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Lie Ideals and Generalized Derivations in Semiprime Rings

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Abstract. Let R be a 2-torsion free ring and L a Lie ideal of R. An additive mapping $F : R \to R$ is called a generalized derivation on R if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y+xd(y)$ holds for all $x, y \in R$. In the present paper we describe the action of generalized derivations satisfying several conditions on Lie ideals of semiprime rings.

Keywords: Derivations, Generalized derivations, Semiprime rings, Lie ideals.

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1. Introduction

Let R be an associative ring with center $Z(R)$. A ring R is said to be n-torsion free if $nx = 0$ implies $x = 0$ for all $x \in R$. For any $x, y \in R$, the symbol $[x, y]$ will represent the commutator $xy - yx$, Recall that a ring R is prime if $aRb = 0$ implies $a = 0$ or $b = 0$ and R is semiprime if $aRa = 0$ yields $a = 0$. An additive mapping $d : R \to R$ is said to be a derivation of R if

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$d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. In particular, for a fixed $a \in R$ the mapping $I_a : R \to R$ given by $I_a(x) = [x, a]$ is a derivation which is called an inner derivation determined by a. In 1991 Bresar [5] introduced the concept of generalized derivation: more precisely an additive mapping $F : R \to R$ is said to be a generalized derivation with associated derivation d if $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. For the sake of convenience, we shall denote by (F, d) a generalized derivation F with associated derivation d. A mapping $f : R \to R$ is known to be centralizing on R if $[f(x), x] \in Z(R)$ for all $x \in R$. In particular, if $[f(x), x] = 0$ for all $x \in R$, then f is said to be commuting on R. We recall that an additive group L of R is said to be a Lie ideal of R if $[L, R] \subseteq L$.

A well known result of Posner [18] states that a prime ring admitting a nonzero centralizing derivation must be commutative. This theorem indicates that the global structure of a ring R is often tightly connected to the behaviour of additive mappings defined on R. Following this line of investigation, several authors studied derivations and generalized derivations acting on appropriate subsets of the ring.

For instance in [19] Quadri et al. prove that if R is a prime ring with a non-zero ideal I and F is a generalized derivation of R such that $F([x, y]) = [x, y]$, for all $x, y \in I$, then R is commutative (Theorem 2.1). Later in [7] Dhara extends all results contained in [19] to semiprime rings.

Further in [10] Gölbaşı and Koç investigate the properties of a prime ring R with a generalized derivation (F, d) acting on a Lie ideal L of R. They prove that if $[F(u), u] \in Z(R)$, for all $u \in L$, then either $d = 0$ or $L \subseteq Z(R)$ (Theorem 3.3). Moreover if $F([u, v]) = [u, v]$, for all $u, v \in L$, then either $d = 0$ of $L \subseteq Z(R)$ (Theorem 3.6).

In this note we will consider a similar situation and extend the cited results to the case of semiprime rings with a generalized derivation (F, d) acting on a Lie ideal. More precisley we prove the following:

Theorem 1. *Let R be a 2-torsion free semiprime ring and L be a non-central Lie ideal of R. Suppose that (F, d) is a generalized derivation of R such that $F[x, y] \in Z(R)$, for all $x, y \in L$. If $d(L) \neq (0)$, then all the following hold simultaneously:*

(1) *$d(R)[L, R] = (0)$ and $[d(R), L] = (0)$;*
(2) *$a[L, R] = (0)$ and $[a, L] = (0)$;*
(3) *$aI = (0)$ and $d(I) = (0)$ (that is $F(I) = (0)$), where I denotes the ideal of R generated by $[L, L]$.*

Theorem 2. *Let R be a 2-torsion free semiprime ring and L be a non-central Lie ideal of R. Suppose R admits a generalized derivation (F, d), defined as $F(x) = ax + d(x)$, for all $x \in R$ and fixed element $a \in R$. If $[F(x), x] \in Z(R)$ for all $x \in L$ and $d(L) \neq (0)$, then all the following hold simultaneously:*

