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An Implicit Difference-ADI Method for the Two-dimensional Space-time Fractional Diffusion Equation

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Abstract. Fractional order diffusion equations are generalizations of classical diffusion equations which are used to model in physics, finance, engineering, etc. In this paper we present an implicit difference approximation by using the alternating directions implicit (ADI) approach to solve the two-dimensional space-time fractional diffusion equation (2DSTFDE) on a finite domain. Consistency, unconditional stability, and therefore first-order convergence of the method are proven. Some numerical examples with known exact solution are tested, and the behavior of the errors are analyzed to demonstrate the order of convergence of the method.

Keywords: Two-dimensional fractional differential equation, Space-time fractional diffusion equation, Implicit difference method, Alternating directions implicit methods.

2000 Mathematics subject classification: 34A08, 65L20, 60J60.

1. Introduction

Fractional derivative and integral are almost as old as their integer-order counterparts [12, 14]. Fractional diffusion equations have recently been used to model problems in physics [11], engineering [2, 1, 18, 19], and finance [6, 8, 16].

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Fractional space derivatives are used to model a particle motion which spreads at a rate that is different than classical model. Because of physical application, the spatial order derivative is usually between one and two. Also temporal (time) fractional derivative may be used to model a particle motion that the time between two jumps is more than usual [19]. Fractional differential equations have been studied by a number of authors, and since analytic closed-form solution for many kinds of fractional differential equations is elusive, a lot of authors have tried to present numerical methods to solve those equations. Zhuang and Liu [25] used an implicit difference approximation for the time fractional diffusion equation and analyzed its error. Ervin and Roop [5] used variational method for fractional advection dispersion equations. Adomian decomposition method developed to derive analytical approximation solution for fractional heat-like and wave-like equations with variable coefficient by Momani [13]. Xu et al. [24] used homotopy method for nonlinear fractional partial differential equations. Podlubny et al. [15] presented a matrix approach to discrete partial differential equations. Tadjeran et al. [21, 22] also developed Crank-Nicolson discretization to solve fractional diffusion equation. An ADI-Euler method was used to solve two-dimensional fractional dispersion equation by Meerschaert et al. [10]. They applied ADI approach for solving two-dimensional fractional dispersion equation, but the order of temporal derivative was not fractional. After that, Liu et al. [26] presented an implicit difference approximation for the two-dimensional space-time fractional diffusion equation. Since the equation is two-dimensional, in this method, a very large linear system of equations with $(N_x - 1)(N_y - 1)$ unknowns should be solved, which is computationally expensive. Therefore, using ADI approach is useful here to decrease computational cost, we should use an efficient method to approximate fractional temporal derivative. Diethelm et al. [3] presented a selection of numerical methods to approximate Caputo fractional derivative. In this paper, in order to use ADI approach, Grünwald formula and unshifted Grünwald formula are used to estimate Caputo derivative. Consistency and unconditional stability are proven for our new method and hence according to Lax's equivalence theorem, the method is convergent.

Consider the two-dimensional space-time fractional diffusion equation

$$\frac{\partial^\alpha u(x,y,t)}{\partial t^\alpha} = d(x,y,t)\frac{\partial^\beta u(x,y,t)}{\partial x^\beta} + e(x,y,t)\frac{\partial^\gamma u(x,y,t)}{\partial y^\gamma} + q(x,y,t). \quad (1.1)$$

On a finite domain $x_L < x < x_H$, $y_L < y < y_H$, with fractional orders $0 < \alpha \leq 1$, $1 < \beta \leq 2$, and $1 < \gamma \leq 2$, the diffusion coefficients $d(x,y,t) > 0$, $e(x,y,t) > 0$. The function $q(x,y,t)$ can be used to represent sources and sinks, with the initial conditions

$$u(x,y,0) = f(x,y),\ x_L \leq x \leq x_H\ ,\ y_L \leq y \leq y_H,$$

and boundary conditions

$$u(x_L, y, t) = 0\ ,\ u(x_H, y, t) = B(x_H, y, t)\ y_L \le y \le y_H\ ,\ t > 0$$
$$u(x, y_L, t) = 0\ ,\ u(x, y_H, t) = B(x, y_H, t)\ x_L \le x \le x_H\ ,\ t > 0$$

The classical diffusion equation is obtained by $\alpha = 1$, $\beta = \gamma = 2$. The values of $1 < \beta < 2$, $1 < \gamma < 2$ lead to a super-diffusion model. The value of $0 < \alpha < 1$ leads to super-slow diffusion model and the value of $\alpha > 1$ leads to super-fast diffusion model [17]. The case $0 < \alpha < 1$ is only considered in this paper.
The spatial fractional derivatives in Eq.(1.1) are Riemman (left) factional derivative. The Riemman fractional derivative for a function $u(x, y, t)$ over the interval $x_L < x < x_H$ is defined as follows:

$$\frac{\partial^\beta u(x,y,t)}{\partial x^\beta} = \frac{1}{\Gamma(m-\beta)} \frac{\partial^m}{\partial x^m} \int_{x_L}^{x} \frac{u(\xi, y, t)}{(x-\xi)^{\beta+1-n}} d\xi, \tag{1.2}$$

