Lie Ideals and Generalized Derivations in Semiprime Rings

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Abstract. Let \( R \) be a 2-torsion free ring and \( L \) a Lie ideal of \( R \). An additive mapping \( F : R \to R \) is called a generalized derivation on \( R \) if there exists a derivation \( d : R \to R \) such that \( F(xy) = F(x)y + xd(y) \) holds for all \( x, y \in R \). In the present paper we describe the action of generalized derivations satisfying several conditions on Lie ideals of semiprime rings.

Keywords: Derivations, Generalized derivations, Semiprime rings, Lie ideals.

2000 Mathematics subject classification: 16W25, 16U80, 16N60.

1. Introduction

Let \( R \) be an associative ring with center \( Z(R) \). A ring \( R \) is said to be \( n \)-torsion free if \( nx = 0 \) implies \( x = 0 \) for all \( x \in R \). For any \( x, y \in R \), the symbol [\( x, y \)] will represent the commutator \( xy - yx \). Recall that a ring \( R \) is prime if \( aRb = 0 \) implies \( a = 0 \) or \( b = 0 \) and \( R \) is semiprime if \( aRa = 0 \) yields \( a = 0 \). An additive mapping \( d : R \to R \) is said to be a derivation of \( R \) if
\[ d(xy) = d(x)y + xd(y) \text{ for all } x, y \in R. \]
In particular, for a fixed \( a \in R \) the mapping \( I_a : R \to R \) given by \( I_a(x) = [x, a] \) is a derivation which is called an inner derivation determined by \( a \). In 1991 Bresar [5] introduced the concept of generalized derivation: more precisely an additive mapping \( F : R \to R \) is said to be a generalized derivation with associated derivation \( d \) if \( F(xy) = F(x)y + xd(y) \) for all \( x, y \in R \). For the sake of convenience, we shall denote by \((F, d)\) a generalized derivation \( F \) with associated derivation \( d \). A mapping \( f : R \to R \) is known to be centralizing on \( R \) if \([f(x), x] \in Z(R)\) for all \( x \in R \). In particular, if \([f(x), x] = 0\) for all \( x \in R \), then \( f \) is said to be commuting on \( R \).
We recall that an additive group \( L \) of \( R \) is said to be a Lie ideal of \( R \) if \([L, R] \subseteq L \).

A well known result of Posner [18] states that a prime ring admitting a nonzero centralizing derivation must be commutative. This theorem indicates that the global structure of a ring \( R \) is often tightly connected to the behaviour of additive mappings defined on \( R \). Following this line of investigation, several authors studied derivations and generalized derivations acting on appropriate subsets of the ring.

For instance in [19] Quadri et al. prove that if \( R \) is a prime ring with a non-zero ideal \( I \) and \( F \) is a generalized derivation of \( R \) such that \( F([x, y]) = [x, y] \), for all \( x, y \in I \), then \( R \) is commutative (Theorem 2.1). Later in [7] Dhara extends all results contained in [19] to semiprime rings.

Further in [10] Gölbaşı and Koç investigate the properties of a prime ring \( R \) with a generalized derivation \((F, d)\) acting on a Lie ideal \( L \) of \( R \). They prove that if \([F(u), u] \in Z(R)\), for all \( u \in L \), then either \( d = 0 \) or \( L \subseteq Z(R) \) (Theorem 3.3). Moreover if \( F([u, v]) = [u, v] \), for all \( u, v \in L \), then either \( d = 0 \) of \( L \subseteq Z(R) \) (Theorem 3.6).

In this note we will consider a similar situation and extend the cited results to the case of semiprime rings with a generalized derivation \((F, d)\) acting on a Lie ideal. More precisley we prove the following:

**Theorem 1.** Let \( R \) be a 2-torsion free semiprime ring and \( L \) be a non-central Lie ideal of \( R \). Suppose that \((F, d)\) is a generalized derivation of \( R \) such that \( F[x, y] \in Z(R) \), for all \( x, y \in L \). If \( d(L) \neq (0) \), then all the following hold simultaneously:

1. \( d(R)[L, R] = (0) \) and \( [d(R), L] = (0) \);
2. \( a[L, R] = (0) \) and \( [a, L] = (0) \);
3. \( aI = (0) \) and \( d(I) = (0) \) (that is \( F(I) = (0) \)), where \( I \) denotes the ideal of \( R \) generated by \([L, L] \).

