Lie Ideals and Generalized Derivations in Semiprime Rings

Vincenzo De Filippis\textsuperscript{a}, Nadeem Ur Rehman\textsuperscript{b*}, Abu Zaid Ansari\textsuperscript{c}

\textsuperscript{a}Department of Mathematics and Computer Science, University of Messina, 98166 Messina, Italy.
\textsuperscript{b}Department of Mathematics, Faculty of Science, Taibah University, Al-Madinah, Al-Munawara, KSA.
\textsuperscript{c}Department of Mathematics, Faculty of Science, Islamic University in Madinah, KSA.

\textbf{E-mail:} defilippis@unime.it
\textbf{E-mail:} rehman100@gmail.com
\textbf{E-mail:} ansari.abuzaid@gmail.com

\textbf{Abstract.} Let $R$ be a 2-torsion free ring and $L$ a Lie ideal of $R$. An additive mapping $F : R \to R$ is called a generalized derivation on $R$ if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$. In the present paper we describe the action of generalized derivations satisfying several conditions on Lie ideals of semiprime rings.

\textbf{Keywords:} Derivations, Generalized derivations, Semiprime rings, Lie ideals.

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1. \textsc{Introduction}

Let $R$ be an associative ring with center $Z(R)$. A ring $R$ is said to be $n$-torsion free if $nx = 0$ implies $x = 0$ for all $x \in R$. For any $x, y \in R$, the symbol $[x, y]$ will represent the commutator $xy - yx$. Recall that a ring $R$ is prime if $aRb = 0$ implies $a = 0$ or $b = 0$ and $R$ is semiprime if $aRa = 0$ yields $a = 0$. An additive mapping $d : R \to R$ is said to be a derivation of $R$ if

*Corresponding Author

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$d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. In particular, for a fixed $a \in R$ the mapping $I_a : R \rightarrow R$ given by $I_a(x) = [x, a]$ is a derivation which is called an inner derivation determined by $a$. In 1991 Bresar [5] introduced the concept of generalized derivation: more precisely an additive mapping $F : R \rightarrow R$ is said to be a generalized derivation with associated derivation $d$ if $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. For the sake of convenience, we shall denote by $(F, d)$ a generalized derivation $F$ with associated derivation $d$. A mapping \( f : R \rightarrow R \) is known to be centralizing on $R$ if \( [f(x), x] \in Z(R) \) for all $x \in R$. In particular, if \( [f(x), x] = 0 \) for all $x \in R$, then $f$ is said to be commuting on $R$. We recall that an additive group $L$ of $R$ is said to be a Lie ideal of $R$ if \([L, R] \subseteq L\).

A well known result of Posner [18] states that a prime ring admitting a nonzero centralizing derivation must be commutative. This theorem indicates that the global structure of a ring $R$ is often tightly connected to the behaviour of additive mappings defined on $R$. Following this line of investigation, several authors studied derivations and generalized derivations acting on appropriate subsets of the ring.

For instance in [19] Quadri et al. prove that if $R$ is a prime ring with a non-zero ideal $I$ and $F$ is a generalized derivation of $R$ such that $F([x, y]) = [x, y]$, for all $x, y \in I$, then $R$ is commutative (Theorem 2.1). Later in [7] Dhara extends all results contained in [19] to semiprime rings.

Further in [10] Gölbaçi and Koç investigate the properties of a prime ring $R$ with a generalized derivation $(F, d)$ acting on a Lie ideal $L$ of $R$. They prove that if \([F(u), v] \in Z(R), \) for all $u \in L$, then either $d = 0$ or $L \subseteq Z(R)$ (Theorem 3.3). Moreover if $F([u, v]) = [u, v]$, for all $u, v \in L$, then either $d = 0$ of $L \subseteq Z(R)$ (Theorem 3.6).