where m is an integer such that $m - 1 < \beta \le m$. In most of the papers and books, the case $L = 0$ is defined as the Riemman-Liouville fractional derivative, and the case $L = -\infty$ is defined as the Liouville fractional derivative. We extend the zero boundary conditions for $x < x_L$, $y < y_L$, so that the Riemman and the Riemman-Liouville forms become equivalent. This definition is left fractional derivative, and right fractional derivative is defined similarly. For different definitions and concepts on fractional derivative, see [12, 14, 17].
Caputo fractional derivative is usually used for time fractional derivative because it leads to integer-order initial conditions and it is important to solve practical problems. Caputo fractional derivative is defined as follows:

$$\frac{\partial^\alpha u(x,y,t)}{\partial t^\alpha} = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{\partial u(x,y,\eta)}{\partial \eta} \frac{\partial \eta}{(t-\eta)^{\alpha+1-n}}, \tag{1.3}$$

where n is an integer such that $n - 1 < \alpha \le n$.
This paper is organized as follows. The implicit difference-ADI method is presented in section 2. In section 3, the stability and convergence of the implicit difference-ADI method are analyzed. In section 4, some numerical examples are given.

2. The Numerical Method

We use the Grünwald finite difference formula to estimate the spatial β-order fractional derivative. It is shown in [9] that the standard Grünwald formula usually results unstable finite difference methods . Therefore we use a right-shifted Grünwald formula to estimate the spatial β-order fractional derivative [9]

$$\frac{\partial^\beta u(x,y,t)}{\partial x^\beta} = \frac{1}{\Gamma(-\beta)} \lim_{N_x \to \infty} \sum_{p=0}^{N_x} \frac{\Gamma(p-\beta)}{\Gamma(p+1)} u(x-(p-1)h, y, t),$$

where N_x is a positive integer such that $h = (x - x_L)/N_x$ and $\Gamma(.)$ is the gamma function. We also define Grünwald weights by:

$$\omega_{\beta,p} = \frac{\Gamma(p-\beta)}{\Gamma(-\beta)\Gamma(p+1)} = (-1)^p \frac{\beta(\beta-1)\ldots(\beta-p+1)}{p!} = (-1)^p \binom{\beta}{p}.$$

We will use the customary notation $t_n = n\Delta t$, $0 \leq t_n \leq T$ for time, $\Delta x = h_x > 0$ is the spatial grid size in x-direction, $\Delta x = (x_H - x_L)/N_x$, with $x_i = x_L + i\Delta x$ for $i = 0, \ldots N_x$; $\Delta y = h_y$ is the spatial grid size in y-direction, $\Delta y = (y_H - y_L)/N_y$, with $y = y_L + j\Delta y$ for $j = 0, \ldots, N_y$. Define $u_{i,j}^n$ as the numerical approximation to $u(x_i, y_j, t_n)$. Similarly, define $d_{i,j}^n = d(x_i, y_j, t_n)$, $e_{i,j}^n = e(x_i, y_j, t_n)$, and $q_{i,j}^n = q(x_i, y_j, t_n)$. The initial conditions are $u_{i,j}^0 = f(x_i, y_j)$. The boundary conditions on the boundary of this region are $u_{0,j}^n = 0$, at $x = x_L$; $u_{N_x,j} = B_{N_x,j}^n = B(x_H, y_j, t_n)$, at $x = x_H$; $u_{i,0}^n = 0$, at $y = y_L$; and $u_{i,N_y}^n = B_{i,N_y}^n = B(x_i, y_H, t_n)$, at $y = y_H$.
Caputo fractional derivative is used for time fractional derivative in Eq.(1.1). It should be mentioned that it is difficult to estimate Caputo fractional derivative because $\partial^n u(x, y, t)/\partial \eta^n$ appears as integrand. For some methods to estimate Caputo fractional derivative see [3, 26]. Lemma 2.1 shows that the Caputo fractional derivative and the Riemman-Liouville fractional derivative coincide if the initial conditions are homogeneous.

Lemma 2.1. *Let $\alpha \geq 0$ and $n-1 < \alpha \leq n$. Assume that f is such that both ${}_aD_x^\alpha f$ (Riemman-Liouville derivative) and ${}_a^CD_x^\alpha f$ (Caputo derivative) exist. Moreover, let $D^k f(a) = 0$ for $k = 0, 1, \ldots, n-1$, then*

$${}_aD_x^\alpha f = {}_a^C D_x^\alpha f.$$

Proof. For proof see [23]. □

According to this lemma, assume that $\partial^k u(x, y, t)/\partial \eta^k = 0$, for $k = 0, 1, \ldots, n-1$ and $n-1 < \alpha \leq n$, therefore we can use approximation methods for Caputo derivatives which were used to estimate Riemman-Liouville derivatives.
$0 < \alpha < 1$ is usually used in Eq.(1.1) for applications and assume that $u(x, y, t) = 0$ then the Grünwald formula (unshifted Grünwald formula) can be used to estimate the fractional derivative at level t_{n+1}

$$\frac{\partial^\alpha u(x_i, y_j, t_{n+1})}{\partial t^\alpha} = \frac{1}{(\Delta t)^\alpha} \sum_{k=0}^{n+1} \omega_{\alpha,k} u(x_i, y_j, t_{n+1} - k\Delta t). \tag{2.1}$$