(1) *$d(R)[L,R] = (0)$ and $[d(R), L] = (0)$;*
(2) *$[a, L] = a[L,R] = (0)$;*
(3) *$aI = d(I) = (0)$ (that is $F(I) = (0)$), where I denotes the ideal of R generated by $[L, L]$.*

***Theorem* 3.** *Let R be a 2-torsion free semiprime ring and L be a non-central Lie ideal of R. Suppose R admits two generalized derivations (F, d) and (G, g). Write $F(x) = ax+d(x)$ and $G(x) = bx+g(x)$, for some $a, b \in U$. If $F([x, y]) = [y, G(x)]$ for all $x, y \in L$, then either*

(1) *$g(L) = (0)$;*
(2) *$d(R)[L,R] = (0)$ and $[d(R), L] = (0)$;*
(3) *$(a + b)[L,R] = (0)$, $[b, L] = (0)$ and $[a, L] = (0)$;*
(4) *$(a+b)I = (0)$ and $d(I) = (0)$, where I denotes the ideal of R generated by $[L, L]$.*

or

(1) *$d(L) = (0)$;*
(2) *$g(R)[L,R] = (0)$ and $[g(R), L] = (0)$;*
(3) *$[b, L] = (0)$ and $a[L, L] = (0)$;*
(4) *$aI = (0)$ and $g(I) = (0)$, where I denotes the ideal of R generated by $[L, L]$.*

or

(1) *$d(R)[L,R] = (0)$ and $[d(R), L] = (0)$;*
(2) *$g(R)[L,R] = (0)$ and $[g(R), L] = (0)$;*
(3) *$[a, L] = (0)$, $[b, L] = (0)$, $b[L,R] = a[L,R] = (0)$;*
(4) *$d(I) = g(I) = (0)$ and $aI = bI = (0)$ (that is $F(I) = G(I) = (0)$), where I denotes the ideal of R generated by $[L, L]$.*

In all that follows let R be a non-commutative semiprime ring, L a non-central Lie ideal of R, U the right Utumi quotient ring of R. We refer the reader to [3] for the definition and the related properties of U.

We begin with the following:

Fact 1.1. Let R be a semiprime ring. Then every generalized derivation F of R is uniquely extended to its right Utumi quotient ring U and assumes the form $F(x) = ax + d(x)$, where $a \in U$ and d is the derivation of U associated with F (see Theorem 4 in [17])

Lemma 1.2. *Let R be a prime ring of characteristic different from 2 and L be a Lie ideal of R. Suppose R admits a nonzero generalized derivation (F, d) such that $F(x)[x, y] = 0$ (or $[x, y]F(x) = 0$) for all $x, y \in L$, then $L \subseteq Z(R)$.*

Proof. Suppose by contradiction that L is not central in R. By [11] (pages 4-5) there exists a non-central ideal I of R such that $0 \neq [I,R] \subseteq L$. By our assumption it follows that $F(x)[x,y]=0$ (or $[x,y]F(x)=0$) for all $x,y\in[I,R]$. Since I and R satisfy the same differential identities (see the main result in [16]), we also have that $F(x)[x,y]=0$ (or $[x,y]F(x)=0$) for all $x,y\in[R,R]$. Let $y_0\in[R,R]$ be such that $y_0\notin Z(R)$ and denote by $\delta:R\to R$ the non-zero inner derivation of R induced by the element y_0. Therefore $F(x)\delta(x)=0$ (or $\delta(x)F(x)=0$) for all $x\in[R,R]$. In light of [6], since $\delta\neq 0$ and $[R,R]$ is not central in R, one has the contradiction that $F=0$. □