In this note we will consider a similar situation and extend the cited results to the case of semiprime rings with a generalized derivation $(F, d)$ acting on a Lie ideal. More precisely we prove the following:

**Theorem 1.** Let $R$ be a 2-torsion free semiprime ring and $L$ be a non-central Lie ideal of $R$. Suppose that $(F, d)$ is a generalized derivation of $R$ such that $F[x, y] \in Z(R)$, for all $x, y \in L$. If $d(L) \neq (0)$, then all the following hold simultaneously:

1. $d(R)[L, R] = (0)$ and $[d(R), L] = (0)$;
2. $a[L, R] = (0)$ and $[a, L] = (0)$;
3. $aI = (0)$ and $d(I) = (0)$ (that is $F(I) = (0)$), where $I$ denotes the ideal of $R$ generated by $[L, L]$.

**Theorem 2.** Let $R$ be a 2-torsion free semiprime ring and $L$ be a non-central Lie ideal of $R$. Suppose $R$ admits a generalized derivation $(F, d)$, defined as $F(x) = ax + d(x)$, for all $x \in R$ and fixed element $a \in R$. If $[F(x), x] \in Z(R)$ for all $x \in L$ and $d(L) \neq (0)$, then all the following hold simultaneously:
(1) \(d(R)[L, R] = (0)\) and \([d(R), L] = (0)\);
(2) \([a, L] = a[L, R] = (0)\);
(3) \(aI = d(I) = (0)\) (that is \(F(I) = (0)\)), where \(I\) denotes the ideal of \(R\) generated by \([L, L]\).

**Theorem 3.** Let \(R\) be a 2-torsion free semiprime ring and \(L\) be a non-central Lie ideal of \(R\). Suppose \(R\) admits two generalized derivations \((F, d)\) and \((G, g)\).

Write \(F(x) = ax + d(x)\) and \(G(x) = bx + g(x)\), for some \(a, b \in U\). If \(F([x, y]) = [y, G(x)]\) for all \(x, y \in L\), then either

1. \(g(L) = (0)\);
2. \(d(R)[L, R] = (0)\) and \([d(R), L] = (0)\);
3. \((a + b)L, R] = (0)\), \([b, L] = (0)\) and \([a, L] = (0)\);
4. \((a + b)L, R] = (0)\) and \(d(I) = (0)\), where \(I\) denotes the ideal of \(R\) generated by \([L, L]\).

or

1. \(d(L) = (0)\);
2. \(g(R)[L, R] = (0)\) and \([g(R), L] = (0)\);
3. \([b, L] = (0)\) and \([a, L] = (0)\);
4. \(aI = (0)\) and \(g(I) = (0)\), where \(I\) denotes the ideal of \(R\) generated by \([L, L]\).

or

1. \(d(R)[L, R] = (0)\) and \([d(R), L] = (0)\);
2. \(g(R)[L, R] = (0)\) and \([g(R), L] = (0)\);
3. \([a, L] = (0)\), \([b, L] = (0)\), \([bL, R] = a[L, R] = (0)\);
4. \(d(I) = g(I) = (0)\) and \(aI = bI = (0)\) (that is \(F(I) = G(I) = (0)\)), where \(I\) denotes the ideal of \(R\) generated by \([L, L]\).

In all that follows let \(R\) be a non-commutative semiprime ring, \(L\) a non-central Lie ideal of \(R\), \(U\) the right Utumi quotient ring of \(R\). We refer the reader to [3] for the definition and the related properties of \(U\).

We begin with the following:

**Fact 1.1.** Let \(R\) be a semiprime ring. Then every generalized derivation \(F\) of \(R\) is uniquely extended to its right Utumi quotient ring \(U\) and assumes the form \(F(x) = ax + d(x)\), where \(a \in U\) and \(d\) is the derivation of \(U\) associated with \(F\) (see Theorem 4 in [17]).

**Lemma 1.2.** Let \(R\) be a prime ring of characteristic different from 2 and \(L\) be a Lie ideal of \(R\). Suppose \(R\) admits a nonzero generalized derivation \((F, d)\) such that \(F(x)[x, y] = 0\) (or \([x, y]F(x) = 0\)) for all \(x, y \in L\), then \(L \subseteq Z(R)\).

