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Higher rank Einstein solvmanifolds

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Abstract. In this paper we study the structure of standard Einstein solvmanifolds of arbitrary rank. Also the validity of a variational method for finding standard Einstein solvmanifolds is proved.

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1. Introduction

General form of standard Einstein solvmanifolds were determined by Jense Heber (see [2]). Later, Gorge Lauret deeply studied this kind of manifolds. Solvable Lie group endowed with the left invariant Riemanian metric is called solvmanifold. Let S be a simply connected Lie group with the corresponding Lie algebra s endowed with the inner product determined by $< .,. >$ and solvable Lie bracket $[.,.]$. We call S a higher rank solvmanifold if

$$s = n \oplus a; \quad n = [s,s], \; a = n^{\perp},$$

where n is a metric nilpotent Lie algebra of dimension k. The codimension n is called the rank of S. The solvable Lie group $(S, [.,.], < .,. >)$ is called standard if a is abelian and it is said to be Einstein if its Ricci tensor $ric_{[.,.]}$ satisfies $ric < .,. >= c < .,. >$, for some $c \in \mathbb{R}$. s is called a metric solvable extension of n if the restriction of the Lie bracket and inner product of s to n coincide respectively with the Lie bracket and inner product of n. Let $\Lambda^2 n^* \otimes n$

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be the vector space of all bilinear skew-symmetric maps from $n \times n$ to n. There is a natural action of $GL(k)$ on $\Lambda^2 n^* \otimes n$ which is given by

$$\phi.\mu(X,Y) = \phi\mu(\phi^{-1}X, \phi^{-1}Y);\ X,Y \in n,\ \phi \in GL(k),\ \mu \in \Lambda^2 n^* \otimes n.$$

Let N denote a simply connected nilpotent Lie group with Lie algebra (n,μ) endowed with the left invariant Riemannian metric $< .\,,\, . >_n$, where μ is a nilpotent Lie algebra on n. The Ricci operator $R_\mu : n \to n$ of N is defined by

$$< R_\mu\, .\,, . >_n = ric < .\,, . >_n \ .$$

This operator is reduced to

$$(1.1) \quad < R_\mu X, Y > = -\frac{1}{2}\sum_{i,j} < \mu(X, X_i), X_j >< \mu(Y, X_i), X_j > + \frac{1}{4}\sum_{i,j} < \mu(X_i, X_j), X >< \mu(X_i, X_j), Y >,$$

for all $X, Y \in n$, where $\{X_1, X_2, ..., X_k\}$ is any orthonormal basis of n. The inner product $< .\,,\, . >_n$ determines an inner product on $\Lambda^2 n^* \otimes n$, denoted by $< .\,,\, . >$ and given by

$$< \mu, \lambda > = \sum_{i,j,k} < \mu(X_i, X_j), X_k >< \lambda(X_i, X_j), X_k > .$$

Also it naturally determines a norm on $\Lambda^2 n^* \otimes n$ defined by

$$\forall \lambda \in \Lambda^2 n^* \otimes n \qquad \|\lambda\| = \sum_{ijk} < \lambda(X_i, X_j), X_v >^2 .$$

Consider the Riemannain function

$$F_k : \Lambda^2 n^* \otimes n \longrightarrow \mathbb{R}, \qquad F_k(\mu) = tr {R_\mu}^2$$

and the sphere S_r given by

$$S_r = \{\mu \in \Lambda^2 n^* \otimes n; \|\mu\|^2 = 2r^2\},$$

for some $r \in \mathbb{R}$. Let $\aleph_k$ be the vector space of all nilpotent Lie brackets on n and $Der(\mu)$ be the Lie algebra of all derivations on n. Then $\mu \in \aleph_k$ is called a Ricci soliton if $R_\mu = cI + D$, for some $D \in Der(\mu)$ and $c \in \mathbb{R}$.

In [6], Jorge Lauret has proved that the standard Einstein solvmanifolds are exactly the critical points of modified scalar curvature function $F_k|_{S_1}$.