For spatial fractional derivatives a right shifted Grünwald formula is used at level t_{n+1}

$$\frac{\partial^\beta u(x_i, y_j, t_{n+1})}{\partial x^\beta} = \frac{1}{(\Delta x)^\beta} \sum_{p=0}^{i+1} \omega_{\beta,p} u(x_i - (p-1)h_x, y_j, t_{n+1}), \tag{2.2}$$

$$\frac{\partial^\gamma u(x_i,y_j,t_{n+1})}{\partial y^\delta}=\frac{1}{(\Delta y)^\gamma}\sum_{q=0}^{j+1}\omega_{\gamma,q}u(x_i,y_j-(q-1)h_y,t_{n+1}). \tag{2.3}$$

If estimations (2.1), (2.2), (2.3) are subsituted into Eq.(1.1) then the resulting implicit difference equations become

$$\begin{aligned}\sum_{k=0}^{n+1}\omega_{\alpha,k}u_{i,j}^{n+1-k} &= \frac{d_{i,j}^{n+1}(\Delta t)^\alpha}{(\Delta x)^\beta}\sum_{p=0}^{i+1}\omega_{\beta,p}u_{i+1-p,j}^{n+1}\\ &\quad+\frac{e_{i,j}^{n+1}(\Delta t)^\alpha}{(\Delta y)^\gamma}\sum_{q=0}^{j+1}\omega_{\gamma,q}u_{i,j+1-q}^{n+1}+(\Delta t)^\alpha q_{i,j}^{n+1},\end{aligned}$$

$$\begin{aligned}u_{i,j}^{n+1}+\sum_{k=1}^{n+1}\omega_{\alpha,k}u_{i,j}^{n+1-k} &= \frac{d_{i,j}^{n+1}(\Delta t)^\alpha}{(\Delta x)^\beta}\sum_{p=0}^{i+1}\omega_{\beta,p}u_{i+1-p,j}^{n+1}\\ &\quad+\frac{e_{i,j}^{n+1}(\Delta t)^\alpha}{(\Delta y)^\gamma}\sum_{q=0}^{j+1}\omega_{\gamma,q}u_{i,j+1-q}^{n+1}+(\Delta t)^\alpha q_{i,j}^{n+1}.\end{aligned} \tag{2.4}$$

Define the finite difference operations as follows:

$$\begin{aligned}L_{\alpha,t}u_{i,j}^{n+1} &= -\sum_{k=1}^{n+1}\omega_{\alpha,k}u_{i,j}^{n+1-k},\\ L_{\beta,x}u_{i,j}^{n+1} &= \frac{d_{i,j}^{n+1}(\Delta t)^\alpha}{(\Delta x)^\beta}\sum_{p=0}^{i+1}\omega_{\beta,p}u_{i+1-p,j}^{n+1},\\ L_{\gamma,y}^{n+1}u_{i,j}^{n+1} &= \frac{e_{i,j}^{n+1}(\Delta t)^\alpha}{(\Delta y)^\gamma}\sum_{q=0}^{j+1}\omega_{\gamma,q}u_{i,j+1-q}^{n+1}.\end{aligned}$$

Eq.(2.4), in operator notation, is as follows:

$$(1-(\Delta t)^\alpha L_{\beta,x}-(\Delta t)^\alpha L_{\gamma,y})u_{i,j}^{n+1}=L_{\alpha,t}u_{i,j}^{n+1}+(\Delta t)^\alpha q_{i,j}^{n+1}. \tag{2.5}$$

The ADI method is used to significantly reduce the computational cost in solving classical multi-dimensional diffusion equations [7]. The ADI method has been used to solve the two-dimensional space fractional diffusion equation [10]. For using ADI method, some perturbations of Eq.(2.5) are used to derive schemes that are specified and solved in one direction at a time, and for this problem the Eq.(2.5) is written in a seperate form

$$(1-(\Delta t)^\alpha L_{\beta,x})(1-(\Delta t)^\alpha L_{\gamma,y})u_{i,j}^{n+1}=L_{\alpha,t}u_{i,j}^{n+1}+(\Delta t)^\alpha q_{i,j}^{n+1}, \tag{2.6}$$

which produces an additional perturbation error as follows:

$$(\Delta t)^{2\alpha}(L_{\beta,x}L_{\gamma,y})u_{i,j}^{n+1}. \tag{2.7}$$

Eq.(2.6) can be divided into two equations, using an intermediate solution $u^*_{i,j}$,

$$(1-(\Delta t)^\alpha L_{\beta,x})u^*_{i,j} = L_{\alpha,t}u^{n+1}_{i,j} + (\Delta t)^\alpha q^{n+1}_{i,j}, \tag{2.8}$$

$$(1-(\Delta t)^\alpha L_{\gamma,y})u^{n+1}_{i,j} = u^*_{i,j}. \tag{2.9}$$

Eq.(2.6) is obtained from equations (2.8) and (2.9) just by multiplying (2.9) by $(1-(\Delta t)^\alpha L_{\beta,x})$ and add the result to (2.8).
The intermediate solution $u^*_{i,j}$ in equations (2.8) and (2.9) is defined to advance the numerical solution $u^n_{i,j}$ at time t_n to the numerical solution $u^{n+1}_{i,j}$ at time t_{n+1}. The implicit difference-ADI algorithm is as follows:

(1) First solve a set of $N_x - 1$ equations in x-direction (for each fixed y_j) to obtain the intermediate solution $u^*_{i,j}$ from Eq.(2.8),
(2) Then change the spatial direction, and solve a set of $N_y - 1$ equations (obtained from Eq.(2.9)) in y-direction (for each fixed x_i) to obtain the solution $u^{n+1}_{i,j}$ by using the intermediate solution $u^*_{i,j}$ from the first step.