Lemma 1.3. *Let R be a 2-torsion free semiprime ring and L be a non-central Lie ideal of R. Suppose R admits a nonzero generalized derivation (F,d), defined as $F(x)=ax+d(x)$, for all $x\in R$ and fixed element $a\in R$. If $F(x)[x,y]=0$ (or $[x,y]F(x)=0$) for all $x,y\in L$, then all the following hold simultaneously:*

(1) $d(R)[L,R]=(0)$ *and* $[d(R),L]=(0)$*;*
(2) $a[L,R]=(0)$ *and* $[a,L]=(0)$*;*
(3) $aI=(0)$ *and* $d(I)=(0)$ *(that is $F(I)=(0)$), where I denotes the ideal of R generated by* $[L,L]$.

Proof. Let P be a prime ideal if R such that $[L,L]\nsubseteq P$.
Assume first that $d(P)\subseteq P$. Then F induces a canonical generalized derivation $\overline{F}$ on $\overline{R}=\frac{R}{P}$. Therefore $\overline{F}(\overline{x})[\overline{x},\overline{y}]=0$ for all $\overline{x},\overline{y}\in\overline{L}$. Moreover $\overline{L}$ is a Lie ideal of $\overline{R}$, such that $[\overline{L},\overline{L}]\neq 0$ since $[L,L]\nsubseteq P$. By Lemma 1.2 it follows that $\overline{F}(\overline{R})=\overline{0}$ that is $aR\subseteq P$, $d(R)\subseteq P$ and $F(R)\subseteq P$.
Assume now that $d(P)\nsubseteq P$, then $\overline{d(P)}\neq\overline{0}$ and $\overline{d(P)R}\neq\overline{0}$. Moreover note that, for any $p\in P$ and $r,s\in R$, $d(pr)s=d(p)rs+pd(r)s$ implies that $d(P)R\subseteq d(PR)R+P$, in particular $\overline{d(P)R}$ is a non-zero right ideal of $\overline{R}$.
Starting from our main assumption and linearizing we have that $F(x)[z,y]+F(z)[x,y]=0$, for all $x,y,z\in L$. For any $p\in P, r\in R, u\in L$, replace x by $[pr,u]$. By computation it follows $[\overline{v},\overline{u}][\overline{z},\overline{y}]=\overline{0}$, for all $\overline{v}\in\overline{d(P)R}$ and $\overline{u},\overline{z},\overline{y}\in\overline{L}$. By using the same argument of Lemma 1.2, since $\overline{L}$ is not central in $\overline{R}$, one has that $\overline{d(P)R}$ is a central right ideal of $\overline{R}$, which implies that $\overline{R}$ is commutative, a contradiction.
Therefore, for any prime ideal P of R, either $aR\subseteq P$, $d(R)\subseteq P$ and $F(R)\subseteq P$ or $[L,L]\subseteq P$. In this last case, by applying Theorem 3 in [15] in the prime ring $\overline{R}$, since $char(\overline{R})\neq 2$ and $[\overline{L},\overline{L}]=\overline{0}$, we have that $\overline{L}$ is central in $\overline{R}$, which means $[L,R]\subseteq P$.
Hence in any case it follows that $d(R)[L,R]=(0)$, $a[R,L]=(0)$ and $[d(R),L]=(0)$.
By $a[R,L]=(0)$ we get $aR[R,L]=(0)$ and so both $aLR[a,L]=(0)$ and $LaR[a,L]=(0)$, that is $[a,L]R[a,L]=(0)$. By the semiprimeness of R it follows $[a,L]=(0)$.
Moreover, if $I=R[L,L]R$ denotes the ideal of R generated by $[L,L]$, it follows that $aI=(0)$ and $d(I)=(0)$, that is $F(I)=(0)$. □

Corollary 1.4. *Let R be a 2-torsion free semiprime ring and L be a non-central Lie ideal of R. Suppose $a \in R$ be such that $ax[x,y] = 0$ for all $x, y \in L$, then $a[L,R] = (0)$, $[a,L] = (0)$ and $aI = (0)$, where I denotes the ideal of R generated by $[L,L]$.*