Theorem 1.1. [6]. *For $\mu \in \aleph_k \cap S_1$, the following statements are equivalent:*

(i) μ is a critical point of $F_k|_{S_1}$.
(ii) μ is a critical point of $F_k|_{GL(k).\mu \cap S_1}$.
(iii) μ admits a rank-one extension which is Einstein.
(iv) μ is a Ricci soliton.

2. Einstein solvmanifolds of rank ≥ 1

In this section, the structure of standard Einstein solvmanifolds is introduced. The next lemma provides some useful properties of solvable Lie bracket and inner product of a solvable Lie group. Then, we extend the Ricci soliton in [4] for any arbitrary rank. We call it multiple Ricci soliton.

Lemma 2.1. [2]. *Let $(s = n \oplus a, [\,.\,,.\,], < .\,,.\, >)$ be a metric solvable extension of $(n, \mu, < .\,,.\, >)$, where for every $0 \neq A \in a$, ad_A is nonzero and symmetric, then*

(i) $< R_{[.,.]}A, B >= -tr(ad_A ad_B)$, for all $A, B \in a$.

(ii) $< R_{[.,.]}A, X >= 0$, for all $A \in a, X \in n$.

(iii) $R_{[.,.]|_n} = -ad_Z|_n + R_\mu$, where $< Z, X >= tr(ad_X)$, for all $X \in n$.

Definition 2.2. $0 \neq \mu \in \aleph_k$ is called a multiple Ricci soliton of degree r if

(a) $R_\mu = c_\mu I + D_\mu;\ D_\mu \in Der(\mu)$, $c_\mu \in \mathbb{R}$.

(b) There are nonzero symmetric derivations D_i, $1 \leq i \leq r$, such that

$$D_\mu = D_1 + D_2 + \ldots + D_r, \quad trD_iD_j = -\delta_{ij}c_\mu trD_i\,.$$

Remark 2.3. If μ is a multiple Ricci soliton of degree r, then μ is a multiple Ricci soliton of degree less than r. Therefore, μ is the critical point of $F_k|_{S_r \cap GL(k).\mu}$.

Using Definition 2.1, we study the structure of standard Einstein solvmanifolds as follows.

Proposition 2.4. *For $0 \neq \mu \in \aleph_k \cap S_r$, the following statements are equivalent:*

(i) μ admits a metric extension which is Einstein.

(ii) μ is a multiple Ricci soliton.

Proof. Let the Lie algebra (n, μ) admit an Einstein metric extension S with corresponding Lie algebra $(s = n \oplus a, [\,.\,,.\,], < .\,,.\, >)$ such that $dim(a) = r$. Let $\{H_1, H_2, \ldots, H_r\}$ be an orthonormal basis for a and Z be the mean curvature vector field for the simply connected Lie group N with Lie algebra n. A straightforward calculation shows that $D_\mu = D_1 + D_2 + \ldots + D_r$, where $D_\mu := ad_Z|_n$ and $D_i = tr(ad_{H_i})ad_{H_i}|_n$. $[\,.\,,.\,]$ is the Lie bracket. Hence, $D_i\mu(.,.) = \mu(., D_i.) + \mu(D_i.,.)$; that is to say, D_i 's are derivations on n. Suppose that D_i 's and D_μ are symmetric (see [2; 4.10]). Let $Z_i = tr(ad_{H_i})H_i$, then Lemma 2.1 implies that

$$trD_iD_j = tr(ad_{Z_i}ad_{Z_j}) = - < R_{[.,.]}Z_i, Z_j >= -c_\mu < Z_i, Z_j >= -\delta_{ij}c_\mu trD_i.$$

Also $R_\mu = c_\mu I + D_\mu$, for some $c_\mu \in \mathbb{R}$. Therefore, μ is a multiple Ricci soliton.

Conversely, let μ be a multiple Ricci soliton i.e.

(a) $R_\mu = c_\mu I + D_\mu; \quad D_\mu \in Der(n, \mu)$, $c_\mu \in \mathbb{R}$.