**Theorem 2.** Let \( R \) be a 2-torsion free semiprime ring and \( L \) be a non-central Lie ideal of \( R \). Suppose \( R \) admits a generalized derivation \((F, d)\), defined as \( F(x) = ax + d(x) \), for all \( x \in R \) and fixed element \( a \in R \). If \([F(x), x] \in Z(R)\) for all \( x \in L \) and \( d(L) \neq (0) \), then all the following hold simultaneously:
(1) \( d(R)[L,R] = (0) \) and \([d(R),L] = (0)\);  
(2) \([a,L] = a[L,R] = (0)\);  
(3) \( aI = d(I) = (0) \) (that is \( F(I) = (0) \)), where \( I \) denotes the ideal of \( R \) generated by \([L,L]\).  

**Theorem 3.** Let \( R \) be a 2-torsion free semiprime ring and \( L \) be a non-central Lie ideal of \( R \). Suppose \( R \) admits two generalized derivations \((F,d)\) and \((G,g)\). Write \( F(x) = ax + d(x) \) and \( G(x) = bx + g(x) \), for some \( a, b \in U \). If \( F([x,y]) = [y,G(x)] \) for all \( x, y \in L \), then either  

1. \( g(L) = (0) \);  
2. \( d(R)[L,R] = (0) \) and \([d(R),L] = (0)\);  
3. \( (a + b)[L,R] = (0), [b,L] = (0) \) and \([a,L] = (0)\);  
4. \( aI = d(I) = (0) \) and \( d(I) = (0) \), where \( I \) denotes the ideal of \( R \) generated by \([L,L]\). 

or  

1. \( d(L) = (0) \);  
2. \( g(R)[L,R] = (0) \) and \([g(R),L] = (0)\);  
3. \( [b,L] = (0) \) and \([a,L,L] = (0)\);  
4. \( aI = (0) \) and \( g(I) = (0) \), where \( I \) denotes the ideal of \( R \) generated by \([L,L]\). 

or  

1. \( d(R)[L,R] = (0) \) and \([d(R),L] = (0)\);  
2. \( g(R)[L,R] = (0) \) and \([g(R),L] = (0)\);  
3. \( [a,L] = (0), [b,L] = (0), b[L,R] = a[L,R] = (0)\);  
4. \( d(I) = g(I) = (0) \) and \( aI = bI = (0) \) (that is \( F(I) = G(I) = (0) \)), where \( I \) denotes the ideal of \( R \) generated by \([L,L]\). 

In all that follows let \( R \) be a non-commutative semiprime ring, \( L \) a non-central Lie ideal of \( R, U \) the right Utumi quotient ring of \( R \). We refer the reader to [3] for the definition and the related properties of \( U \).

We begin with the following:

**Fact 1.1.** Let \( R \) be a semiprime ring. Then every generalized derivation \( F \) of \( R \) is uniquely extended to its right Utumi quotient ring \( U \) and assumes the form \( F(x) = ax + d(x) \), where \( a \in U \) and \( d \) is the derivation of \( U \) associated with \( F \) (see Theorem 4 in [17]).

**Lemma 1.2.** Let \( R \) be a prime ring of characteristic different from 2 and \( L \) be a Lie ideal of \( R \). Suppose \( R \) admits a nonzero generalized derivation \((F,d)\) such that \( F(x)[x,y] = 0 \) (or \([x,y]F(x) = 0\) for all \( x, y \in L \), then \( L \subseteq Z(R) \)).

