions, mappings and sets. It is well-known that every convex, proper and lower semicontinuous function is the upper envelope of a set of affine functions. Therefore, affine functions play a crucial role in classical convex analysis. In abstract convexity, the role of the set of affine functions is taken by an alternative set H of functions, and their upper envelopes constitute the set of abstract convex functions. Different choices of the set H generate variants of the classical concepts, and have shown important applications in global optimization (see [28, 29, 30, 31]). Moreover, if a family of functions is abstract convex for a specific choice of H, then we can use some key ideas of convex analysis in order to gain new insight on these functions. On the other hand, by using an alternative set for affine functions, we indentify those facts in classical convex analysis which depend on the specific properties of affine functions.

Abstract convexity has mainly been used for the study of point-to-point functions. Examples of its use in the analysis of multifunctions can be found in works of Levin [13, 14], who focused in the study of abstract cyclical monotonicity, and also, Penot [24] by using a framework of generalized convexity showed the existence of a convex representation of a maximal monotone operator by a convex function which is invariant with respect to the Fenchel conjugacy. Recently, Burachik and Rubinov [3] studied semi-continuity properties of abstract monotone operators. Roughly speaking the study of monotone operators reduces to the study of the convexification of the coupling functions, restricted to monotone sets. Convexity is sometimes a restrictive assumption, and therefore based on the works [3, 5, 14, 15, 23, 24, 34] the problem arises to generalize the theory of monotone operators via abstract convexity.

Recently, a theory of monotone operators has been developed in the framework of abstract convexity (see [8, 18, 19]). Indeed, in [8] a generalization of Fenchel duality theorem in the framework of abstract convexity and also in [19] some criteria for maximal abstract monotonicity have been given. The Rockafellar's surjectivity theorem is one of the most important results to investigate maximal monotone operators in reflexive Banach spaces. The necessary condition of Rockafellar's surjectivity theorem has been extended to abstract convex framework in ([19], Theorem 4.3). In this paper, we are going to show that this condition can be also sufficient. On the other words, although the "abstract duality pairing" (which is defined by (2.2)) is symmetric, the theory presented in this paper would specialize to if we consider the set of all continuous linear functionals as the set of elementary functions.

One of the most important questions in this environment is when the sum of two maximal abstract monotone operators is maximal. In general, this is not true even in the classical sense without a qualification assumption. Some results for the sum formula in reflexive Banach spaces have been shown in [2, 7, 25]. In this paper, we shall show that there is an example, which shows that the sum of two maximal abstract monotone operators is also maximal.

The structure of this paper is as follows: In section 2, we provide some preliminary definitions and results related to abstract convexity and abstract monotonicity. In section 3, we give an example in which the sum of two maximal abstract monotone operators is maximal. Finally, in section 4, we prove that the necessary condition in ([19], Theorem 4.3) is also sufficient. In fact, we show that Rockafellar's surjectivity holds in the framework of abstract monotonicity.

2. Preliminaries

Let X and Y be two sets. Recall (see [6]) that a set valued mapping (multifunction) from X to Y is a mapping $F : X \longrightarrow 2^Y$, where 2^Y represents the collection of all subsets of Y. We define the domain and graph of F by

$$\text{dom }(F) := \{x \in X : F(x) \neq \emptyset\},$$

and

$$G(F) := \{(x, y) \in X \times Y : y \in F(x)\},$$

respectively. The inverse of F is the set valued mapping $F^{-1} : Y \longrightarrow 2^X$ defined by

$$F^{-1}(y) := \{x \in X : y \in F(x)\}.$$

Now, let X be a set and L be a set of real valued functions $l : X \longrightarrow \mathbb{R}$, which will be called abstract linear. For each $l \in L$ and $c \in \mathbb{R}$, consider the shift $h_{l,c}$ of l on the constant c :

$$h_{l,c}(x) := l(x) - c, \quad (x \in X).$$

The function $h_{l,c}$ is called L-affine. Recall (see [27]) that the set L is called a set of abstract linear functions if $h_{l,c} \notin L$ for all $l \in L$ and all $c \in \mathbb{R} \setminus \{0\}$. The set of all L-affine functions will be denoted by H_L. If L is the set of abstract linear functions, then $h_{l,c} = h_{l_0,c_0}$ if and only if $l = l_0$ and $c = c_0$.

If L is a set of abstract linear functions, then the mapping $(l, c) \longrightarrow h_{l,c}$ is a one-to-one correspondence. In this case, we identify $h_{l,c}$ with (l, c), in other words, we consider an element $(l, c) \in L \times \mathbb{R}$ as a function defined on X by $x \longrightarrow l(x) - c$ $(x \in X)$.

A function $f : X \longrightarrow (-\infty, +\infty]$ is called proper if $\text{dom} f \neq \emptyset$, where $\text{dom} f$ is defined by

$$\text{dom} f := \{x \in X : f(x) < +\infty\}.$$

Let $\mathcal{F}(X)$ be the set of all functions $f : X \longrightarrow (-\infty, +\infty]$ and the function $-\infty$.