(b) There are symmetric derivations D_i, $1 \le i \le r$, such that

$$D_\mu = D_1 + D_2 + \ldots + D_r, \;\; trD_iD_j = -\delta_{ij}c_\mu trD_i.$$

Let $(n, \mu, < .,. >_n)$ be a Lie algebra with orthonormal basis $\{X_1, X_2, \ldots, X_k\}$. We define Lie algebra s with a simply connected Lie group S as follows

$$s = n \oplus \sum_i \mathbb{R}Z_i,$$

endowed with the inner product $< .,. >$ defined by

$$< Z_i, Z_j >= \delta_{ij}trD_i, \;\; < Z_i, n >= 0, \;\; < .,. > |_{n\times n} =< .,. >_n .$$

Also, Lie bracket $[\,.\,,.\,]$ is defined by

$$[Z_i, Z_j] = 0, \;\; [Z_i, X_j] = -[X_j, Z_i] = D_iX_j, \;\; [\,.\,,.\,]|_{n\times n} = \mu.$$

Clearly $[\,.\,,.\,]$ is a Lie bracket, since D_i 's are derivations. $\{D_\mu X_1, D_\mu X_2, ..., D_\mu X_k\}$ is a linearly independent set which generates a subalgebra of $[s, s]$. Therefore, $n = [s, s]$. μ is nilpotent hence $[\,.\,,.\,]$ is a solvable Lie bracket. Finally using Lemma 2.1, we have

$$< R_{[.,.]}Z_i, Z_j >= -tr(D_iD_j) = \delta_{ij}c_\mu trD_i = c_\mu < Z_i, Z_j >, \;\; < R_{[.,.]}Z_i, n >= 0,$$

$$< R_{[.,.]}X_i, X_j >=< (-D_\mu + R_\mu)X_i, X_j >=< c_\mu X_i, X_j >= c_\mu < X_i, X_j >,$$

which implies that $< .,. >_s$ is a Einstein metric. This completes the proof. $\square$

Using Proposition 2.1, we get a higher rank Einstein solvmanifold as the direct sum of the Lie algebras.

Proposition 2.5. *If nonzero nilpotent Lie brackets μ_1 and μ_2 are Ricci solitons, then $\mu = \mu_1 \oplus \mu_2$ is a multiple Ricci soliton of degree 2.*

Proof. μ_1 and μ_2 are Ricci solitons i.e.

$$R_{\mu_i} = c_{\mu_i}I + D_{\mu_i}; \quad D_{\mu_i} \in Der(\mu_i), \; c_{\mu_i} \in \mathbb{R}, \; i = 1, 2. \tag{2.1}$$

Up to isometry and scaling we can determine norms of μ_1 and μ_2 such that $c_{\mu_1} = c_{\mu_2}$. Set

$$D_\mu = \begin{bmatrix} D_{\mu_1} & 0 \\ 0 & D_{\mu_2} \end{bmatrix},$$

$$R_\mu = \begin{bmatrix} R_{\mu_1} & 0 \\ 0 & R_{\mu_2} \end{bmatrix},$$

Then $R_\mu = c_{\mu_1}I + D_\mu$, $R_\mu = R_{\mu_1} \oplus R_{\mu_2}$ and $D_\mu = D_{\mu_1} \oplus D_{\mu_2}$. Also, by Theorem 1.3, $trD_{\mu_i}D_{\mu_j} = -\delta_{ij}c_{\mu_1}trD_{\mu_i}$; $i = 1, 2$. Therefore μ is a multiple Ricci soliton which admits a 2-rank Einstein solvable extension. $\square$

Corollary 2.6. *If nonzero nilpotent Lie brackets μ_i 's, $1 \le i \le r$, are Ricci solitons, then $\mu = \mu_1 \oplus \mu_2 \oplus ... \oplus \mu_r$ is a multiple Ricci soliton of degree r which admits an Einstein solvable extension of rank r.*

Remark 2.7. There exist 31 Ricci soliton nonzero Lie algebras of dimension 6 (see [7]), which by direct sum of them, we can obtain a lot of multiple Ricci soliton nilpotent Lie algebras.

3. Standard methods

The goal of this section is to present certain results from [4] and [6]. In view of [4], Jorge Lauret has used a variational method for finding standard Einstein solvmanifolds. We will demonstrate this method in Theorem 3.1. We first give some preliminaries.