**Proof.** Suppose by contradiction that \( L \) is not central in \( R \). By [11] (pages 4-5) there exists a non-central ideal \( I \) of \( R \) such that \( 0 \neq [I,R] \subseteq L \). By our
assumption it follows that \( F(x)[x, y] = 0 \) (or \([x, y]F(x) = 0\)) for all \( x, y \in [I, R] \). Since \( I \) and \( R \) satisfy the same differential identities (see the main result in [16]), we also have that \( F(x)[x, y] = 0 \) (or \([x, y]F(x) = 0\)) for all \( x, y \in [R, R] \). Let \( y_0 \in [R, R] \) be such that \( y_0 \notin Z(R) \) and denote by \( \delta : R \to R \) the non-zero inner derivation of \( R \) induced by the element \( y_0 \). Therefore \( F(x)\delta(x) = 0 \) (or \( \delta(x)F(x) = 0 \)) for all \( x \in [R, R] \). In light of [6], since \( \delta \neq 0 \) and \([R, R] \) is not central in \( R \), one has the contradiction that \( F = 0 \).

**Lemma 1.3.** Let \( R \) be a 2-torsion free semiprime ring and \( L \) be a non-central Lie ideal of \( R \). Suppose \( R \) admits a nonzero generalized derivation \((F, d)\), defined as \( F(x) = ax + d(x) \), for all \( x \in R \) and fixed element \( a \in R \). If \( F(x)[x, y] = 0 \) (or \( [x, y]F(x) = 0 \)) for all \( x, y \in L \), then all the following hold simultaneously:

1. \( d(R)[L, R] = (0) \) and \([d(R), L] = (0)\);
2. \( a[L, R] = (0) \) and \([a, L] = (0)\);
3. \( aI = (0) \) and \( d(I) = (0) \) (that is \( F(I) = (0) \)), where \( I \) denotes the ideal of \( R \) generated by \([L, L]\).

**Proof.** Let \( P \) be a prime ideal if \( R \) such that \([L, L] \notin P\). Assume first that \( d(P) \subseteq P \). Then \( F \) induces a canonical generalized derivation \( \overline{F} \) on \( \overline{R} = \frac{R}{P} \). Therefore \( \overline{F}([\overline{x}, \overline{y}]_P) = 0 \) for all \( \overline{x}, \overline{y} \in \overline{L} \). Moreover \( \overline{L} \) is a Lie ideal of \( \overline{R} \), such that \([L, L] \neq 0 \) since \([L, L] \notin P \). By Lemma 1.2 it follows that \( \overline{F}(R) = 0 \) that is \( aR \subseteq P \), \( d(R) \subseteq P \) and \( F(R) \subseteq P \).

Assume now that \( d(P) \notin P \), then \( d(P) \neq 0 \) and \( d(P)\overline{R} \neq 0 \). Moreover note that, for any \( p \in P \) and \( r, s \in R \), \( d(pr)s = d(p)rs + pd(r)s \) implies that \( d(P)R \subseteq d(PR)R + P \), in particular \( d(P)\overline{R} \) is a non-zero right ideal of \( \overline{R} \).

Starting from our main assumption and linearizing we have that \( F(x)[z, y] + F(z)[x, y] = 0 \), for all \( x, y, z \in L \). For any \( p \in P \), \( r \in R \), \( u \in L \), replace \( x \) by \([pr, u] \). By computation it follows \( [\overline{p}, \overline{r}][\overline{x}, \overline{y}] = 0 \), for all \( \overline{p} \in d(P)\overline{R} \) and \( \overline{x}, \overline{y} \in \overline{L} \). By using the same argument of Lemma 1.2, since \( \overline{L} \) is not central in \( \overline{R} \), one has that \( d(P)\overline{R} \) is a central right ideal of \( \overline{R} \), which implies that \( \overline{R} \) is commutative, a contradiction.

Therefore, for any prime ideal \( P \) of \( R \), either \( aR \subseteq P \), \( d(R) \subseteq P \) and \( F(R) \subseteq P \) or \([L, L] \subseteq P \). In this last case, by applying Theorem 3 in [15] in the prime ring \( \overline{R} \), since \( char(\overline{R}) \neq 2 \) and \([L, L] = 0 \), we have that \( \overline{L} \) is central in \( \overline{R} \), which means \([L, R] \subseteq P \).

Hence in any case it follows that \( d(R)[L, R] = (0) \), \( a[R, L] = (0) \) and \([d(R), L] = (0)\).