To solve the first step in Eq.(2.8), the boundary conditions for the intermediate solution $u^*_{i,j}$ should be defined carefully, and these conditions should be adopted to Eq.(2.9). Boundary conditions are usually given on the boundary of the rectangular region $x_L < x < x_H$, $y_L < y < y_H$, so the boundary conditions for intermediate solution u^* can be obtained. The left and the bottom boundary conditions for $u^n_{i,j}$ are zero, and the right boundary condition $u^{n+1}_{N_x,j} = B^{n+1}_{N_x,j}$ is used to compute the boundary values for u^* as

$$u^*_{N_x,j} = (1-(\Delta t)^\alpha L_{\gamma,y})B^{n+1}_{N_x,j}, \tag{2.10}$$

that is used in solving the sets of equations defined by Eq.(2.8).
In example 4.1 it is shown that because the implicit difference-ADI method needs zero initial conditions, we should first change the initial conditions to zero.

3. Consistency and Stability of the Implicit Difference-ADI Method

In this section, we demonstrate that the implicit difference-ADI method for the two-dimensional space-time fractional diffusion equation (1.1) is not only consistent, but also unconditionally stable. Therefore according to the Lax's equivalence theorem, this method will be actually convergent. It is shown in the following theorem that this method is consistent and has truncation error $O(\Delta x) + O(\Delta y) + O(\Delta t)$.

Theorem 3.1. *Let $0 < \alpha < 1$, $1 < \beta < 2$, $1 < \gamma < 2$, the solution of Eq.(1.1) is unique, and its temporal partial derivative up to order $\alpha + 1$ and spatial partial derivatives up to order r are in $L(\mathbb{R}^3)$, and its spatial partial derivative up to order $r-1$ are zero at infinity, where $r > \alpha + \beta + 3$. Then the implicit* *difference-ADI method defined by (2.6) for solving (1.1) is consistent, and its truncation error has the order* $O(\Delta x) + O(\Delta y) + O(\Delta t)$.

Proof. We only use Grünwald formula (shifted and unshifted) to estimate fractional derivatives. The $O(\Delta x)$, $O(\Delta y)$, $O(\Delta t)$ for the truncation error of the fractional derivatives is proven in [9]. It should be proven that the additional perturbation error of (2.7) has the truncation error $O(\Delta x) + O(\Delta y)$, it was proven in [10] that $(L_{\beta,x}L_{\gamma,y})u_{i,j}^{n+1}$ converges to the mixed fractional derivative of order $O(\Delta x) + O(\Delta y)$.
Therefore, the implicit difference-ADI method has truncation error of the form $O(\Delta x) + O(\Delta y) + O(\Delta t)$. □

We define the following notations:

$$D_{i,j}^{n+1} = \frac{d_{i,j}^{n+1}(\Delta t)^{\alpha}}{(\Delta x)^{\beta}},$$
$$E_{i,j}^{n+1} = \frac{e_{i,j}^{n+1}(\Delta t)^{\alpha}}{(\Delta x)^{\gamma}}.$$

In the following, two theorems are proven to show the stability of the implicit difference-ADI method. One of them shows the stability of each one-dimensional system defined by equations (2.8) and (2.9). The proof of the first theorem is similar to theorem 3.2 in [10].

Theorem 3.2. *Let* $0 < \alpha < 1$, $1 < \beta < 2$, $1 < \gamma < 2$, *then each one-dimensional implicit system that presented in the equations (2.8) and (2.9) is unconditionally stable.*

Proof. At each gridpoint y_k, for $k = 1, \dots, N_y-1$, the linear system of equations can be written as follows:

$$A_k U_k^* = -\omega_{\alpha,1} I U_k^n - \omega_{\alpha,2} I U_k^{n-1} - \cdots - \omega_{\alpha,n+1} I U_k^0 + (\Delta t)^{\alpha} Q_k^{n+1}, \qquad (3.1)$$

where results from Eq.(2.8) and we have

$$U_k^* = [u_{1,k}^*, u_{2,k}^*, \dots, u_{N_x-1,k}^*]^T,$$
$$U_k^* = [u_{1,k}^n, u_{2,k}^n, \dots, u_{N_x-1,k}^n]^T,$$

and according to the boundary conditions from Eq.(2.10) we get

$$Q_k^{n+1} = [q_{1,k}^{n+1}, q_{k,2}^{n+1}, \dots, q_{N_x-1,k}^{n+1} + D_{N_x-1,k}^{n+1}\omega_{\alpha,0}(1-(\Delta t)^{\alpha}L_{\gamma,y})B_{N_x,k}^{n+1}]^T,$$

and $A_k = [A_{i,j}]$ is the $(N_x - 1) \times (N_x - 1)$ matrix of coefficients resulting from Eq.(3.1) where the matrix entries are resulting from ith row defined by Eq.(2.8). For example, for $i = 1$ we have