Lemma 3.1. *(Lagrange multiplier theorem)* [1]. *Let P and M be smooth manifolds and $g : M \longrightarrow P$ be a smooth submersion. Let $f : M \longrightarrow \mathbb{R}$ be C^r, $m \in M$ and $p \in P$ such that $m \in g^{-1}(p)$, then the following statements are equivalent:*
(i) m is a critical point of $f|_{g^{-1}(p)}$.
(ii) There are $\lambda \in {T_p}^* M$ such that $T_m f = \lambda \circ T_m g$.

The vector space $\aleph_k$ is $GL(k)-$invariant, so we can refine Theorem 1.1 more accurately as follows.

Lemma 3.2. *For $0 \neq \mu \in \aleph_k$ and $\psi \in GL(k)$ the following statements are equivalent:*
(i) $\psi.\mu$ is a Ricci soliton.
(ii) $\psi.\mu$ is a critical point of $F_k|_{S_r \cap GL(k).\mu}$.
(iii) ψ is a solution of the following system of equations:

$$\begin{cases} \|\phi.\mu\|^2 = 2r^2 \\ \frac{\partial F_k(\phi.\mu)}{\partial \phi_{ij}} = t\frac{\partial(\|\phi.\mu\|)}{\partial \phi_{ij}} \end{cases}$$

where $t \in \mathbb{R}$ and $\phi \in GL(k)$.

Proof. Let $g(\lambda) = \frac{1}{2}\|\lambda\|^2$ be a function on $\Lambda^2 n^* \otimes n$ and use Theorem 1.1 and Lemma 3.1 .

It is possible that the above system of equations is not solvable, hence we assume that for some $k \in \mathbb{N}$ and every $\phi \in GL(k)$ there exists $\mu \in \aleph_k$ such that $\phi.\mu$ isn't a Ricci soliton. □

Notation 3.3. *Suppose that $DGL(k) := \{\phi \in GL(k); \phi \text{ is diagonal}\}$, For any $\mu \in \aleph_k \subseteq \Lambda^2 n^* \otimes n$, set*

$$\mu(X_i, X_j) = \sum_v c_{ijv} X_v, \quad \phi = diag(\phi_1, \phi_2, ..., \phi_k) \ ,$$

then

$$(\phi.\mu)(X_i, X_j) = \sum_{x_{ijv} \neq 0} x_{ijv} X_v; \quad x_{ijv} = \frac{\phi_v c_{ijv}}{\phi_i \phi_j}$$

and for any i, j and v such that $< \mu(X_i, X_j), X_v > \neq 0$, the diagonal elements of $R_{\phi.\mu}$ are equal to

$$(R_{\phi.\mu})_{ii} = -\frac{1}{2}(-x_{ijv}^2 + \sum_{\substack{rst \\ x_{rst} \neq x_{ijv}}} \delta^{rst,i} x_{rst}^2),$$

$$(R_{\phi.\mu})_{jj} = -\frac{1}{2}(-x_{ijv}^2 + \sum_{\substack{rst \\ x_{rst} \neq x_{ijv}}} \delta^{rst,j} x_{rst}^2),$$

$$(R_{\phi.\mu})_{vv} = \frac{1}{2}(x_{ijv}^2 + \sum_{\substack{rst \\ x_{rst} \neq x_{ijv}}} \delta^{rst,v} x_{rst}^2),$$

where $\delta^{rst,i}$, $\delta^{rst,j}$ and $\delta^{rst,v}$ are equal to 0, 1 or -1 .