Recall (see [27]) that a function $f \in \mathcal{F}(X)$ is called H-convex ($H = L$, *or* $H = H_L$) if

$$f(x) = \sup\{h(x) : h \in \text{supp } (f, H)\}, \quad \forall\, x \in X,$$

where

$$\text{supp}\ \ (f, H) := \{h \in H : h \leq f\}$$

is called the support set of the function f, and $h \leq f$ if and only if $h(x) \leq f(x)$ for all $x \in X$.
Note that if X is a locally convex Hausdorff topological vector space and L is the set of all real valued continuous linear functionals defined on X, then $f : X \longrightarrow (-\infty, +\infty]$ is an L-convex function if and only if f is lower semi-continuous and sublinear. Also, f is an H_L-convex function if and only if f is lower semi-continuous and convex.
Now, we consider the coupling function $\langle ., . \rangle : X \times L \longrightarrow \mathbb{R}$ defined by $\langle x, l \rangle := l(x)$ for all $x \in X$ and all $l \in L$. For a function $f \in \mathcal{F}(X)$, define the Fenchel-Moreau L-conjugate f_L^* of f (see [27]) by

$$f_L^*(l) := \sup_{x \in X}(l(x) - f(x)), \quad l \in L.$$

Similarly, the Fenchel-Moreau X-conjugate g_X^* of an extended real valued function g defined on L is given by

$$g_X^*\ (x) := \sup_{l \in L}(l(x) - g(l)), \quad x \in X.$$

The function $f_{L,X}^{**} := (f_L^*)_X^*$ is called the second conjugate (or biconjugate) of f, and by definition we have

$$f_{L,X}^{**}(x) := \sup_{l \in L}(l(x) - f_L^*(l)), \quad x \in X.$$

A set $C \subset \mathcal{F}(X)$ is called additive if for $f_1,\ f_2 \in C$, then $f_1 + f_2 \in C$.
If X is a set on which an addition $+$ is defined, then we say that a function $f \in \mathcal{F}(X)$ is additive if

$$f(x + y) = f(x) + f(y), \quad \forall\ x,\ y \in X.$$

Let $f : X \longrightarrow (-\infty, +\infty]$ be a function and $x_0 \in dom f$. Recall (see [27]) that an element $l \in L$ is called an L-subgradient of f at x_0 if

$$f(x) \geq f(x_0) + l(x) - l(x_0), \quad \forall\ x \in X.$$

The set $\partial_L f(x_0)$ of all L-subgradients of f at x_0 is called the L-subdifferential of f at x_0. The subdifferential $\partial_L f(x_0)$ (see [[27], Proposition 1.2]) is non-empty if and only if $x_0 \in dom f$ and

$$f(x_0) = \max\{h(x_0) : h \in \text{supp}\ \ (f, H_L)\}.$$

Now, assume that X is a set and L is a set of real valued abstract linear functions $l : X \longrightarrow \mathbb{R}$ defined on X, with the coupling function $\langle ., . \rangle : X \times L \longrightarrow \mathbb{R}$ defined by $\langle x, l \rangle := l(x)$ for all $x \in X$ and all $l \in L$. In the following, we present some definitions and properties of abstract monotone operators (see [8, 18, 14, 24]).
(i) A set valued mapping $T : X \longrightarrow 2^L$ is called L-monotone operator (or, abstract monotone operator) if

$$l(x) - l(x') - l'(x) + l'(x') \geq 0 \tag{2.1}$$

for all $x,\ x' \in X$ and all $l \in Tx, l' \in Tx'$.
It is worth to note that if X is a Banach space with dual space X^* and $L := X^*$, then T is a monotone operator in the classical sense.

(ii) A set valued mapping $T: X \longrightarrow 2^L$ is called maximal L-monotone (or maximal abstract monotone) if T is L-monotone and $T = T'$ for any L-monotone operator $T': X \longrightarrow 2^L$ such that $G(T) \subseteq G(T')$.

In the following, we present some results which have been obtained in [8].
Let X be a set with an operation $+$ having the following properties:
(A_1) $x + y \in X, \quad \forall\, x,\ y \in X.$
(A_2) *There exists a unique element* $0 \in X$ *such that* $0 + x = x + 0 = x, \quad \forall\, x \in X.$
(A_3) *For each* $x \in X$ *there exists a unique element* $-x \in X$ *such that* $x + (-x) = (-x) + x = 0.$
Let L be a set of real valued additive abstract linear functions defined on X. Assume that L is equipped with the point-wise operation $+$ of functions such that $(L, +)$ satisfies the properties (A_1), (A_2) and (A_3), where for each $l \in L$, define $(-l)(x) := -l(x)$ for all $x \in X$, and define the function $0 \in L$ by $0(x) := 0$ for all $x \in X$. We consider the coupling function $\langle ., . \rangle : X \times L \longrightarrow \mathbb{R}$ defined by $\langle x, l \rangle := l(x)$ for all $x \in X$ and all $l \in L$.

Remark 2.1. *Note that for each* $l \in L$, *we have* $l(0) = 0$. *Moreover,* $l(-x) = -l(x)$ *for all* $x \in X$ *and all* $l \in L$. *Indeed, assume that* $l \in L$ *and* $x \in X$ *are arbitrary. Then*

$$0 = l(0) = l(x + (-x)) = l(x) + l(-x),$$

and hence $l(-x) = -l(x)$ *for all* $x \in X$ *and all* $l \in L$.