$$-D_{1,k}^{n+1}\omega_{\beta,2}u_{0,k}^* + (1 - D_{1,k}^{n+1}\omega_{\beta,1})u_{1,k}^* - D_{1,k}^{n+1}\omega_{\beta,0}u_{2,k}^* =$$
$$-\sum_{s=1}^{n+1}\omega_{\alpha,s}u_{1,k}^{n+1-s} + q_{1,k}^{n+1}(\Delta t)^{\alpha},$$

also, for $i = N_x - 1$ the equation is as follows:

$$-D^{n+1}_{N_x-1,k}\omega_{\beta,N_x}u^*_{0,k} - D^{n+1}_{N_x-1,k}\omega_{\beta,N_x-1}u^*_{1,k} - \ldots + (1 - D^{n+1}_{N_x-1,k}\omega_{\beta,1})u^*_{N_x-1,k}$$
$$-D^{n+1}_{N_x-1,k}\omega_{\beta,0}u^*_{N_x,k} = -\sum_{s=1}^{n+1}\omega_{\alpha,s}u^{n+1-s}_{N_x-1,k} + q^{n+1}_{N_x-1,k}(\Delta t)^{\alpha}.$$

Then, the entries $A_{i,j}$, for $i = 1, \ldots, N_x - 1$, $j = 1, \ldots, N_x - 1$ are defined by

$$A_{i,j} = \begin{cases} -D^{n+1}_{i,k}\omega_{\beta,i-j+1}, & j \leq i-1, \\ 1 - D^{n+1}_{i,k}\omega_{\beta,1}, & j = i, \\ -D^{n+1}_{i,k}\omega_{\beta,0}, & j = i+1, \\ 0, & j > i+1. \end{cases}$$

To prove the stability, it will be shown that the spectral radius of each matrix A_k^{-1} is less than one. We will show that every eigenvalue of the matrix A_k has a magnitude larger than 1, using the Gershgorin theorem.
It is easy to see that $-\omega_{\beta,1} > \sum_{l=0,l\neq 1}^{N}\omega_{\beta,k}$, because $\omega_{\beta,1} = -\beta$, and for $1 < \beta < 2$ and $j \neq 1$ we have $\omega_{\beta,j} > 0$. Substituting $z = -1$ into $(1+z)^{\beta} = \sum_{l=0}^{\infty}\binom{\beta}{l}z^l$ yields $\sum_{l=0}^{\infty}\omega_{\beta,l} = 0$, and therefore $-\omega_{\beta,1} > \sum_{k=0,k\neq 1}^{N}\omega_{\beta,k}$. Using Gershgorin theorem, the eigenvalues of the matrix A_k are in the disks centered at $A_{i,i} = 1 - D^{n+1}_{i,k}\omega_{\beta,1} = 1 + D^{n+1}_{i,k}\beta$, with radius

$$r_i = \sum_{l=1,l\neq i}^{N_x-1}|A_{i,l}| \leq \sum_{l=1,l\neq i}^{i+1}D^{n+1}_{i,k}\omega_{\beta,i-l+1} < D^{n+1}_{i,k}\beta.$$

So every eigenvalue λ of the matrix A_k has a real part larger than 1, and hence a magnitude larger than 1. Therefore, the spectral radius of each matrix A_k^{-1} is less than one.
The stability proof of the second step is also similar to theorem 3.2 in [10], but we need the coefficient matrix to prove the next theorem. When we change the direction of sweeping to obtain u^{n+1} from u^*, we should solve the linear system of equations defined by $C_k U_k^{n+1} = U_k^*$ that result from Eq.(2.9) at the fixed grid point x_k, where

$$U_k^* = [u^*_{k,1}, u^*_{k,2}, \ldots, u^*_{k,N_y-1}]^T,$$
$$U_k^{n+1} = [u^{n+1}_{k,1}, u^{n+1}_{k,2}, \ldots, u^{n+1}_{k,N_y-1}]^T,$$

and $C_k = [C_{i,j}]$ is the coefficient matrix at the grid point x_k for $k = 1, \ldots, N_x - 1$. The entries of the matrix C_k are defined from (2.9), for $i = 1, \ldots, N_y - 1$, $j = 1, \ldots, N_y - 1$ as follows:

$$C_{i,j} = \begin{cases} -E^{n+1}_{k,i}\omega_{\gamma,i-j+1}, & j \leq i-1, \\ 1 - E^{n+1}_{k,i}\omega_{\gamma,1}, & j = i, \\ -E^{n+1}_{k,i}\omega_{\gamma,0}, & j = i+1, \\ 0, & j > i+1. \end{cases}$$

Similar argument results that C_k has spectral radius less than one. □

Eq.(2.6) can be written in the matrix form

$$MNU^{n+1} = -\omega_{\alpha,1}IU^n - \omega_{\alpha,2}IU^{n-1} - \cdots - \omega_{\alpha,n+1}IU^0 + R^{n+1}, \qquad (3.2)$$

where the matrices M, N represent the operators $(1-(\Delta t)^\alpha L_{\beta,x})$, $(1-(\Delta t)^\alpha L_{\gamma,y})$, and

$$U^n = [u^n_{1,1}, \ldots, u^n_{N_x-1,1}, u^n_{1,2}, \ldots, u^n_{N_x-1,2}, \ldots, u^n_{1,N_y-1}, \ldots, u^n_{N_x-1,N_y-1}]^T,$$