By \( a[R, L] = (0) \) we get \( aR[R, L] = (0) \) and so both \( aLR[a, L] = (0) \) and \( aR[a, L] = (0) \), that is \([a, L]R[a, L] = (0) \). By the semiprimeness of \( R \) it follows \([a, L] = (0) \).

Moreover, if \( I = R[L, L]R \) denotes the ideal of \( R \) generated by \([L, L]\), it follows that \( aI = (0) \) and \( d(I) = (0) \), that is \( F(I) = (0) \). □
Corollary 1.4. Let $R$ be a 2-torsion free semiprime ring and $L$ be a non-central Lie ideal of $R$. Suppose $a \in R$ be such that $ax[y] = 0$ for all $x, y \in L$, then $a[L, R] = (0)$, $[a, L] = (0)$ and $aI = (0)$, where $I$ denotes the ideal of $R$ generated by $[L, L]$.

Theorem 1.5. Let $R$ be a 2-torsion free semiprime ring and $L$ be a non-central Lie ideal of $R$. Suppose that $(F, d)$ is a generalized derivation of $R$ such that $F(x, y) \in Z(R)$, for all $x, y \in L$. If $d(L) \neq (0)$, then all the following hold simultaneously:

1. $d(R)[L, R] = (0)$ and $[d(R), L] = (0)$;
2. $a[L, R] = (0)$ and $[a, L] = (0)$;
3. $aI = (0)$ and $d(I) = (0)$ (that is $F(I) = (0)$), where $I$ denotes the ideal of $R$ generated by $[L, L]$.

Proof. Assume first that $R$ is prime and denote $V = [L, L]$. Hence we have $F(V) \subseteq Z(R)$. As a consequence of Lemma 2 in [9] we conclude that either $F = 0$ or $V \subseteq Z(R)$. In the first case we have the contradiction $d = 0$, and in the other case one has $L \subseteq Z(R)$ (see Lemma 2 in [12]), a contradiction again. Let now $P$ be a prime ideal of $R$ such that $[L, L] \not\subseteq P$.

Assume first that $d(P) \subseteq P$. Then $F$ induces a canonical generalized derivation $\overline{F}$ on $\overline{R} = \frac{R}{P}$. Therefore $\overline{F}(\overline{x}, \overline{y}) \in Z(\overline{R})$ for all $\overline{x}, \overline{y} \in \overline{L}$. Moreover $\overline{L}$ is a Lie ideal of $\overline{R}$, such that $[\overline{L}, \overline{L}] \neq 0$ since $[L, L] \not\subseteq P$.

By previous argument it follows that $\overline{F(\overline{L})} = 0$ that $\overline{d(R)} \subseteq P$ and $\overline{F(R)} \subseteq P$.

Assume now that $d(P) \not\subseteq P$, then $\overline{d(P)} \neq 0$ and $\overline{d(P)R} \neq 0$. We remark again that $\overline{d(P)R}$ is a non-zero right ideal of $\overline{R}$.

Starting from our main assumption and linearizing we have that

\[ F(x)y + F(x)z + xd(y) + xd(z) - F(y)x - F(z)x - yd(x) - zd(x) \in Z(R), \forall x, y, z \in L. \]

For any $p, p', p'' \in P, r, s \in R, u, v \in L$, replace $y$ by $[pr, u]$ and $z$ by $[[p's, v], p'']$.

By computation it follows

\[ \overline{\pi} - [\overline{\pi}, \overline{\pi}] \in Z(\overline{R}) \]

that is

\[ \overline{\pi}[\overline{\pi}, \overline{\pi}] \in Z(\overline{R}) \]

for all $\overline{\pi} \in \overline{d(P)R}$ and $\overline{\pi}, \overline{\pi} \in \overline{L}$. As above denote $\overline{\nabla} = [\overline{L}, \overline{L}]$, which is a Lie ideal for $\overline{R}$, and $\delta$ is the derivation of $\overline{R}$ induced by $\overline{L}$. Hence we have $\delta(\overline{\nabla}) \subseteq Z(\overline{R})$. Again as a consequence of Lemma 2 in [9] it follows that either $\delta = 0$ or $\overline{\nabla} \subseteq Z(\overline{R})$. Since $\overline{R}$ is not commutative, then there exists some $\overline{\pi} \in \overline{R}$ which is not central. Thus $\overline{\nabla} \subseteq Z(\overline{R})$, and $\overline{\nabla} \subseteq Z(\overline{R})$ follows from Lemma 2 in [12].