Let $K \subseteq X \times L$ be any non-empty set such that K satisfies the properties (A_1), (A_2) and (A_3), where $-(x, l) := (-x, -l)$ and $0 := (0, 0) \in K$. Define $L^* := \{(l, x) \in L \times X : (x, l) \in K\} \subseteq L \times X$. It is clear that L^* satisfies the properties (A_1), (A_2) and (A_3). Define the coupling function $\langle ., . \rangle_* : K \times L^* \longrightarrow \mathbb{R}$ by

$$\langle (x', l'), (l, x) \rangle_* := l(x') + l'(x), \quad \forall\, (x', l') \in K;\ \forall\, (l, x) \in L^*. \tag{2.2}$$

One can consider an element $(l, x) \in L^*$ as a function defined on K by

$$(l, x)(x', l') := \langle (x', l'), (l, x) \rangle_*, \quad \forall\, (x', l') \in K,$$

and an element $(x, l) \in K$ as a function defined on L^* by

$$(x, l)(l', x') := \langle (x, l), (l', x') \rangle_*, \quad \forall\, (l', x') \in L^*.$$

Note that the coupling function $\langle ., . \rangle_*$ is symmetric, that is

$$\langle (x', l'), (l, x) \rangle_* = \langle (x, l), (l', x') \rangle_*, \quad \textit{for all } (x', l') \in K, \textit{ and all } (l, x) \in L^*.$$

It is easy to check that L^* and K are sets of real valued abstract linear functions. Indeed, if there exist $(l_0, x_0) \in L^*$ and $c_0 \in \mathbb{R} \setminus \{0\}$ such that $h_{(l_0, x_0), c_0} \in L^*$, where $h_{(l_0, x_0), c_0} := (l_0, x_0) - c_0$, then $h_{(l_0, x_0), c_0} = (l, x)$ for some $(l, x) \in L^*$. It follows that

$$l_0(x') + l'(x_0) - c_0 = l(x') + l'(x), \quad \forall\, (x', l') \in K. \tag{2.3}$$

Since $(0, 0) \in K$, put $x' = 0$ and $l' = 0$ in (2.3). Thus, we have $c_0 = 0$. This is a contradiction, because $c_0 \neq 0$. Hence, $h_{(l, x), c} \notin L^*$ for all $(l, x) \in L^*$ and all $c \in \mathbb{R} \setminus \{0\}$. Therefore, L^* is a set of abstract linear functions. By a similar argument, K is also a set of abstract linear functions.

Denote by

$$\mathcal{P}(H_{L^*}) := \{h : K \longrightarrow (-\infty, +\infty] : h \textit{ is a proper } H_{L^*} - \textit{convex function}\}$$

the set of all proper H_{L^*}-convex functions defined on K. Define the transpose operator $t : K \longrightarrow L^*$ by $t(x,l) := (l,x)$ for all $(x,l) \in K$.
In the sequel, we shall use the following assumption which has been introduced in [8].
Assumption (D): Assume that there exists a function $\gamma \in \mathcal{P}(H_{L^*})$ such that
(i) $0 \leq \gamma < +\infty$ on K,
(ii) $\gamma^*_{L^*} \circ t = \gamma$ on K,
(iii) $\langle ., . \rangle + \gamma \geq 0$ on K,
(iv) If $\gamma(x,l) = 0$, then $(x,l) = (0,0)$.
(v) $\gamma(-(x,l)) = \gamma(x,l)$ for all $(x,l) \in K$.
Notice that, in the case when X is a Banach space with the dual space X^* and $L := X^*$, the function γ defined by $\gamma(x,x^*) := \frac{1}{2}\left(\|x\|^2 + \|x^*\|^2\right)$ satisfies the Assumption (D). There are examples of function γ which satisfies Assumption (D) in the case of abstract convexity, for more details see [8].

3. Some Results on Abstract Monotonicity

In this section, we give an example in which the sum of two maximal abstract monotone operators is also maximal. Indeed, It has been shown in [18] that the abstract subdifferentials of IPH functions are maximal abstract monotone operators. We shall show that the sum of two abstract subdifferentials of IPH functions is maximal.
Let X be a topological vector space. We assume that X is equipped with a closed convex pointed cone S (the latter means that $S \cap (-S) = \{0\}$). We say $x \leq y$ or $y \geq x$ if and only if $y - x \in S$.
Recall that the function $p : X \longrightarrow [-\infty, +\infty]$ is IPH if p is an increasing and positively homogeneous function (the latter means that $p(\lambda x) = \lambda p(x)$ for all $x \in X$ and all $\lambda > 0$).
Now, consider the function $l : X \times X \longrightarrow [0, +\infty]$ defined by

$$l(x,y) := \max\{\lambda \geq 0 : \lambda y \leq x\}, \quad (x,\ y \in X),$$

(with the convention $\max \emptyset := 0$).
This function was introduced and examined in [20]. The following results for the function l have been proved in ([20], Proposition 3.1). In fact, for every $x,\ y,\ x',\ y' \in X,\ \textit{and}\ \gamma > 0$, one has