and the vector R^{n+1} contains the forcing term and the boundary conditions. The matrix M is a block diagonal matrix, with $(N_y-1)\times(N_y-1)$ blocks. The non-zero blocks are $(N_x-1)\times(N_x-1)$ super-triangular A_k matrices resulting from Eq.(2.8). Then, the matrix M can be written as

$$M = diag(A_1, A_2, \ldots, A_{N_y-1}).$$

The matrix N is a block super-triangular matrix with $(N_y-1)\times(N_y-1)$ blocks in which the non-zero blocks are $(N_x-1)\times(N_x-1)$ diagonal matrices C_k resulting from Eq.(2.9). If $N = [N_{i,j}]$, where each $N_{i,j}$ is an $(N_x-1)\times(N_x-1)$ matrix, we can write $N_{i,j}$ as follows:

$$N_{i,j} = \begin{cases} 0, & j > i+1, \\ diag((C_1)_{i,j}, (C_2)_{i,j}, \ldots, (C_{N_x-1})_{i,j}), & j \le i+1, \end{cases}$$

where $(C_k)_{i,j}$ is (i,j)th entry of matrix C_k.
We need the matrices M and N to commute in order to show unconditional stability. The commutativity assumption refers to commutativity assumption for the operators $(1-(\Delta t)^\alpha L_{\beta,x})$, $(1-(\Delta t)^\alpha L_{\beta,x})$. The requirement for the commutativity of these two operators is also a common assumption in establishing stability and convergence of the ADI methods in the classical (i.e. $\alpha = 1$, $\beta = \gamma = 2$) two-dimensional equation (see [4, 10]).
We will present the following lemma to prove the stability theorem.

Lemma 3.3. *Let $0 < \alpha < 1$, then it can be shown that*

(1) $-1 < \omega_{\alpha,j} < 0$, *for* $j = 1, 2, \ldots$,
(2) $\forall N \ge 1$,

$$0 < -\sum_{j=1}^{N} \omega_{\alpha,j} < 1.$$

Theorem 3.4. *Let $0 < \alpha < 1$, $1 < \beta < 2$, $1 < \gamma < 2$. Assuming that the matrices M, N commute then the implicit difference-ADI method, presented in (2.6), is unconditionally stable.*

Proof. It was shown that $M = diag(A_1, A_2, \ldots, A_{N_y-1})$, according to Gershgorin theorem, the eigenvalues of the matrix M are in the union of the Gershgorin disks for the matrices A_k. According to the proof of theorem 3.2, every eigenvalue of the matrix M has a real-part, and a magnitude larger than 1. Hence the magnitude of every eigenvalue of the inverse matrix M^{-1} is less

than 1, and so the spectral radius of matrix M^{-1} is less than 1. We had similar argument about matrix C_k in theorem 3.2. Similarly, the eigenvalues of the matrix N are in the union of the Gershgorin disks for the matrices C_k, therefore the spectral radius of the matrix N^{-1} is also less than 1.
Let us assume an error ε^0 in U^0 then according to Eq.(3.2) an error ε^{n+1} in U^{n+1} follows as

$$\varepsilon^{n+1} = -\omega_{\alpha,1}(MN)^{-1}\varepsilon^n - \omega_{\alpha,2}(MN)^{-1}\varepsilon^{n-1} - \cdots - \omega_{\alpha,n+1}(MN)^{-1}\varepsilon^0.$$

Since the matrices M, N commute, we have

$$\begin{aligned} \|\varepsilon^{n+1}\|_\infty \leq & \ |\omega_{\alpha,1}|\|M^{-1}\|\|N^{-1}\|\|\varepsilon^n\| + |\omega_{\alpha,2}|\|M^{-1}\|\|N^{-1}\|\|\varepsilon^{n-1}\| \\ & + \cdots + |\omega_{\alpha,n+1}|\|M^{-1}\|\|N^{-1}\|\|\varepsilon^0\|, \end{aligned} \tag{3.3}$$

where $\|.\|$ is infinity norm. Since spectral radius of the inverse matrices M^{-1}, N^{-1} are less than 1, according to the second Gershgorin theorem [20], $\|M^{-1}\| \leq 1$, $\|N^{-1}\| \leq 1$. We use mathematical induction to prove $\|\varepsilon^n\| \leq \|\varepsilon^0\|$. Subsitute $n = 0$ in Eq.(3.3)

$$\|\varepsilon^1\| \leq |\omega_{\alpha,1}|\|M^{-1}\|\|N^{-1}\|\|\varepsilon^0\| \leq \|\varepsilon^0\|.$$

Let $\|\varepsilon^k\| \leq \|\varepsilon^0\|$, for $k = 1, \ldots, n$, then Eq. (3.3) can be written as follows

$$\begin{aligned} \|\varepsilon^{n+1}\| & \leq |\omega_{\alpha,1}|\|\varepsilon^n\| + |\omega_{\alpha,2}|\|\varepsilon^{n-1}\| + \cdots + |\omega_{\alpha,n+1}|\|\varepsilon^0\| \\ & \leq |\omega_{\alpha,1}|\|\varepsilon^0\| + |\omega_{\alpha,2}|\|\varepsilon^0\| + \cdots + |\omega_{\alpha,n+1}|\|\varepsilon^0\| \\ & = \sum_{l=1}^{n+1} \omega_{\alpha,l}\|\varepsilon^0\| \end{aligned}$$

Now we have $\|\varepsilon^{n+1}\| \leq \|\varepsilon^0\|$ by using lemma 3.3. This proves stability of implicit difference-ADI method. □

We note that zero boundary conditions on the left and the bottom of the rectangular domain are necessary and we extend this zero condition, because in this case the Riemman-Liouville and the Liouville definition coincide and all the results about consistency and the order of Grünwald formula already proven for Liouville case [9, 10, 22] are applicable. It is interesting to solve this problem without non-zero boundary condition [21].