**Proof.** Suppose by contradiction that \(L\) is not central in \(R\). By [11] (pages 4-5) there exists a non-central ideal \(I\) of \(R\) such that \(0 \neq [I, R] \subseteq L\). By our
assumption it follows that $F(x)[x, y] = 0$ (or $[x, y]F(x) = 0$) for all $x, y \in [I, R]$. Since $I$ and $R$ satisfy the same differential identities (see the main result in [16]), we also have that $F(x)[x, y] = 0$ (or $[x, y]F(x) = 0$) for all $x, y \in [R, R]$. Let $y_0 \in [R, R]$ be such that $y_0 \notin Z(R)$ and denote by $\delta : R \to R$ the non-zero inner derivation of $R$ induced by the element $y_0$. Therefore $F(x)\delta(x) = 0$ (or $\delta(x)F(x) = 0$) for all $x \in [R, R]$. In light of [6], since $\delta \neq 0$ and $[R, R]$ is not central in $R$, one has the contradiction that $F = 0$.

Lemma 1.3. Let $R$ be a 2-torsion free semiprime ring and $L$ be a non-central Lie ideal of $R$. Suppose $R$ admits a nonzero generalized derivation $(F, d)$, defined as $F(x) = ax + d(x)$, for all $x \in R$ and fixed element $a \in R$. If $F(x)[x, y] = 0$ (or $[x, y]F(x) = 0$) for all $x, y \in L$, then all the following hold simultaneously:

1. $d(R)[L, R] = (0)$ and $[d(R), L] = (0)$;
2. $a[L, R] = (0)$ and $[a, L] = (0)$;
3. $aI = (0)$ and $d(I) = (0)$ (that is $F(I) = (0)$), where $I$ denotes the ideal of $R$ generated by $[L, L]$.

Proof. Let $P$ be a prime ideal if $R$ such that $[L, L] \nsubseteq P$.

Assume first that $d(P) \subseteq P$. Then $F$ induces a canonical generalized derivation $\overline{F}$ on $\overline{R} = \frac{R}{P}$. Therefore $\overline{F}(x, y) = 0$ for all $x, y \in \overline{L}$. Moreover $\overline{L}$ is a Lie ideal of $\overline{R}$, such that $[\overline{L}, \overline{L}] \neq 0$ since $[L, L] \nsubseteq P$. By Lemma 1.2 it follows that $\overline{F}(\overline{R}) = 0$ that is $a\overline{R} \subseteq P$, $d(\overline{R}) \subseteq P$ and $F(\overline{R}) \subseteq P$.

Assume now that $d(P) \nsubseteq P$, then $d(\overline{P}) = \overline{0}$ and $d(P)\overline{R} \neq 0$. Moreover note that, for any $p \in P$ and $r, s \in R$, $d(pr)s = d(p)rs + pd(r)s$ implies that $d(P)\overline{R} \subseteq d(PR)\overline{R} + P$, in particular $d(\overline{P})\overline{R}$ is a non-zero right ideal of $\overline{R}$.

Starting from our main assumption and linearizing we have that $F(x)[z, y] + F(z)[x, y] = 0$, for all $x, y, z \in L$. For any $p \in P$, $r \in R, u \in L$, replace $x$ by $[p, r, u]$. By computation it follows $[\overline{p}, \overline{r}][\overline{z}, \overline{y}] = 0$, for all $\overline{p} \in d(\overline{P})\overline{R}$ and $\overline{z}, \overline{y} \in \overline{L}$. By using the same argument of Lemma 1.2, since $\overline{L}$ is not central in $\overline{R}$, one has that $d(\overline{P})\overline{R}$ is a central right ideal of $\overline{R}$, which implies that $\overline{R}$ is commutative, a contradiction.

Therefore, for any prime ideal $P$ of $R$, either $aR \subseteq P$, $d(R) \subseteq P$ and $F(R) \subseteq P$ or $[L, L] \subseteq P$. In this last case, by applying Theorem 3 in [15] in the prime ring $\overline{R}$, since $char(\overline{R}) \neq 2$ and $[\overline{L}, \overline{L}] = 0$, we have that $\overline{L}$ is central in $\overline{R}$, which means $[L, R] \subseteq P$.

Hence in any case it follows that $d(R)[L, R] = (0)$, $a[R, L] = (0)$ and $[d(R), L] = (0)$.

By $a[R, L] = (0)$ we get $aR[R, L] = (0)$ and so both $aLR[a, L] = (0)$ and $LaR[a, L] = (0)$, that is $[a, L]R[a, L] = (0)$. By the semiprimeness of $R$ it follows $[a, L] = (0)$.