$$l(\gamma x, y) = \gamma l(x,y), \tag{3.1}$$

$$l(x, \gamma y) = \frac{1}{\gamma} l(x,y), \tag{3.2}$$

$$l(x,y) = +\infty \implies y \in -S, \tag{3.3}$$

$$l(x,x) = 1 \iff x \notin -S, \tag{3.4}$$

$$x \in S,\ y \in -S \implies l(x,y) = +\infty, \tag{3.5}$$

$$x \leq x' \implies l(x,y) \leq l(x',y), \tag{3.6}$$

$$y \leq y' \implies l(x,y) \geq l(x,y'). \tag{3.7}$$

Let $L_S := \{l_y \ : \ y \in X\backslash(-S)\}$, where for each $y \in X\backslash(-S)$, define $l_y : X \longrightarrow [0, +\infty]$ by $l_y(x) := l(x, y)$ for all $x \in X$. Note that l_y is an IPH function for each $y \in X$, and every non-negative IPH function is L_S-convex. Also, one has supp $(p, L_S) = \{l_y \in L_S : \ p(y) \geq 1\}$ (for more details see [20]). By the next lemma, $\partial_{L_S} p$ is characterized.

Lemma 3.1. *(*[17]*, Theorem 3.3) Let $p : X \to [0, +\infty]$ be an IPH function and $p(x) \neq 0, +\infty$, then*

$$\partial_{L_S} p(x) = \{l_y \in L_S \quad : \quad l_y(x) = p(x), \quad p(y) = 1\}.$$

Consider the set $L = L_S \cup \{0\}$. Trivially, if $p : X \to [0, +\infty]$ is an IPH function, then p is L-convex. Also, $\partial_L p(x) \neq \emptyset$ for every $x \in X$ with $p(x) \neq +\infty$. Indeed, assume that $p(x) \neq 0$ (note that in this case we have $p(x) > 0$, and hence $x \notin -S$). Then, by (3.2), (3.4) and Lemma 3.1, $l_{\frac{x}{p(x)}} \in \partial_L p(x)$. If $p(x) = 0$, then $0 \in \partial_L p(x)$.

In the following, we show that $\partial_L p$ is a maximal L-monotone operator, although the proof is similar to that of Theorem 3.1 in [18], which is obtained with respect to the set L_S.

Theorem 3.2. *Let $p : X \to [0, +\infty)$ be an IPH function and $L = L_S \cup \{0\}$. Then, $\partial_L p$ is a maximal L-monotone operator.*

Proof. First, we are going to show that $\partial_L p$ is L-monotone. To this end, consider $(x, l), (x_0, l_0) \in G(\partial_L p)$. It is easy to see that $\partial_L p(x) = \{l \in L \ : \ l(x) = p(x), \ l(t) \leq p(t), \ \forall \, t \in X\}$. So we have

$$l(t) \leq p(t), \ l(x) = p(x); \quad l_0(t) \leq p(t), \ l_0(x_0) = p(x_0) \ (\forall \, t \in X).$$

Therefore,

$$l(x) - l(x_0) - l_0(x) + l_0(x_0) = (p(x) - l_0(x)) + (p(x_0) - l(x_0)) \geq 0.$$

Hence, $\partial_L p$ is an L-monotone operator.
For maximality of $\partial_L p$, suppose that (x_0, l_0) is monotonically related to $G(\partial_L p)$, so

$$l(x) - l(x_0) - l_0(x) + l_0(x_0) \geq 0 \quad (\forall \, l \in \partial_L p(x), \ \forall \, x \in X). \tag{3.8}$$

Let $\lambda > 0$ and $x = \lambda x_0$. Then, as $\lambda \to +\infty$, we obtain $l_0(x_0) \leq l(x_0)$. If $\lambda \to 0$, we conclude that $l_0(x_0) \geq l(x_0)$. Therefore, $l(x_0) = l_0(x_0)$ for each $l \in \partial_L p(x_0)$, and hence, since $l(x_0) = p(x_0)$, we deduce that $l_0(x_0) = p(x_0)$.
On the other hand, let $x \in X$ be arbitrary. Replace x by λx in (3.8), so as $\lambda \to +\infty$ we get

$$l_0(x) \leq l(x) \quad (\forall \, l \in \partial_L p(x), \ \forall \, x \in X).$$

Since $l(x) \leq p(x)$ for all $x \in X$ and all $l \in \partial_L p(x)$), we can obtain $l_0(x) \leq p(x)$. This, together with the fact that $l_0(x_0) = p(x_0)$ implies that $l_0 \in \partial_L p(x_0)$, which completes the proof. □

In the following, we investigate the main result of this section.