Theorem 1.5. *Let R be a 2-torsion free semiprime ring and L be a non-central Lie ideal of R. Suppose that (F,d) is a generalized derivation of R such that $F[x,y] \in Z(R)$, for all $x, y \in L$. If $d(L) \neq (0)$, then all the following hold simultaneously:*

1. *$d(R)[L,R] = (0)$ and $[d(R),L] = (0)$;*
2. *$a[L,R] = (0)$ and $[a,L] = (0)$;*
3. *$aI = (0)$ and $d(I) = (0)$ (that is $F(I) = (0)$), where I denotes the ideal of R generated by $[L,L]$.*

Proof. Assume first that R is prime and denote $V = [L,L]$. Hence we have $F(V) \subseteq Z(R)$. As a consequence of Lemma 2 in [9] we conclude that either $F = 0$ or $V \subseteq Z(R)$. In the first case we have the contradiction $d = 0$, and in the other case one has $L \subseteq Z(R)$ (see Lemma 2 in [12]), a contradiction again. Let now P be a prime ideal of R such that $[L,L] \nsubseteq P$.
Assume first that $d(P) \subseteq P$. Then F induces a canonical generalized derivation $\overline{F}$ on $\overline{R} = \frac{R}{P}$. Therefore $\overline{F}([\overline{x},\overline{y}]) \in Z(\overline{R})$ for all $\overline{x},\overline{y} \in \overline{L}$. Moreover $\overline{L}$ is a Lie ideal of $\overline{R}$, such that $[\overline{L},\overline{L}] \neq 0$ since $[L,L] \nsubseteq P$. By previous argument it follows that $\overline{F}(\overline{R}) = \overline{0}$ that $d(R) \subseteq P$ and $F(R) \subseteq P$.
Assume now that $d(P) \nsubseteq P$, then $\overline{d(P)} \neq \overline{0}$ and $\overline{d(P)R} \neq \overline{0}$. We remark again that $\overline{d(P)R}$ is a non-zero right ideal of $\overline{R}$.
Starting from our main assumption and linearizing we have that

$$F(x)y + F(x)z + xd(y) + xd(z) - F(y)x - F(z)x - yd(x) - zd(x) \in Z(R),\ \forall x, y, z \in L.$$

For any $p, p', p'' \in P, r, s \in R, u, v \in L$, replace y by $[pr, u]$ and z by $[[p's, v], p'']$. By computation it follows

$$\overline{x}[\overline{t},\overline{u}] - [\overline{t},\overline{u}]\overline{x} \in Z(\overline{R})$$

that is

$$\Big[\overline{x},[\overline{t},\overline{u}]\Big] \in Z(\overline{R})$$

for all $\overline{t} \in \overline{d(P)R}$ and $\overline{u},\overline{x} \in \overline{L}$. As above denote $\overline{V} = [\overline{L},\overline{L}]$, which is a Lie ideal for $\overline{R}$, and δ is the derivation of $\overline{R}$ induced by $\overline{t}$. Hence we have $\delta(\overline{V}) \subseteq Z(\overline{R})$. Again as a consequence of Lemma 2 in [9] it follows that either $\delta = 0$ or $\overline{V} \subseteq Z(\overline{R})$. Since $\overline{R}$ is not commutative, then there exists some $\overline{t} \in \overline{R}$ which is not central. Thus $\overline{V} \subseteq Z(\overline{R})$, and $\overline{L} \subseteq Z(\overline{R})$ follows from Lemma 2 in [12].
Therefore, for any prime ideal P of R, either $d(R) \subseteq P$ and $F(R) \subseteq P$ or $[L,L] \subseteq P$. In this last case, by applying Theorem 3 in [15] in the prime ring $\overline{R}$, since $char(\overline{R}) \neq 2$ and $[\overline{L},\overline{L}] = \overline{0}$, we conclude that $\overline{L}$ is central in $\overline{R}$, which

means $[L,R] \subseteq P$.
Hence in any case it follows that $d(R)[L,R] = (0)$, $a[R,L] = (0)$ and $[d(R),L] = (0)$. Finally we obtain the required conclusions by following the same argument as in Lemma 1.3. □