Moreover, if $I = R[L, L]R$ denotes the ideal of $R$ generated by $[L, L]$, it follows that $aI = (0)$ and $d(I) = (0)$, that is $F(I) = (0)$.

$\square$
Corollary 1.4. Let \( R \) be a 2-torsion free semiprime ring and \( L \) be a non-central Lie ideal of \( R \). Suppose \( a \in R \) be such that \( ax[x, y] = 0 \) for all \( x, y \in L \), then \( a[L, R] = (0) \), \( [a, L] = (0) \) and \( aI = (0) \), where \( I \) denotes the ideal of \( R \) generated by \([L, L]\).

Theorem 1.5. Let \( R \) be a 2-torsion free semiprime ring and \( L \) be a non-central Lie ideal of \( R \). Suppose that \( (F, d) \) is a generalized derivation of \( R \) such that \( F[x, y] \in Z(R) \), for all \( x, y \in L \). If \( d(L) \neq (0) \), then all the following hold simultaneously:

1. \( d(R)[L, R] = (0) \) and \([d(R), L] = (0)\);
2. \( a[L, R] = (0) \) and \([a, L] = (0)\);
3. \( aI = (0) \) and \( d(I) = (0) \) (that is \( F(I) = (0)\)), where \( I \) denotes the ideal of \( R \) generated by \([L, L]\).

Proof. Assume first that \( R \) is prime and denote \( V = [L, L] \). Hence we have \( F(V) \subseteq Z(R) \). As a consequence of Lemma 2 in \([9]\) we conclude that either \( F = 0 \) or \( V \subseteq Z(R) \). In the first case we have the contradiction \( d = 0 \), and in the other case one has \( L \subseteq Z(R) \) (see Lemma 2 in \([12]\)), a contradiction again. Let now \( P \) be a prime ideal of \( R \) such that \([L, L] \not\subseteq P\).

Assume first that \( d(P) \subseteq P \). Then \( F \) induces a canonical generalized derivation \( \overline{F} \) on \( \overline{R} = \frac{R}{P} \). Therefore \( \overline{F}(\overline{[x, y]}) \in Z(\overline{R}) \) for all \( \overline{x}, \overline{y} \in \overline{L} \). Moreover \( \overline{L} \) is a Lie ideal of \( \overline{R} \), such that \([\overline{L}, \overline{L}] \neq 0\) since \([L, L] \not\subseteq P\). By previous argument it follows that \( \overline{F}(\overline{R}) = \overline{0} \) that \( d(R) \subseteq P \) and \( F(R) \subseteq P \).

Assume now that \( d(P) \not\subseteq P \), then \( \overline{d}(P) \not= \overline{0} \) and \( \overline{d}(P) \overline{R} \neq \overline{0} \). We remark again that \( \overline{d}(P) \overline{R} \) is a non-zero right ideal of \( \overline{R} \).

Starting from our main assumption and linearizing we have that

\[
F(x)y + F(x)z + xd(y) + xd(z) - F(y)x - F(z)x - yd(x) - zd(x) \in Z(R), \ \forall x, y, z \in L.
\]

For any \( p, p', p'' \in P, r, s \in R, u, v \in L \), replace \( y \) by \([pr, u] \) and \( z \) by \([[p's], v], p''\].

By computation it follows

\[
\pi[t, \pi] - [t, \pi]\pi \in Z(\overline{R})
\]

that is

\[
[\pi, [t, \pi]] \in Z(\overline{R})
\]

for all \( t \in \overline{d}(P)\overline{R} \) and \( \pi, \pi \in \overline{L} \). As above denote \( \overline{V} = [\overline{L}, \overline{L}] \), which is a Lie ideal for \( \overline{R} \), and \( \delta \) is the derivation of \( \overline{R} \) induced by \( \overline{t} \). Hence we have \( \delta(\overline{V}) \subseteq Z(\overline{R}) \). Again as a consequence of Lemma 2 in \([9]\) it follows that either \( \delta = 0 \) or \( \overline{V} \subseteq Z(\overline{R}) \). Since \( \overline{R} \) is not commutative, then there exists some \( \overline{t} \in \overline{R} \) which is not central. Thus \( \overline{V} \subseteq Z(\overline{R}) \), and \( \overline{L} \subseteq Z(\overline{R}) \) follows from Lemma 2 in \([12]\).