Theorem 3.3. *Let $p_1, p_2 : X \to [0, +\infty)$ be two IPH functions and $L = L_S \cup \{0\}$. Then $\partial_L p_1 + \partial_L p_2$ is a maximal L-monotone operator.*

Proof. First, we claim that $\partial_L(p_1+p_2)(x_0) \subset \partial_L p_1(x_0) + \partial_L p_2(x_0)$ for all $x_0 \in X$. Let $l \in \partial_L(p_1+p_2)(x_0)$ and $l \neq 0$. So there exists $l_{y_0} \in L_S$ such that $l = l_{y_0}$. Assume that $(p_1+p_2)(x_0) \neq 0$, it follows from Lemma 3.1 that $(p_1+p_2)(x_0) = l_{y_0}(x_0)$ and $(p_1+p_2)(y_0) = 1$.
Now, consider the possible three cases.
Case (i): Assume that $p_1(y_0) \neq 0 \neq p_2(y_0)$. Put, $y_1 = \frac{y_0}{p_1(y_0)}$ and $y_2 = \frac{y_0}{p_2(y_0)}$. Let $x \in X$ be arbitrary. Then, by (3.2), we have

$$\begin{aligned} l_{y_1}(x) + l_{y_2}(x) &= p_1(y_0)l_{y_0}(x) + p_2(y_0)l_{y_0}(x) \\ &= ((p_1+p_2)(y_0))l_{y_0}(x) \\ &= l_{y_0}(x). \end{aligned}$$

Since $p_1(y_1) = 1 = p_2(y_2)$, it follows that $l_{y_1} \in \text{supp}\ (p_1, L)$ and $l_{y_2} \in \text{supp}\ (p_2, L)$, which means that $l_{y_1}(x) \leq p_1(x)$ and $l_{y_2}(x) \leq p_2(x)$ for all $x \in X$.
Now, assume that $l_{y_1}(x_0) < p_1(x_0)$. Then, $l_{y_0}(x_0) = l_{y_1}(x_0) + l_{y_2}(x_0) < p_1(x_0) + p_2(x_0) = l_{y_0}(x_0)$. This is a contradiction. So, $l_{y_1}(x_0) = p_1(x_0)$. By a similar argument we have $l_{y_2}(x_0) = p_2(x_0)$. Therefore, by Lemma 3.1, $l_{y_1} \in \partial_L p_1(x_0)$ and $l_{y_2} \in \partial_L p_2(x_0)$. Hence, $l_{y_0} = l_{y_1} + l_{y_2} \in \partial_L p_1(x_0) + \partial_L p_2(x_0)$.

Case (ii): Suppose that $p_1(y_0) = 0$. So, $p_2(y_0) = 1$ and $l_{y_0} \in \text{supp}\ (p_2, L)$, which implies that

$$l_{y_0}(x_0) \leq p_2(x_0) \leq p_1(x_0) + p_2(x_0) = l_{y_0}(x_0).$$

Thus, $p_2(x_0) = l_{y_0}(x_0)$ and $p_1(x_0) = 0$. This, together with the fact that $p_2(y_0) = 1$ implies that $l_{y_0} \in \partial_L p_2(x_0)$. Moreover, since $p_1(x_0) = 0$, it follows that $0 \in \partial_L p_1(x_0)$. Hence, $l_{y_0} = 0 + l_{y_0} \in \partial_L p_1(x_0) + \partial_L p_2(x_0)$.

Case (iii): Assume that $p_2(y_0) = 0$. This is similar to the case (ii).
Note that since $p_1(y_0) + p_2(y_0) = 1$, then $p_1(y_0)$ and $p_2(y_0)$ can not be vanished simultaneous.

if $(p_1+p_2)(x_0) = 0$ then $\partial_L(p_1+p_2)(x_0) = \{l_y;\ l_y(x_0) = 0,\ (p_1+p_2)(y) \geq 1\} \cup \{0\}$. Use this fact and the above cases except in case (i) put $y_1 = \frac{y_0(p_1(y_0)+p_2(y_0))}{p_1(y_0)}$ and $y_2 = \frac{y_0(p_1(y_0)+p_2(y_0))}{p_2(y_0)}$, then we get the desirable result.
Now, assume that $l = 0$, which means that $p_1(x_0) + p_2(x_0) = 0$. Thus, $p_1(x_0) = 0 = p_2(x_0)$. This implies that $0 \in \partial_L p_1(x_0) \cap \partial_L p_2(x_0)$. Therefore, $0 = 0 + 0 \in \partial_L p_1(x_0) + \partial_L p_2(x_0)$. Hence, the claim proved in all cases.
Due to Theorem 3.2, $\partial_L(p_1+p_2)$ is a maximal L-monotone operator because p_1+p_2 is an IPH function. On the other hand, by the above one has $G(\partial_L(p_1+p_2)) \subset G(\partial_L p_1 + \partial_L p_2)$. Then, by L-monotonicity of $\partial_L p_1 + \partial_L p_2$, we conclude that $\partial_L(p_1+p_2) = \partial_L p_1 + \partial_L p_2$. Hence, $\partial_L p_1 + \partial_L p_2$ is a maximal L-monotone operator. □

4. Rockafellar's Surjectivity Theorem in the Framework of Abstract Monotonicity

In this section, we present a generalization of Rockafellar's surjectivy theorem in the framework of abstract monotonicity. In fact, in ([19] Theorem 4.3), a necessary condition for this generalization has been shown. We shall show that the sufficiency

of this condition can be obtained.
Throughout this section, let X, L be as in section 2 and $K := X \times L$, $L^* := L \times X$ and $\langle ., . \rangle_*$ be the coupling function defined by (2.2).
Let γ be a function defined on K which satisfies Assumption (D). In [8] and [19], by using the function γ, some suitable results have been obtained in the case of abstract monotonicity. We shall concentrate on Assumption (D) to obtain more results.