In the sequel we will use the following known result:

Lemma 1.6. *Let R be a 2-torsion free semiprime ring, L a Lie ideal of R such that $L \nsubseteq Z(R)$. Let $a \in L$ be such that $aLa = 0$, then $a = 0$.*

Remark 1.7. If R is a prime ring of characteristic different from 2, $a \in R$ and L is a non-central Lie ideal of R such that $[a,L] \subseteq Z(R)$, then $a \in Z(R)$.

Proof. Denote by $\delta : R \to R$ the inner derivation of R induced by the element $a \in R$. Since $[[a,x],r] = 0$ for all $x \in L$ and $r \in R$, a fortiori we have $[a,x]_2 = 0$, that is $[\delta(x),x] = 0$, for all $x \in L$. Thus, by [14] it follows $\delta = 0$, that is $a \in Z(R)$. □

Theorem 1.8. *Let R be a 2-torsion free semiprime ring and L be a non-central Lie ideal of R. Suppose R admits a generalized derivation (F,d), defined as $F(x) = ax + d(x)$, for all $x \in R$ and fixed element $a \in R$. If*

$$[F(x),x] \in Z(R) \text{ for all } x \in L. \tag{1.1}$$

and $d(L) \neq (0)$, then all the following hold simultaneously:

(1) $d(R)[L,R] = (0)$ *and* $[d(R),L] = (0)$*;*
(2) $[a,L] = a[L,R] = (0)$*;*
(3) $aI = d(I) = (0)$ *(that is $F(I) = (0)$), where I denotes the ideal of R generated by $[L,L]$.*

Proof. Let P be a prime ideal of R such that $[L,L] \nsubseteq P$.
Assume first that $d(P) \subseteq P$. Then F induces a canonical generalized derivation $\overline{F}$ on $\overline{R} = \frac{R}{P}$. Therefore $[\overline{F}(\overline{x}),\overline{x}] \in Z(\overline{R})$ for all $\overline{x} \in \overline{L}$. Moreover $\overline{L}$ is a Lie ideal of $\overline{R}$, such that $[\overline{L},\overline{L}] \neq 0$ since $[L,L] \nsubseteq P$. Since $[L,L] \nsubseteq P$, a fortiori we get $\overline{L}$ is not central in $\overline{R}$. Therefore, by Theorem 3.3 in [10], it follows that $\overline{d}(\overline{R}) = \overline{0}$ that is $d(R) \subseteq P$.
Assume now that $d(P) \nsubseteq P$, then $\overline{d(P)} \neq \overline{0}$ and $\overline{d(P)R} \neq \overline{0}$. By using similar argument as in Lemma 1.3, $R\overline{d(P)}$ is a non-zero right ideal of $\overline{R}$.
Linearizing (1.1) and using (1.1), we obtain

$$[F(x),y] + [F(y),x] \in Z(R) \text{ for all } x,y \in L. \tag{1.2}$$

Now, replace y by $[rp,u]$, for $r \in R$, $p \in P$ and $u \in L$ and use (1.2) to get

$$[\overline{F}(\overline{[rp,u]}),\overline{x}] \in Z(\overline{R}). \tag{1.3}$$

Moreover, since $F(r) = ar + d(r)$, for all $r \in R$, by (1.3) it follows

$$[\overline{d}(\overline{[rp,u]}),\overline{L}] \subseteq Z(\overline{R}). \tag{1.4}$$