Therefore, for any prime ideal \( P \) of \( R \), either \( d(R) \subseteq P \) and \( F(R) \subseteq P \) or \([L, L] \subseteq P \). In this last case, by applying Theorem 3 in \([15]\) in the prime ring \( \overline{R} \), since \( char(\overline{R}) \neq 2 \) and \([L, L] = 0 \), we conclude that \( \overline{L} \) is central in \( \overline{R} \), which
means $[L, R] \subseteq P$.
Hence in any case it follows that $d(R)[L, R] = (0)$, $a[R, L] = (0)$ and $[d(R), L] = (0)$. Finally we obtain the required conclusions by following the same argument as in Lemma 1.3. □

In the sequel we will use the following known result:

**Lemma 1.6.** Let $R$ be a 2-torsion free semiprime ring, $L$ a Lie ideal of $R$ such that $L \not\subseteq Z(R)$. Let $a \in L$ be such that $aLa = 0$, then $a = 0$.

**Remark 1.7.** If $R$ is a prime ring of characteristic different from 2, $a \in R$ and $L$ is a non-central Lie ideal of $R$ such that $[a, L] \subseteq Z(R)$, then $a \in Z(R).

**Proof.** Denote by $\delta : R \to R$ the inner derivation of $R$ induced by the element $a \in R$. Since $[[a, x], r] = 0$ for all $x \in L$ and $r \in R$, a fortiori we have $[a, x]_2 = 0$, that is $[\delta(x), x] = 0$, for all $x \in L$. Thus, by [14] it follows $\delta = 0$, that is $a \in Z(R)$. □

**Theorem 1.8.** Let $R$ be a 2-torsion free semiprime ring and $L$ be a non-central Lie ideal of $R$. Suppose $R$ admits a generalized derivation $(F, d)$, defined as $F(x) = ax + d(x)$, for all $x \in R$ and fixed element $a \in R$. If

$$[F(x), x] \in Z(R) \text{ for all } x \in L. \quad (1.1)$$

and $d(L) \neq (0)$, then all the following hold simultaneously:

1. $d(R)[L, R] = (0)$ and $[d(R), L] = (0)$;
2. $[a, L] = a[L, R] = (0)$;
3. $aI = d(I) = (0)$ (that is $F(I) = (0)$), where $I$ denotes the ideal of $R$ generated by $[L, L]$.

**Proof.** Let $P$ be a prime ideal of $R$ such that $[L, L] \not\subseteq P$. Assume first that $d(P) \subseteq P$. Then $F$ induces a canonical generalized derivation $\overline{F}$ on $\overline{R} = \frac{R}{P}$. Therefore $[\overline{F}(\overline{x}), \overline{x}] \in Z(\overline{R})$ for all $\overline{x} \in \overline{L}$. Moreover $\overline{L}$ is a Lie ideal of $\overline{R}$, such that $[\overline{L}, \overline{L}] \neq 0$ since $[L, L] \not\subseteq P$. Since $[L, L] \not\subseteq P$, a fortiori we get $\overline{L}$ is not central in $\overline{R}$. Therefore, by Theorem 3.3 in [10], it follows that $\overline{d}(\overline{R}) = 0$ that is $d(R) \subseteq P$.

Assume now that $d(P) \not\subseteq P$, then $\overline{d}(\overline{P}) \neq 0$ and $\overline{d}(\overline{P})\overline{R} \neq 0$. By using similar argument as in Lemma 1.3, $\overline{Rd(P)}$ is a non-zero right ideal of $\overline{R}$.