Lemma 3.4. [6]. *Let $(n, \mu, < .,. >)$ be a Lie algebra and $P_\mu = Sym(n) \cap Der(n)$, then $R_\mu \perp P_\mu$ with inner product $tr(AB)$ on $Sym(n) \times Sym(n)$.*

Theorem 3.5. *(Lauret theory) For every $\phi \in GL(k)$ and $\mu \in \aleph_k$ if $\phi.\mu \in S_r$ and R_μ is diagonal, then for any i, j and v such that $< \mu(X_i, X_j), X_v > \neq 0$, the following statements are equivalent:*

(i) $R_{\psi.\mu} = c_{\psi.\mu} I + D_{\psi.\mu};\;\; D_{\psi.\mu} \in Der(\psi.\mu)$

(ii) $c_{\psi.\mu}$ and a_{ijv}^2's are solutions of the system

$$\begin{cases} \sum_{i,j,v} x_{ijv}^2 = r^2 \\ \frac{\partial F_k(\phi.\mu)}{\partial u_{ijv}}|_{x_{ijv}:=a_{ijv}} = -c_{\psi.\mu} \end{cases}$$

where $(\phi.\mu)(X_i, X_j) = \sum_{x_{ijv} \neq 0} x_{ijv} X_v$, $u_{ijv} = x_{ijv}^2$, $\psi.\mu = \phi.\mu|_{x_{ijv}:=a_{ijv}}$.

Proof. By Lemma 3.2 and the chain rule, it is easy to see that $\psi.\mu$ is a Ricci soliton if and only if variation t and a_{ijv}^2's are solutions of the system

$$\begin{cases} \sum_{i,j,v} x_{ijv}^2 = r^2 \\ \frac{\partial F_k(\phi.\mu)}{\partial u_{ijv}}|_{x_{ijv}:=a_{ijv}} = t \end{cases}$$

Now we shall obtain the Lagrangian coefficient. By Lemma 3.3 it is easy to see that $trR_\mu{}^2 = c_\mu trR_\mu$. Also $trR_\mu = -\frac{1}{2}\|\mu\|^2$. Thus $F_k = -c_\mu r^2$. Consequently $\frac{\partial F_k(\phi.\mu)}{\partial u_{ijv}}|_{x_{ijv}:=a_{ijv}} = -c_\mu$. □

Finally, we exhibit a rank-two Einstein solvmanifold of dimension 8 and a rank-three Einstein solvmanifold of dimension 15.

Example 3.6. *Let $\mu = \mu_1 \oplus \mu_2$, where $\mu_1(X_1, X_2) = X_5$, $\mu_2(X_3, X_4) = X_6$ and $\phi = diag(\phi_1, \phi_2, ..., \phi_6)$, then*

$$\phi.\mu_1(X_1, X_2) = \frac{\phi_5}{\phi_1\phi_2} X_5 \;\; and \;\; \phi.\mu_2(X_3, X_4) = \frac{\phi_6}{\phi_3\phi_4} X_6 \,.$$

Let $x := \frac{\phi_5}{\phi_1\phi_2}$, $y := \frac{\phi_6}{\phi_3\phi_4}$. *Using Theorem* 3.1*, it is easy to see* $x^2 = y^2 = 1$. *Therefore* $D_{\mu_1} = diag(1,1,0,0,2,0)$ *and* $D_{\mu_2} = diag(0,0,1,1,0,2)$. *If* $(n_1, \mu_1, < . , . >_1)$ *and* $(n_2, \mu_2, < . , . >_2)$ *are nilradical Lie algebras, define the Lie algebra* s *with simply connected Lie group* S *using the following direct sum*

$$s = \mathbb{R}Z_1 \oplus n_1 \oplus \mathbb{R}Z_2 \oplus n_2$$

endowed with the inner product $< .,. >$ *defined by*

$$< Z_i, Z_j >= 4\delta_{ij}, < Z_i, n >= 0, < . , . > |_{n_i \times n_i} =< . , . >_i; \;\; 1 \leq i, j \leq 2\,,$$

where the Lie bracket $[\,.\,,.\,]$ *on* s *is defined by*

$$\begin{array}{ll} [Z_1, X_1] = -[X_1, Z_1] = X_1, & [Z_1, X_2] = -[X_2, Z_1] = X_2, \\ [Z_1, X_5] = -[X_5, Z_1] = 2X_5, & [Z_2, X_3] = -[X_3, Z_2] = X_3, \\ [Z_2, X_4] = -[X_4, Z_2] = X_4, & [Z_2, X_6] = -[X_6, Z_2] = 2X_6, \end{array}$$

$$[Z_i, Z_j] = 0, \;\; [.,.]_s|_{n_i \times n_i} = \mu_i; \;\; 1 \leq i, j \leq 2$$

and it is equal to zero otherwise. It is easy to see that $R_{[.,.]_s} = -\frac{3}{2} I_{8\times 8}$ *which implies that* S_μ *is Einstein of rank* 2.