By the primeness of $\overline{R}$ and Remark 1.7, one has that $\overline{d}(\overline{[rp,u]}) \in Z(\overline{R})$. On the other hand, an easy computation shows that $\overline{d}(\overline{[rp,u]}) = [\overline{rd(p)}, \overline{u}]$, which implies $[\overline{Rd(P)}, \overline{L}] \subseteq Z(\overline{R})$. Once again by Remark 1.7, we have $\overline{Rd(P)} \subseteq Z(\overline{R})$. Since $\overline{Rd(P)}$ is a non-zero right ideal of $\overline{R}$, it follows $[\overline{R}, \overline{R}] = (0)$, which contradicts with $[\overline{L}, \overline{L}] \neq (0)$.

The previous argument shows that, for any prime ideal P of R, either $[L,L] \subseteq P$ or $d(R) \subseteq P$. Thus $d(R)[L,L] \subseteq \cap P_i = (0)$. Hence, by Lemma 1.3 and since $L \nsubseteq Z(R)$, we finally get the required conclusions:

(1) $d(R)[L,R] = (0)$ and $[d(R),L] = (0)$;
(2) $a[L,R] = (0)$ and $[a,L] = (0)$;
(3) $aI = d(I) = (0)$, where I denotes the ideal of R generated by $[L,L]$.

□

Theorem 1.9. *Let R be a 2-torsion free semiprime ring and L be a non-central Lie ideal of R. Suppose R admits two generalized derivations (F,d) and (G,g). Write $F(x) = ax + d(x)$ and $G(x) = bx + g(x)$, for some $a,b \in U$. If $F([x,y]) = [y, G(x)]$ for all $x,y \in L$, then either*

(1) $g(L) = (0)$;
(2) $d(R)[L,R] = (0)$ *and* $[d(R),L] = (0)$;
(3) $(a+b)[L,R] = (0)$, $[b,L] = (0)$ *and* $[a,L] = (0)$;
(4) $(a+b)I = (0)$ *and* $d(I) = (0)$, *where I denotes the ideal of R generated by $[L,L]$.*

or

(1) $d(L) = (0)$;
(2) $g(R)[L,R] = (0)$ *and* $[g(R),L] = (0)$;
(3) $[b,L] = (0)$ *and* $a[L,L] = (0)$;
(4) $aI = (0)$ *and* $g(I) = (0)$, *where I denotes the ideal of R generated by $[L,L]$.*

or

(1) $d(R)[L,R] = (0)$ *and* $[d(R),L] = (0)$;
(2) $g(R)[L,R] = (0)$ *and* $[g(R),L] = (0)$;
(3) $[a,L] = (0)$, $[b,L] = (0)$, $b[L,R] = a[L,R] = (0)$;
(4) $d(I) = g(I) = (0)$ *and* $aI = bI = (0)$ *(that is $F(I) = G(I) = (0)$), where I denotes the ideal of R generated by $[L,L]$.*

Proof. Assume first $g(L) = (0)$, then $F([x,y]) = [y,bx]$ for all $x,y \in L$. Thus

$$a[x,y] + d([x,y]) = b[y,x] \tag{1.5}$$

for all $x,y \in L$, that is $(a+b)[x,y] + d([x,y]) = 0$ for all $x,y \in L$. Therefore, applying Theorem 1.5, one has

(1) $d(R)[L,R] = (0)$ and $[d(R),L] = (0)$;
(2) $(a+b)[L,R] = (0)$ and $[a+b,L] = (0)$;

(3) $(a+b)I = (0)$ and $d(I) = (0)$, where I denotes the ideal of R generated by $[L, L]$.