Linearizing (1.1) and using (1.1), we obtain

$$[F(x), y] + [F(y), x] \in Z(R) \text{ for all } x, y \in L. \quad (1.2)$$

Now, replace $y$ by $[rp, u]$, for $r \in R$, $p \in P$ and $u \in L$ and use (1.2) to get

$$[\overline{F}([rp, u]), \overline{x}] \in Z(\overline{R}). \quad (1.3)$$

Moreover, since $F(r) = ar + d(r)$, for all $r \in R$, by (1.3) it follows

$$[\overline{d}([rp, u]), \overline{L}] \subseteq Z(\overline{R}). \quad (1.4)$$
By the primeness of \( R \) and Remark 1.7, one has that \( \overline{d([rp,u])} \in Z(R) \). On the other hand, an easy computation shows that \( \overline{d([rp,u])} = \overline{rd(p),v} \), which implies \( [rd(P),L] \subseteq Z(R) \). Once again by Remark 1.7, we have \( [rd(P) \subseteq Z(R) \). Since \( rd(P) \) is a non-zero right ideal of \( R \), it follows \( [R, R] = (0) \), which contradicts with \( [L, L] \neq (0) \).

The previous argument shows that, for any prime ideal \( P \) of \( R \), either \( [L, L] \subseteq P \) or \( d(R) \subseteq P \). Thus \( d(R)[L, L] \subseteq P \). Hence, by Lemma 1.3 and since \( L \nsubseteq Z(R) \), we finally get the required conclusions:

1. \( d(R)[L, R] = (0) \) and \( [d(R), L] = (0) \);
2. \( (a+b)[L, R] = (0) \) and \( [a+b, L] = (0) \);
3. \( aI = d(I) = (0) \), where \( I \) denotes the ideal of \( R \) generated by \( [L, L] \).

\[ \square \]

**Theorem 1.9.** Let \( R \) be a 2-torsion free semiprime ring and \( L \) be a non-central Lie ideal of \( R \). Suppose \( R \) admits two generalized derivations \((F,d)\) and \((G,g)\). Write \( F(x) = ax + d(x) \) and \( G(x) = bx + g(x) \), for some \( a,b \in U \). If \( F([x,y]) = [y,G(x)] \) for all \( x,y \in L \), then either

1. \( g(L) = (0) \);
2. \( d(R)[L, R] = (0) \) and \( [d(R), L] = (0) \);
3. \( (a+b)[L, R] = (0) \), \( [b, L] = (0) \) and \( [a, L] = (0) \);
4. \( aI = d(I) = (0) \), where \( I \) denotes the ideal of \( R \) generated by \( [L, L] \).

or

1. \( d(L) = (0) \);
2. \( g(R)[L, R] = (0) \) and \( [g(R), L] = (0) \);
3. \( [b, L] = (0) \) and \( a[L, L] = (0) \);
4. \( aI = (0) \) and \( g(I) = (0) \), where \( I \) denotes the ideal of \( R \) generated by \( [L, L] \).

or

1. \( d(R)[L, R] = (0) \) and \( [d(R), L] = (0) \);
2. \( g(R)[L, R] = (0) \) and \( [g(R), L] = (0) \);
3. \( [a, L] = (0) \), \( [b, L] = (0) \), \( b[L, R] = a[L, R] = (0) \);
4. \( d(I) = g(I) = (0) \) and \( aI = bI = (0) \) (that is \( F(I) = G(I) = (0) \)), where \( I \) denotes the ideal of \( R \) generated by \( [L, L] \).

**Proof.** Assume first \( g(L) = (0) \), then \( F([x,y]) = [y, bx] \) for all \( x,y \in L \). Thus

\[
a[x,y] + d([x,y]) = b[y, x]
\]

for all \( x,y \in L \), that is \( (a+b)[x,y] + d([x,y]) = 0 \) for all \( x,y \in L \). Therefore, applying Theorem 1.5, one has

1. \( d(R)[L, R] = (0) \) and \( [d(R), L] = (0) \);
2. \( (a+b)[L, R] = (0) \) and \( [a+b, L] = (0) \);
(3) \((a+b)I = (0)\) and \(d(I) = (0)\), where \(I\) denotes the ideal of \(R\) generated by \([L,L]\).

In particular \(d([L,L]) = (0)\) and \(a[x,y] = -b[x,y]\) for all \(x,y \in L\), so that (1.5) reduces to \((by-yb)x = 0\), for all \(x,y \in L\), that is \([b,L]L = (0)\). Hence by Lemma 1.6, we have \([b,L] = (0)\) and so also \([a,L] = (0)\).