Proposition 4.1. *Assume that γ satisfies Assumption (D). Then,*

$$\gamma(x,0) \le \max\{\gamma(x,l), \gamma(x,-l)\} \quad (\forall\, x \in X,\ \forall\, l \in L),$$

and

$$\gamma(0,l) \le \max\{\gamma(x,l), \gamma(-x,l)\} \quad (\forall\, x \in X,\ \forall\, l \in L).$$

Proof. We only prove the first inequality and the second one is similar. Let $x \in X$ and $l \in L$ be arbitrary. Put $A := \{l'(x) - \gamma(x', l') \ : \ x' \in X,\ l' \in L\}$ and $B := \{\pm l(x') + l'(x) - \gamma(x', l') \ : \ x' \in X,\ l' \in L\}$. Trivially, $\sup A \le \sup B$. So we have

$$\begin{aligned}
\gamma(x,0) &= \sup_{(x',l') \in X \times L} (l'(x) - \gamma(x', l')) \\
&\le \sup_{(x',l') \in X \times L} (\pm l(x') + l'(x) - \gamma(x', l')) \\
&= \max\{\sup_{(x',l') \in X \times L} (l(x') + l'(x) - \gamma(x', l')), \sup_{(x',l') \in X \times L} (-l(x') + l'(x) - \gamma(x', l'))\} \\
&= \max\{\gamma(x,l), \gamma(x,-l)\}.
\end{aligned}$$

Hence, the proof is complete. □

In the rest of this section, we replace part (v) of Assumption (D) by (v') which is defined as follows:

(v') $\gamma(-x,l) = \gamma(x,l) = \gamma(x,-l)$ for all $x \in X$ and all $l \in L$,
in this case, we denote (D) by (D$'$).
Clearly, (v') implies (v), and so all results which have been obtained based on Assumption (D) in [8] and [19] still hold for Assumption (D$'$).

Remark 4.2. *By (v') and Proposition 4.1, we conclude that $\gamma(x,0) \le \gamma(x,l)$ and $\gamma(0,l) \le \gamma(x,l)$ for all $x \in X$ and all $l \in L$.*

Lemma 4.3. *Assume that γ satisfies Assumption (D$'$). Then the following assertions are true.*
(i) *We have*

$$\gamma(0,l) = \sup_{x' \in X} (l(x') - \gamma(x', 0)) \quad (\forall\, l \in L).$$

(ii) *We have*

$$\gamma(x,0) = \sup_{l' \in L} (l'(x) - \gamma(0, l')) \quad (\forall\, x \in X).$$

Proof. The proof of (ii) is similar to that of (i), so we only prove (i). It is clear that

$$\sup_{x'\in X}(l(x')-\gamma(x',0))\le\sup_{(x',l')\in X\times L}(l(x')-\gamma(x',l')).$$

For the converse inequality, since $\gamma(x',0)\le\gamma(x',l')$ for all $(x',l')\in X\times L$, then $l(x')-\gamma(x',0)\ge l(x')-\gamma(x',l')$, which implies that

$$\sup_{x'\in X}(l(x')-\gamma(x',0))\ge\sup_{(x',l')\in X\times L}(l(x')-\gamma(x',l')).$$

Hence, the proof is complete. □

Assume that Assumption (D′) holds for the function γ. We are going to define a new function $\bar\gamma$, which not only satisfies Assumption (D′) but also has the following property

$$\bar\gamma(x,l)=\bar\gamma(x,0)+\bar\gamma(0,l)\quad(\forall\ (x,l)\in X\times L).$$

Define the function $\bar\gamma$ as follows

$$\bar\gamma: X\times L\to[0,+\infty];\quad \bar\gamma(x,l):=\gamma(x,0)+\gamma(0,l),\ \forall\ (x,l)\in X\times L. \tag{4.1}$$

Theorem 4.4. *Suppose that Assumption (D′) holds for the function γ. Let the function $\bar\gamma$ be defined by (4.1). Then Assumption (D′) also holds for $\bar\gamma$. Moreover, $\bar\gamma(x,l)=\bar\gamma(x,0)+\bar\gamma(0,l)$ for all $x\in X$ and all $l\in L$.*