Therefore, for any prime ideal $P$ of $R$, either $d(R) \subseteq P$ and $F(R) \subseteq P$ or $[L, L] \subseteq P$. In this last case, by applying Theorem 3 in [15] in the prime ring $\overline{R}$, since $char(\overline{R}) \neq 2$ and $[\overline{L}, \overline{L}] = 0$, we conclude that $\overline{L}$ is central in $\overline{R}$, which
means \([L, R] \subseteq P\).
Hence in any case it follows that \(d(R)[L, R] = (0)\), \(a[R, L] = (0)\) and \([d(R), L] = (0)\). Finally we obtain the required conclusions by following the same argument as in Lemma 1.3.

In the sequel we will use the following known result:

**Lemma 1.6.** Let \(R\) be a 2-torsion free semiprime ring, \(L\) a Lie ideal of \(R\) such that \(L \not\subseteq Z(R)\). Let \(a \in L\) be such that \(aL = 0\), then \(a = 0\).

**Remark 1.7.** If \(R\) is a prime ring of characteristic different from 2, \(a \in R\) and \(L\) is a non-central Lie ideal of \(R\) such that \([a, L] \subseteq Z(R)\), then \(a \in Z(R)\).

**Proof.** Denote by \(\delta : R \to R\) the inner derivation of \(R\) induced by the element \(a \in R\). Since \([a, x], r\] = 0 for all \(x \in L\) and \(r \in R\), a fortiori we have \([a, x]_2 = 0\), that is \([\delta(x), x] = 0\), for all \(x \in L\). Thus, by [14] it follows \(\delta = 0\), that is \(a \in Z(R)\).

**Theorem 1.8.** Let \(R\) be a 2-torsion free semiprime ring and \(L\) be a non-central Lie ideal of \(R\). Suppose \(R\) admits a generalized derivation \((F, d)\), defined as \(F(x) = ax + d(x)\), for all \(x \in R\) and fixed element \(a \in R\). If \([F(x), x] \in Z(R)\) for all \(x \in L\). \(\text{(1.1)}\)

and \(d(L) \neq (0)\), then all the following hold simultaneously:

1. \(d(R)[L, R] = (0)\) and \([d(R), L] = (0)\);
2. \([a, L] = a[L, R] = (0)\);
3. \(aI = d(I) = (0)\) (that is \(F(I) = (0)\)), where \(I\) denotes the ideal of \(R\) generated by \([L, L]\).

**Proof.** Let \(P\) be a prime ideal of \(R\) such that \([L, L] \not\subseteq P\).
Assume first that \((d(P) \subseteq P\). Then \(F\) induces a canonical generalized derivation \(\mathcal{F}\) on \(\overline{R} = \frac{R}{P}\). Therefore \([\mathcal{F}([x], [x])] \in Z(\overline{R})\) for all \([x] \in L\). Moreover \(L\) is a Lie ideal of \(\overline{R}\), such that \([L, L] \not\subseteq P\) since \([L, L] \not\subseteq P\). Since \([L, L] \not\subseteq P\), a fortiori we get \(L\) is not central in \(\overline{R}\). Therefore, by Theorem 3.3 in [10], it follows that \(\overline{L(\overline{R})} = \overline{0}\) that is \(d(R) \subseteq P\).
Assume now that \((d(P) \not\subseteq P\), then \(\overline{d(P)} \neq \overline{0}\) and \(\overline{d(P)R} \neq \overline{0}\). By using similar argument as in Lemma 1.3, \(\overline{Rd(P)}\) is a non-zero right ideal of \(\overline{R}\).
Linearizing \((1.1)\) and using \((1.1)\), we obtain \([F(x), y] + [F(y), x] \in Z(R)\) for all \(x, y \in L\). \((1.2)\)