Let now \(d(L) = (0)\), then \(a[x,y] = [y,G(x)]\) for all \(x,y \in L\). In this case, for \(x = y\), we have \([G(y),y] = 0\) and by Theorem 1.8 the following hold:

1. \(g(R)[L,R] = (0)\) and \([g(R),L] = (0)\);
2. \([b,L] = (0)\), \(b[L,R] = (0)\) and \([a,L] = (0)\);
3. \(bI = (0)\) and \(g(I) = (0)\), where \(I\) denotes the ideal of \(R\) generated by \([L,L]\).

Moreover, since \([[L,L],R] \subseteq [L,L]\), we also have \(0 = a[[L,L],R] = aR[L,L]\), which implies \(aI = (0)\).

Assume finally that both \(g(L) \neq (0)\) and \(d(L) \neq (0)\). Once again for \(x = y \in L\) we have \([G(x),x] = 0\) for any \(x \in L\). Thus by Theorem 1.8, we have that all the following hold:

1. \(g(R)[L,R] = (0)\) and \([g(R),L] = (0)\);
2. \([b,L] = (0)\) and \(b[L,R] = (0)\);
3. \(bI = (0)\) and \(g(I) = (0)\), where \(I\) denotes the ideal of \(R\) generated by \([L,L]\).

Hence by the main assumption it follows that \((a+b)[x,y] + d([x,y]) = 0\), for all \(x,y \in L\). Denote \(H(x) = (a-b)x + d(x)\), then \(H(u) = 0\) for all \(u \in [L,L]\).

Finally, by applying Theorem 1.5, one has

1. \(d(R)[L,R] = (0)\) and \([d(R),L] = (0)\);
2. \((a+b)[L,R] = (0)\) and \([a,L] = (0)\);
3. \((a+b)I = (0)\) and \(d(I) = (0)\), where \(I\) denotes the ideal of \(R\) generated by \([L,L]\).

Note that, since both \(bI = (0)\) and \((a+b)I = (0)\), we are done.

We conclude our paper with some applications to generalized derivations acting on ideals of semiprime rings:

**Theorem 1.10.** Let \(R\) be a 2-torsion free semiprime ring and \(I\) be a non-central ideal of \(R\). Suppose \(R\) admits a generalized derivation \((F,d)\), defined as \(F(x) = ax + d(x)\), for all \(x \in R\) and fixed element \(a \in R\). If \([F(x),x] = 0\) for all \(x \in I\), then either \(d(I) = 0\) or \(R\) contains a non-zero central ideal.

**Proof.** By Theorem 1.8, we have that if \(d(I) \neq (0)\) then \([d(R),I] = (0)\). Hence, by applying Main Theorem in [13], it follows that \(R\) must contain a non-zero central ideal.

**Corollary 1.11.** Let \(R\) be a 2-torsion free semiprime ring \(F\) a generalized derivation of \(R\). If \([F(x),x] = 0\) for all \(x \in R\), then either \(R\) contains a
non-zero central ideal or there exists $\lambda \in Z(R)$ such that $F(x) = \lambda x$, for all $x \in R$.

**Theorem 1.12.** Let $R$ be a 2-torsion free semiprime ring and $I$ be a non-central ideal of $R$. Suppose $R$ admits two generalized derivations $(F, d)$ and $(G, g)$. Write $F(x) = ax + d(x)$ and $G(x) = bx + g(x)$, for some $a, b \in U$. If $F([x, y]) = [y, G(x)]$ for all $x, y \in L$, then either $d(I) = g(I) = (0)$ or $R$ contains a non-zero central ideal.

**Proof.** Assume either $d(I) \neq 0$ or $g(I) \neq 0$. Thus, by Theorem 1.9 respectively we have that either $[d(R), I] = (0)$ or $[g(R), I] = (0)$. In any case, again by [13], $R$ must contain some non-zero central ideals. \[\square\]

**Corollary 1.13.** Let $R$ be a 2-torsion free semiprime ring and $F, G$ two generalized derivations of $R$. If $F([x, y]) = [y, G(x)]$ for all $x, y \in R$, then either $R$ contains a non-zero central ideal or there exist $\lambda \in Z(R)$ such that $F(x) = G(x) = \lambda x$, for all $x \in R$.

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**References**