4. Numerical Results

We solve two fractional differential equations, see [26], and compare the numerical results, and also solve another example to show convergence of the presented method.

4.1. **Example 1.** Consider the fractional diffusion equation

$$\frac{\partial^{0.5}u(x,y,t)}{\partial t^{0.5}} = d(x,y,t)\frac{\partial^{1.2}u(x,y,t)}{\partial x^{1.2}} + e(x,y,t)\frac{\partial^{1.8}u(x,y,t)}{\partial y^{1.8}} + q(x,y,t), \quad (4.1)$$

on finite domain $0 < x < 1$, $0 < y < 1$, for $0 \le t \le 1$. The diffusion coefficients and the forcing function are as follows

$$d(x,y,t) = \frac{\Gamma(2.8)}{\Gamma(4)}x^{1.2},$$
$$e(x,y,t) = \frac{\Gamma(2.2)}{\Gamma(4)}y^{1.8},$$
$$q(x,y,t) = x^3y^3\left(\frac{8}{3\Gamma(0.5)}t^{1.5} - 2t^2 - 2\right),$$

with initial conditions $u(x,y,0) = x^3y^3$, and the boundary conditions

$$u(0,y,t) = u(x,0,t) = 0,$$
$$u(1,y,t) = (t^2+1)y^3,$$
$$u(x,1,t) = (t^2+1)x^3.$$

The exact solution is $u(x,y,t) = (t^2+1)x^3y^3$.
Since the implicit difference-ADI method needs zero initial conditions, we assume that

$$v(x,y,t) = u(x,y,t) - u(x,y,0) = u(x,y,t) - x^3y^3, \quad (4.2)$$

and by using the Riemman-Liouville derivative formula [14]

$${}_0D_x^{\alpha}x^{\nu} = \frac{\Gamma(1+\nu)}{\Gamma(1+\nu-\alpha)}x^{\nu-\alpha},$$

we have

$$\frac{\partial^{0.5}v(x,y,t)}{\partial t^{0.5}} = \frac{\partial^{0.5}u(x,y,t)}{\partial t^{0.5}},$$
$$\frac{\partial^{1.2}v(x,y,t)}{\partial x^{1.2}} = \frac{\partial^{1.2}u(x,y,t)}{\partial x^{1.2}} - \frac{\Gamma(4)}{\Gamma(2.8)}x^{1.8}y^3,$$
$$\frac{\partial^{1.8}v(x,y,t)}{\partial y^{1.8}} = \frac{\partial^{1.8}u(x,y,t)}{\partial y^{1.8}} - \frac{\Gamma(4)}{\Gamma(2.2)}x^3y^{1.2}.$$

Substituting $v(x,y,t)$ for $u(x,y,t)$ results in the following differential equation with zero initial conditions

$$\frac{\partial^{0.5}v(x,y,t)}{\partial t^{0.5}} = d(x,y,t)\frac{\partial^{1.2}v(x,y,t)}{\partial x^{1.2}} + e(x,y,t)\frac{\partial^{1.8}v(x,y,t)}{\partial y^{1.8}} + q(x,y,t), \quad (4.3)$$

with the new diffusion coefficients and the new forcing function

$$d(x,y,t) = \frac{\Gamma(2.8)}{\Gamma(4)} x^{1.2},$$
$$e(x,y,t) = \frac{\Gamma(2.2)}{\Gamma(4)} y^{1.8},$$
$$q(x,y,t) = x^3 y^3 \left(\frac{8}{3\Gamma(0.5)} t^{1.5} - 2t^2 \right),$$

with the boundary conditions

$$v(0,y,t) = v(x,0,t) = 0,$$
$$v(1,y,t) = t^2 y^3,$$
$$v(x,1,t) = t^2 x^3.$$

Table 1 shows the maximum absolute numerical error, at time t=1.

TABLE 1. Maximum absolute numerical error by using the implicit difference-ADI method for Example 1 at time t=1

Δt	$\Delta x = \Delta y$	*Maximum error*
$\frac{1}{10}$	$\frac{1}{10}$	8.43962×10^{-3}
$\frac{1}{20}$	$\frac{1}{20}$	8.34223×10^{-3}
$\frac{1}{40}$	$\frac{1}{40}$	7.23612×10^{-3}
$\frac{1}{100}$	$\frac{1}{100}$	$5.40802 \times 10^{z}-3$

TABLE 2. Maximum absolute numerical error by using the implicit difference approximation [26] for Example 1 at time t=1

Δt	$\Delta x = \Delta y$	*Maximum error*
$\frac{1}{10}$	$\frac{1}{10}$	9.93756×10^{-2}
$\frac{1}{20}$	$\frac{1}{20}$	7.14376×10^{-2}
$\frac{1}{40}$	$\frac{1}{40}$	4.23914×10^{-2}
$\frac{1}{100}$	$\frac{1}{100}$	1.87382×10^{-2}

Example 1 was solved in [26] by applying the implicit difference approximation. The numerical result in table 1 can be compared with table 2. Accuracy of results obtained by the method of this paper is better than those reported in [26].