Now, replace \(y\) by \([rp, u]\), for \(r \in R\), \(p \in P\) and \(u \in L\) and use \((1.2)\) to get \([F([rp, u]), [x]] \in Z(\overline{R})\). \((1.3)\)
Moreover, since \(F(r) = ar + d(r)\), for all \(r \in R\), by \((1.3)\) it follows \([\overline{d([rp, u])}, [L]] \subseteq Z(\overline{R})\). \((1.4)\)
By the primeness of $\mathcal{R}$ and Remark 1.7, one has that $\mathcal{d}(\bar{rp},u) \in Z(\mathcal{R})$. On the other hand, an easy computation shows that $\mathcal{d}(\bar{rp},u) = [rd(p),u]$, which implies $[\mathcal{R}d(P),L] \subseteq Z(\mathcal{R})$. Once again by Remark 1.7, we have $\mathcal{R}d(P) \subseteq Z(\mathcal{R})$. Since $\mathcal{R}d(P)$ is a non-zero right ideal of $\mathcal{R}$, it follows $[\mathcal{R},\mathcal{R}] = (0)$, which contradicts with $[\mathcal{L},\mathcal{L}] \neq (0)$.

The previous argument shows that, for any prime ideal $P$ of $\mathcal{R}$, either $[L,L] \subseteq P$ or $d(R) \subseteq P$. Thus $d(R)[L,L] \subseteq \cap P = (0)$. Hence, by Lemma 1.3 and since $L \not\subseteq Z(\mathcal{R})$, we finally get the required conclusions:

1. $d(R)[L,R] = (0)$ and $[d(R),L] = (0)$;
2. $(a+b)[L,R] = (0)$ and $[a+ b,L] = (0)$;
3. $aI = d(I) = (0)$, where $I$ denotes the ideal of $\mathcal{R}$ generated by $[L,L]$.

\[ \Box \]

**Theorem 1.9.** Let $\mathcal{R}$ be a 2-torsion free semiprime ring and $L$ be a non-central Lie ideal of $\mathcal{R}$. Suppose $\mathcal{R}$ admits two generalized derivations $F,d$ and $(G,g)$. Write $F(x) = ax + d(x)$ and $G(x) = bx + g(x)$, for some $a,b \in U$. If $F([x,y]) = [y,G(x)]$ for all $x,y \in L$, then either

1. $g(L) = (0)$;
2. $d(R)[L,R] = (0)$ and $[d(R),L] = (0)$;
3. $(a+b)[L,R] = (0)$, $[b,L] = (0)$ and $[a,L] = (0)$;
4. $(a+b)I = (0)$ and $d(I) = (0)$, where $I$ denotes the ideal of $\mathcal{R}$ generated by $[L,L]$.

or

1. $d(L) = (0)$;
2. $g(R)[L,R] = (0)$ and $[g(R),L] = (0)$;
3. $[b,L] = (0)$ and $a[L,L] = (0)$;
4. $aI = (0)$ and $g(I) = (0)$, where $I$ denotes the ideal of $\mathcal{R}$ generated by $[L,L]$.

or

1. $d(R)[L,R] = (0)$ and $[d(R),L] = (0)$;
2. $g(R)[L,R] = (0)$ and $[g(R),L] = (0)$;
3. $[a,L] = (0)$, $[b,L] = (0)$, $b[L,R] = a[L,R] = (0)$;
4. $d(I) = g(I) = (0)$ and $aI = bI = (0)$ (that is $F(I) = G(I) = (0)$), where $I$ denotes the ideal of $\mathcal{R}$ generated by $[L,L]$.

**Proof.** Assume first $g(L) = (0)$, then $F([x,y]) = [y,bx]$ for all $x,y \in L$. Thus

\[ a[x,y] + d([x,y]) = b[y,x] \quad (1.5) \]

for all $x,y \in L$, that is $(a + b)[x,y] + d([x,y]) = 0$ for all $x,y \in L$. Therefore, applying Theorem 1.5, one has

1. $d(R)[L,R] = (0)$ and $[d(R),L] = (0)$;
2. $(a+b)[L,R] = (0)$ and $[a+b,L] = (0)$;
v*(a+b)*I = (0) and d(I) = (0), where I denotes the ideal of R generated by [L, L].