In particular $d([L, L]) = (0)$ and $a[x, y] = -b[x, y]$ for all $x, y \in L$, so that (1.5) reduces to $(by - yb)x = 0$, for all $x, y \in L$, that is $[b, L]L = (0)$. Hence by Lemma 1.6, we have $[b, L] = (0)$ and so also $[a, L] = (0)$

Let now $d(L) = (0)$, then $a[x, y] = [y, G(x)]$ for all $x, y \in L$. In this case, for $x = y$, we have $[G(y), y] = 0$ and by Theorem 1.8 the following hold:

(1) $g(R)[L, R] = (0)$ and $[g(R), L] = (0)$;
(2) $[b, L] = (0)$, $b[L, R] = (0)$ and $a[L, L] = (0)$;
(3) $bI = (0)$ and $g(I) = (0)$, where I denotes the ideal of R generated by $[L, L]$.

Moreover, since $[[L, L], R] \subseteq [L, L]$, we also have $0 = a[[L, L], R] = aR[L, L]$, which implies $aI = (0)$.

Assume finally that both $g(L) \neq (0)$ and $d(L) \neq (0)$. Once again for $x = y \in L$ we have $[G(x), x] = 0$ for any $x \in L$. Thus by Theorem 1.8, we have that all the following hold:

(1) $g(R)[L, R] = (0)$ and $[g(R), L] = (0)$;
(2) $[b, L] = (0)$ and $b[L, R] = (0)$;
(3) $bI = (0)$ and $g(I) = (0)$, where I denotes the ideal of R generated by $[L, L]$.

Hence by the main assumption it follows that $(a+b)[x, y] + d([x, y]) = 0$, for all $x, y \in L$. Denote $H(x) = (a - b)x + d(x)$, then $H(u) = 0$ for all $u \in [L, L]$. Finally, by applying Theorem 1.5, one has

(1) $d(R)[L, R] = (0)$ and $[d(R), L] = (0)$;
(2) $(a + b)[L, R] = (0)$ and $[a, L] = (0)$;
(3) $(a+b)I = (0)$ and $d(I) = (0)$, where I denotes the ideal of R generated by $[L, L]$.

Note that, since both $bI = (0)$ and $(a + b)I = (0)$, we are done. $\square$

We conclude our paper with some applications to generalized derivations acting on ideals of semiprime rings:

Theorem 1.10. *Let R be a 2-torsion free semiprime ring and I be a non-central ideal of R. Suppose R admits a generalized derivation (F, d), defined as $F(x) = ax + d(x)$, for all $x \in R$ and fixed element $a \in R$. If $[F(x), x] = 0$ for all $x \in I$, then either $d(I) = 0$ or R contains a non-zero central ideal.*

Proof. By Theorem 1.8, we have that if $d(I) \neq 0$ then $[d(R), I] = (0)$. Hence, by applying Main Theorem in [13], it follows that R must contain a non-zero central ideal. $\square$

Corollary 1.11. *Let R be a 2-torsion free semiprime ring F a generalized derivation of R. If $[F(x), x] = 0$ for all $x \in R$, then either R contains a* *non-zero central ideal or there exists $\lambda \in Z(R)$ such that $F(x) = \lambda x$, for all $x \in R$.*

Theorem 1.12. *Let R be a 2-torsion free semiprime ring and I be a non-central ideal of R. Suppose R admits two generalized derivations (F, d) and (G, g). Write $F(x) = ax + d(x)$ and $G(x) = bx + g(x)$, for some $a, b \in U$. If $F([x, y]) = [y, G(x)]$ for all $x, y \in L$, then either $d(I) = g(I) = (0)$ or R contains a non-zero central ideal.*

Proof. Assume either $d(I) \neq 0$ or $g(I) \neq 0$. Thus, by Theorem 1.9 respectively we have that either $[d(R), I] = (0)$ or $[g(R), I] = (0)$. In any case, again by [13], R must contain some non-zero central ideals. □

Corollary 1.13. *Let R be a 2-torsion free semiprime ring and F, G two generalized derivations of R. If $F([x, y]) = [y, G(x)]$ for all $x, y \in R$, then either R contains a non-zero central ideal or there exist $\lambda \in Z(R)$ such that $F(x) = G(x) = \lambda x$, for all $x \in R$.*

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