Example 3.7. *Let* $\mu = \mu_1 \oplus \mu_2 \oplus \mu_3$*, where* μ_1 *and* μ_2 *are Lie algebras given in Example* 1 *and* $\mu_3 \in \aleph_6$ *is given by*

$$\mu_3(X_7, X_i) = X_{i+1}; \; 8 \leq i \leq 11\,.$$

Every $\phi.\mu_3 \in DGL(6).\mu_3 \cap S_{\sqrt{\frac{30}{13}}}$ *is equal to*

$$\phi.\mu_3(X_7, X_i) = a_{7,i,i+1} X_{i+1}; \; i = 8, 9, 10, 11\,.$$

By Lauret theory it is easy to see that a critical point of F_6 *restricted to the leaf* $\sum_{7<i<12} a^2_{7,i,i+1} = \frac{20}{13}$ *is equal to* $\mu_3\{a_{7,i,i+1}\}$ *where*

$$a^2_{7,8,9} = \frac{12}{13}, \;\; a^2_{7,9,10} = \frac{9}{13}, \;\; a^2_{7,10,11} = \frac{3}{13}, \;\; a^2_{7,11,12} = \frac{9}{13}, \;\; c_{\psi.\mu_3} = -\frac{3}{2}$$

and

$$D_{\mu_3(a_{7,i,i+1})} = diag(0,0,0,0,0,0, \frac{15}{26}, \frac{42}{26}, \frac{36}{26}, \frac{51}{26}, \frac{48}{26}, \frac{51}{26})\,.$$

Let $(n_3, \mu_3(a_{7,i,i+1}), < . , . >_3)$ *be a Ricci soliton. Define Lie algebra* s *with simply connected Lie group* S *using the following direct sum*

$$s = \mathbb{R}Z_1 \oplus n_1 \oplus \mathbb{R}Z_2 \oplus n_2 \oplus \mathbb{R}Z_3 \oplus n_3$$

endowed with the inner product $< .,. >$ *which is defined by*

$$< Z_1, Z_1 >= 4, < Z_2, Z_2 >= 4, < Z_3, Z_3 >= \frac{243}{13}, < . , . >|_{n_i \times n_i} =< . , . >_i$$

and it is equal to zero otherwise. Lie bracket $[\,.\,,.\,]$ *on* S *defined by*

$$\begin{array}{ll} [Z_1, X_1] = -[X_1, Z_1] = X_1, & [Z_1, X_2] = -[X_2, Z_1] = X_2, \\ [Z_1, X_5] = -[X_5, Z_1] = 2X_5, & [Z_2, X_3] = -[X_3, Z_2] = X_3, \\ [Z_2, X_4] = -[X_4, Z_2] = X_4, & [Z_2, X_6] = -[X_6, Z_2] = 2X_6, \end{array}$$

$$[Z_3, X_7] = -[X_7, Z_3] = \tfrac{15}{26} X_7, \qquad [Z_3, X_8] = -[X_8, Z_3] = \tfrac{42}{26} X_8,$$

$$[Z_3, X_9] = -[X_9, Z_3] = \tfrac{36}{26} X_9, \qquad [Z_3, X_{10}] = -[X_{10}, Z_3] = \tfrac{51}{26} X_{10},$$

$$[Z_3, X_{11}] = -[X_{11}, Z_3] = \tfrac{48}{26} X_{11}, \quad [Z_3, X_{12}] = -[X_{12}, Z_3] = \tfrac{51}{26} X_{12},$$

$$[.,.]|_{n_i \times n_i} = \mu_{n_i}; \quad i = 1, 2$$

and otherwise is equal to zero. It is easy to check that $R_{[.,.]} = -\frac{3}{2} I_{15\times 15}$ *which implies that* S_μ *is Einstein of rank 3.*

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