In particular d([L, L]) = (0) and a[y, x] = b[x, y] for all x, y ∈ L, so that (1.5) reduces to (by − yb)x = 0, for all x, y ∈ L, that is [b, L]L = (0). Hence by Lemma 1.6, we have b[L, L] = (0) and so also a[a, L] = (0).

Let now d(L) = (0), then a[x, y] = [y, G(x)] for all x, y ∈ L. In this case, for x = y, we have G(y), y] = 0 and by Theorem 1.8 the following hold:

(1) g(R)[L, R] = (0) and [g(R), L] = (0);
(2) [b, L] = (0), b[L, R] = (0) and a[L, L] = (0);
(3) bI = (0) and g(I) = (0), where I denotes the ideal of R generated by [L, L].

Moreover, since [[L, L], R] ⊆ [L, L], we also have 0 = a[[L, L], R] = aR[L, L], which implies aI = (0).

Assume finally that both g(L) ≠ (0) and d(L) ≠ (0). Once again for x = y ∈ L we have [G(x), x] = 0 for any x ∈ L. Thus by Theorem 1.8, we have that all the following hold:

(1) g(R)[L, R] = (0) and [g(R), L] = (0);
(2) [b, L] = (0) and b[L, R] = (0);
(3) bI = (0) and g(I) = (0), where I denotes the ideal of R generated by [L, L].

Hence by the main assumption it follows that (a + b)[x, y] + d([x, y]) = 0, for all x, y ∈ L. Denote H(x) = (a − b)x + d(x), then H(u) = 0 for all u ∈ [L, L].

Finally, by applying Theorem 1.5, one has

(1) d(R)[L, R] = (0) and [d(R), L] = (0);
(2) (a + b)[L, R] = (0) and [a, L] = (0);
(3) (a + b)I = (0) and d(I) = (0), where I denotes the ideal of R generated by [L, L].

Note that, since both bI = (0) and (a + b)I = (0), we are done. □

We conclude our paper with some applications to generalized derivations acting on ideals of semiprime rings:

Theorem 1.10. Let R be a 2-torsion free semiprime ring and I be a non-central ideal of R. Suppose R admits a generalized derivation (F, d), defined as F(x) = ax + d(x), for all x ∈ R and fixed element a ∈ R. If [F(x), x] = 0 for all x ∈ I, then either d(I) = 0 or R contains a non-zero central ideal.

Proof. By Theorem 1.8, we have that if d(I) ≠ (0) then [d(R), I] = (0). Hence, by applying Main Theorem in [13], it follows that R must contain a non-zero central ideal. □

Corollary 1.11. Let R be a 2-torsion free semiprime ring F a generalized derivation of R. If [F(x), x] = 0 for all x ∈ R, then either R contains a
non-zero central ideal or there exists $\lambda \in Z(R)$ such that $F(x) = \lambda x$, for all $x \in R$.

**Theorem 1.12.** Let $R$ be a 2-torsion free semiprime ring and $I$ be a non-central ideal of $R$. Suppose $R$ admits two generalized derivations $(F,d)$ and $(G,g)$. Write $F(x) = ax + d(x)$ and $G(x) = bx + g(x)$, for some $a,b \in U$. If $F([x,y]) = [y,G(x)]$ for all $x,y \in L$, then either $d(I) = g(I) = (0)$ or $R$ contains a non-zero central ideal.

**Proof.** Assume either $d(I) \neq 0$ or $g(I) \neq 0$. Thus, by Theorem 1.9 respectively we have that either $[d(R),I] = (0)$ or $[g(R),I] = (0)$. In any case, again by [13], $R$ must contain some non-zero central ideals. □

**Corollary 1.13.** Let $R$ be a 2-torsion free semiprime ring and $F,G$ two generalized derivations of $R$. If $F([x,y]) = [y,G(x)]$ for all $x,y \in R$, then either $R$ contains a non-zero central ideal or there exist $\lambda \in Z(R)$ such that $F(x) = G(x) = \lambda x$, for all $x \in R$.

**ACKNOWLEDGMENTS**

The authors wishes to thank the referees for their valuable comments, suggestions.

**REFERENCES**