Proof. It is easy to see that $\bar\gamma(x,l)=\gamma(x,0)+\gamma(0,l)=\bar\gamma(x,0)+\bar\gamma(0,l)$, and also $\bar\gamma$ satisfies (i), (iii), (iv), (v') and (vi) of Assumption (D′).
Now, we show that $\bar\gamma^*_{L^*}\circ t=\bar\gamma$. By definition of $\bar\gamma^*_{L^*}$ and Lemma 4.3 we have

$$\begin{aligned}
(\bar\gamma^*_{L^*}\circ t)(x,l) &= \bar\gamma^*(l,x)\\
&= \sup_{(x',l')\in X\times L}(l'(x)+l(x')-\bar\gamma(x',l'))\\
&= \sup_{(x',l')\in X\times L}(l'(x)+l(x')-\gamma(x',0)-\gamma(0,l'))\\
&= \sup_{x'\in X,\ l'\in L}(l'(x)-\gamma(0,l')+l(x')-\gamma(x',0))\\
&= \sup_{l'\in L}(l'(x)-\gamma(0,l'))+\sup_{x'\in X}(l(x')-\gamma(x',0))\\
&= \gamma(x,0)+\gamma(0,l)\\
&= \bar\gamma(x,l),
\end{aligned}$$

which completes the proof. □

Remark 4.5. *Suppose that γ satisfies Assumption (D′). According to Theorem 4.4, the function $\bar\gamma$ also satisfies Assumption (D′) and has the property*

$$\bar\gamma(x,l)=\bar\gamma(x,0)+\bar\gamma(0,l)\quad(\forall\ x\in X,\ \forall\ l\in L).$$

Therefore one can replace the function γ by $\bar\gamma$. So, we define Assumption (D″) as follows:
There exists a function γ satisfying Assumption (D′) and has the property:

$$\gamma(x,l)=\gamma(x,0)+\gamma(0,l)\quad(\forall\ x\in X,\ \forall\ l\in L).$$

Assume that Assumption (D$''$) holds for the function γ. Consider the function $\varphi : X \to [0, +\infty]$ defined by $\varphi(x) := \gamma(x, 0)$ for all $x \in X$, and the function $\psi : L \longrightarrow [0, +\infty]$ defined by $\psi(l) := \gamma(0, l)$ for all $l \in L$.

In the following, we give some properties of the functions φ and ψ.

Proposition 4.6. *Let φ and ψ be defined as the above. Then*
(i) $\varphi^*_L(l) = \psi(l)$ for all $l \in L$,
(ii) $\psi^*_X(x) = \varphi(x)$ for all $x \in X$,
(iii) φ and ψ are abstract convex functions with respect to L and X, respectively.

Proof. (i). By Lemma 4.3(i) we have

$$\begin{aligned}\psi(l) &= \gamma(0, l) \\ &= \sup_{x' \in X} (l(x') - \gamma(x', 0)) \\ &= \sup_{x' \in X} (l(x') - \varphi(x')) \\ &= \varphi^*_L(l).\end{aligned}$$

(ii) is similar to (i), and (iii) is clear. Hence, the proof is complete. □

The following theorem has a crucial role to obtain our main results.

Theorem 4.7. *Let the function φ be defined as the above and $x \in X$ be arbitrary. Then*

$$\partial_L \varphi(x) = \{l \in L \ : \ l(x) = \varphi(x) + \psi(l) = \gamma(x, l)\}.$$

Proof. Let $l \in \partial_L \varphi(x)$ be arbitrary. By definition we have $l(t) - \varphi(t) \leq l(x) - \varphi(x)$ for all $t \in X$. Then, $\varphi^*_L(l) = \sup_{t \in X}(l(t) - \varphi(t)) \leq l(x) - \varphi(x)$. By Proposition 4.6, one has $\varphi^*_L(l) = \psi(l)$. Hence,

$$\psi(l) + \varphi(x) \leq l(x). \tag{4.2}$$

Moreover,

$$\begin{aligned}\gamma(x, l) &= \gamma^*_{L^*}(l, x) \\ &= \sup_{(x', l') \in X \times L} (l'(x) + l(x') - \gamma(x', l')) \\ &\geq l'(x) + l(x') - \gamma(x', l') \quad (\forall \ (x', l') \in X \times L).\end{aligned}$$

Replace x' and l' by x and l, respectively, we obtain $\gamma(x, l) \geq l(x)$. This, together with (4.2) implies that $l(x) = \psi(l) + \varphi(x)$. Thus,

$$\partial_L \varphi(x) \subseteq \{l \in L \ : \ l(x) = \varphi(x) + \psi(l) = \gamma(x, l)\}.$$

The converse inclusion is obvious. Indeed, let $l \in L$ be such that $l(x) = \gamma(x, l)$ and let $t \in X$ be arbitrary. Thus

$$l(t) - \varphi(t) \leq \sup_{t \in X}(l(t) - \varphi(t)) = \varphi^*(l) = \psi(l) = l(x) - \varphi(x).$$

So, $l \in \partial_L \varphi(x)$, which completes the proof. □

Define the set valued mapping $d : X \longrightarrow 2^L$ by

$$d(x) := \{l \in L : \gamma(x, -l) = l(x)\}, \quad \forall\, x \in X. \tag{4.3}$$

Also, define the set valued mapping $-d : X \longrightarrow 2^L$ by $(-d)(x) := -d(x)$ for each $x \in X$.
Therefore, we have

$$(-d)(x) = \{l \in L : \gamma(x, l) = -l(x)\}, \quad \forall\, x \in X. \tag{4.4}$$

It is easy to check that $d(-x) = -d(x)$ for each $x \in X$.