4.2. **Example 2.** Consider the fractional diffusion equation

$$\frac{\partial^{0.4} u(x,y,t)}{\partial t^{0.4}} = d(x,y,t) \frac{\partial^2 u(x,y,t)}{\partial x^2} + e(x,y,t), \frac{\partial^2 u(x,y,t)}{\partial y^2} + q(x,y,t), \quad (4.4)$$

on finite domain $0 < x < 1$, $0 < y < 1$, for $0 \le t \le 1$. The diffusion coefficients and the forcing function are as follows

$$\begin{aligned} d(x,y,t) &= \frac{2t^{1.6}}{\pi^2}\Gamma(0.6), \\ e(x,y,t) &= \frac{t^{1.6}}{12\pi^2\Gamma(0.6)} \\ q(x,y,t) &= \frac{25t^{1.6}}{12\Gamma(0.6)}(t^2+2)\sin\pi x\cos\pi y, \end{aligned}$$

with initial conditions $u(x,y,0) = \sin\pi x \cos\pi y$, and the boundary conditions

$$\begin{aligned} u(0,y,t) = u(x,0,t) = 0, \\ u(1,y,t) = u(x,1,t) = 0. \end{aligned}$$

The exact solution for this fractional diffusion equation is $u(x,y,t) = (t^2+2)\sin\pi x\cos\pi y$.

TABLE 3. Maximum absolute numerical error by using the implicit difference-ADI for Example 2 at time t=1

Δt	$\Delta x = \Delta y$	*Maximum error*
$\frac{1}{16}$	$\frac{1}{4}$	2.96627×10^{-2}
$\frac{1}{64}$	$\frac{1}{8}$	4.85530×10^{-3}
$\frac{1}{100}$	$\frac{1}{10}$	2.25561×10^{-3}
$\frac{1}{400}$	$\frac{1}{10}$	2.47028×10^{-4}

Table 3 shows that accuracy of this method is better than [26], table 4. The order of spatial derivatives are 2 in this example, so the shifted Grünwald formula changes to central difference formula and the numerical results show this.

TABLE 4. Maximum absolute numerical error by using the implicit difference approximation [26] for Example 2 at time t=1

Δt	$\Delta x = \Delta y$	*Maximum error*
$\frac{1}{16}$	$\frac{1}{4}$	5.39188×10^{-2}
$\frac{1}{64}$	$\frac{1}{8}$	1.30699×10^{-2}
$\frac{1}{100}$	$\frac{1}{10}$	8.26645×10^{-3}
$\frac{1}{400}$	$\frac{1}{20}$	1.67537×10^{-3}

4.3. **Example 3.** Consider the fractional diffusion equation

$$\frac{\partial^{0.5}u(x,y,t)}{\partial t^{0.5}} = d(x,y,t)\frac{\partial^{1.8}u(x,y,t)}{\partial x^{1.8}} + e(x,y,t)\frac{\partial^{1.6}u(x,y,t)}{\partial y^{1.6}} + q(x,y,t), \quad (4.5)$$

on finite domain $0 < x < 1$, $0 < y < 1$, for $0 \le t \le 1$. The diffusion coefficients and the forcing function are as follows

$$d(x,y,t) = \frac{\Gamma(2.2)}{6}x^{2.8}y,$$
$$e(x,y,t) = \frac{2}{\Gamma(4.6)}y^{2.6}x,$$
$$q(x,y,t) = t^{0.5}x^3y^{3.6}E_{1,1.5}(t) - 2(e^t - 1)x^4y^{4.6},$$

with initial conditions $u(x,y,0) = 0$, and the boundary conditions

$$u(0,y,t) = u(x,0,t) = 0,$$
$$u(1,y,t) = (e^t - 1)y^{3.6},$$
$$u(x,1,t) = (e^t - 1)x^3.$$

The exact solution for this fractional diffusion equation is $u(x,y,t) = (e^t - 1)x^3y^{3.6}$.

Table 5. Maximum absolute numerical error by using the implicit difference-ADI for Example 3 at time t=1

Δt	$\Delta x = \Delta y$	*Maximum error*
$\frac{1}{10}$	$\frac{1}{10}$	1.54478×10^{-2}
$\frac{1}{20}$	$\frac{1}{20}$	1.46362×10^{-2}
$\frac{1}{40}$	$\frac{1}{40}$	1.20455×10^{-2}
$\frac{1}{80}$	$\frac{1}{80}$	9.43800×10^{-3}
$\frac{1}{100}$	$\frac{1}{100}$	8.66339×10^{-3}

These numerical examples show convergence of the implicit difference-ADI method as was proven.

5. Conclusions

In this paper, the implicit difference method for the two-dimensional space-time fractional diffusion equation has been presented, and the alternating direction implicit approach was used to decrease computational cost. We proved that our method is consistent and unconditionally stable, and so convergent. This method can be used to solve time fractional, or space fractional, and space-time fractional differential equations with appropriate computational cost. Additionally we need zero initial conditions to use Grünwald formula, and also zero boundary conditions on the left and the bottom of the rectangular domain.

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