Note that the above set valued mappings d and $-d$ were introduced in [19], and also the necessary condition for abstract Rockafellar's surjectivity theorem ([19], Theorem 4.3) is based on the function d.

Proposition 4.8. *Suppose that the Assumption (D$''$) holds for the function γ, and d is the function defined by (4.3). Then, $d(x) = \{l \in L \ : \ \gamma(x, l) = l(x)\} = \partial_L \varphi(x)$ for all $x \in X$.*

Proof. The result follows from Assumption (D$''$) and Theorem 4.7. □

In the following, we give an example of a function d such that dom$\,(d) = X$. Moreover, d and d^{-1} are single-valued.

Example 4.9. *Let $X := \mathbb{Q}$ be the set of all rational numbers endowed with the ordinary addition. Now, for each $x \in X$, define the function $l_x : X \longrightarrow \mathbb{R}$ by $l_x(y) := xy$ for all $y \in X$. Let $L := \{l_x : x \in X\}$. It is easy to check that L is a set of real valued additive abstract linear functions. Let $K := X \times L$ and $L^* := L \times X$. Define the function $\gamma : K \longrightarrow (-\infty, +\infty]$ by*

$$\gamma(x, l_y) := \frac{1}{2}(x^2 + y^2), \quad \forall\, x,\ y \in X.$$

Therefore, It is not difficult to show that γ satisfies Assumption (D$''$). Define the function $d : X \longrightarrow 2^L$ by

$$d(x) := \{l_y \in L : \ \gamma(x, l_y) = l_y(x)\}, \quad \forall\, x \in X.$$

Therefore, for each $x \in X$, we have

$$\begin{aligned} d(x) &= \{l_y \in L : \gamma(x, l_y) = l_y(x)\} \\ &= \{l_y \in L : \frac{1}{2}(x^2 + y^2) = xy\} \\ &= \{l_y \in L : (x - y)^2 = 0\} \\ &= \{l_y \in L : x = y\} \\ &= \{l_x\}. \end{aligned}$$

This implies that dom$\,(d) = X$, and also, d is single-valued. Moreover, one has $d^{-1}(l_y) = \{y\}$ for each $y \in X$, that is, dom$\,(d^{-1}) = L$ and d^{-1} is single-valued.

The following theorem gives us a sufficient condition for Rockafellar's surjectivity theorem in the framework of abstract monotonicity.

Theorem 4.10. *Suppose that Assumption (D'') holds for the function γ. Let $T : X \longrightarrow 2^L$ be an abstract monotone operator such that $R(T+d) = L$. Assume that d and d^{-1} are single valued. Then, T is a maximal abstract monotone operator.*

Proof. First note that since $d(x)$ is a singleton for all $x \in X$, we denote the element of this singleton by $d(x)$.
Let (x_0, l_0) be monotonically related to $G(T)$. Since $l_0 + d(x_0) \in L$, then there exists $x \in D(T)$ such that $l_0 + d(x_0) \in (T+d)(x)$. This implies that there exists $l \in T(x)$ such that

$$l_0 + d(x_0) = l + d(x). \tag{4.5}$$

Also, we have

$$\begin{aligned} 0 &= \langle x_0 - x, (l_0 + d(x_0)) - (l + d(x)) \rangle \\ &= \langle x_0 - x, l_0 - l \rangle + \langle x_0 - x, d(x_0) - d(x) \rangle. \end{aligned}$$

Since (x_0, l_0) is monotonically related to $G(T)$, we have $\langle x_0 - x, l_0 - l\rangle \geq 0$. Moreover, d is monotone. Thus, one has $\langle x_0 - x, d(x_0) - d(x)\rangle \geq 0$. Therefore, we conclude that

$$d(x_0)(x_0) - d(x_0)(x) - d(x)(x_0) + d(x)(x) = 0. \tag{4.6}$$

Now, we are going to show that $d(x_0) \in \partial_L \varphi(x)$. To this end, in view of (4.6) we obtain

$$\begin{aligned} d(x_0)(x) &= d(x_0)(x_0) + d(x)(x) - d(x)(x_0) \\ &= \varphi(x_0) + \psi(d(x_0)) + \varphi(x) + \psi(d(x)) - d(x)(x_0) \\ &\geq \varphi(x_0) + \psi(d(x_0)) + \varphi(x) + \psi(d(x)) - \gamma(x_0, d(x)) \\ &= \varphi(x) + \psi(d(x_0)) \\ &= \gamma(x, d(x_0)). \end{aligned}$$

Thus, $d(x_0)(x) \geq \gamma(x, d(x_0))$. Clearly, $d(x_0)(x) \leq \gamma(x, d(x_0))$, and hence $d(x_0)(x) = \gamma(x, d(x_0))$. According to Theorem 4.7 and Proposition 4.3, we conclude that $d(x_0) \in \partial_L \varphi(x) = \{d(x)\}$. Therefore, $d(x_0) = d(x)$. So, by (4.5), we have $l = l_0$. Since d^{-1} is single valued, then $x = x_0$ and $(l_0, x_0) = (l, x) \in G(T)$. Hence, T is maximal. □

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