Lie Ideals and Generalized Derivations in Semiprime Rings

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Abstract. Let $R$ be a 2-torsion free ring and $L$ a Lie ideal of $R$. An additive mapping $F : R \to R$ is called a generalized derivation on $R$ if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$. In the present paper we describe the action of generalized derivations satisfying several conditions on Lie ideals of semiprime rings.

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1. Introduction

Let $R$ be an associative ring with center $Z(R)$. A ring $R$ is said to be $n$-torsion free if $nx = 0$ implies $x = 0$ for all $x \in R$. For any $x, y \in R$, the symbol $[x, y]$ will represent the commutator $xy - yx$. Recall that a ring $R$ is prime if $aRb = 0$ implies $a = 0$ or $b = 0$ and $R$ is semiprime if $aRa = 0$ yields $a = 0$. An additive mapping $d : R \to R$ is said to be a derivation of $R$ if

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\[ d(xy) = d(x)y + xd(y) \] for all \( x, y \in R \). In particular, for a fixed \( a \in R \) the mapping \( I_a : R \to R \) given by \( I_a(x) = [x, a] \) is a derivation which is called an inner derivation determined by \( a \). In 1991 Bresar [5] introduced the concept of generalized derivation: more precisely an additive mapping \( F : R \to R \) is said to be a generalized derivation with associated derivation \( d \) if \( F(xy) = F(x)y + xd(y) \) for all \( x, y \in R \). For the sake of convenience, we shall denote by \( (F, d) \) a generalized derivation \( F \) with associated derivation \( d \). A mapping \( f : R \to R \) is known to be centralizing on \( R \) if \( [f(x), x] \in Z(R) \) for all \( x \in R \). If \( f \) is said to be commuting on \( R \). We recall that an additive group \( L \) is known to be centralizing on \( R \) if \( L, R \subseteq L \).

A well known result of Posner [18] states that a prime ring admitting a nonzero centralizing derivation must be commutative. This theorem indicates that the global structure of a ring \( R \) is often tightly connected to the behaviour of additive mappings defined on \( R \). Following this line of investigation, several authors studied derivations and generalized derivations acting on appropriate subsets of the ring.

For instance in [19] Quadri et al. prove that if \( R \) is a prime ring with a non-zero ideal \( I \) and \( F \) is a generalized derivation of \( R \) such that \( F([x, y]) = [x, y] \), for all \( x, y \in I \), then \( R \) is commutative (Theorem 2.1). Later in [7] Dhara extends all results contained in [19] to semiprime rings.

Further in [10] Gölbaşı and Koç investigate the properties of a prime ring \( R \) with a generalized derivation \((F, d)\) acting on a Lie ideal \( L \) of \( R \). They prove that if \( [F(u), u] \in Z(R) \), for all \( u \in L \), then either \( d = 0 \) or \( L \subseteq Z(R) \) (Theorem 3.3). Moreover if \( F([u, v]) = [u, v] \), for all \( u, v \in L \), then either \( d = 0 \) of \( L \subseteq Z(R) \) (Theorem 3.6).

In this note we will consider a similar situation and extend the cited results to the case of semiprime rings with a generalized derivation \((F, d)\) acting on a Lie ideal. More precisley we prove the following:

**Theorem 1.** Let \( R \) be a 2-torsion free semiprime ring and \( L \) be a non-central Lie ideal of \( R \). Suppose that \((F, d)\) is a generalized derivation of \( R \) such that \( F(x, y) \in Z(R) \), for all \( x, y \in L \). If \( d(L) \neq (0) \), then all the following hold simultaneously:

1. \( d(R)[L, R] = (0) \) and \( [d(R), L] = (0) \);
2. \( a[L, R] = (0) \) and \( [a, L] = (0) \);
3. \( aI = (0) \) and \( d(I) = (0) \) (that is \( F(I) = (0) \)), where \( I \) denotes the ideal of \( R \) generated by \([L, L]\).

**Theorem 2.** Let \( R \) be a 2-torsion free semiprime ring and \( L \) be a non-central Lie ideal of \( R \). Suppose \( R \) admits a generalized derivation \((F, d)\), defined as \( F(x) = ax + d(x) \), for all \( x \in R \) and fixed element \( a \in R \). If \( [F(x), x] \in Z(R) \) for all \( x \in L \) and \( d(L) \neq (0) \), then all the following hold simultaneously:
(1) $d(R)[L, R] = (0)$ and $[d(R), L] = (0)$;
(2) $[a, L] = a[L, R] = (0)$;
(3) $aI = d(I) = (0)$ (that is $F(I) = (0)$), where $I$ denotes the ideal of $R$ generated by $[L, L]$.

**Theorem 3.** Let $R$ be a 2-torsion free semiprime ring and $L$ be a non-central Lie ideal of $R$. Suppose $R$ admits two generalized derivations $(F, d)$ and $(G, g)$. Write $F(x) = ax + d(x)$ and $G(x) = bx + g(x)$, for some $a, b \in U$. If $F([x, y]) = [y, G(x)]$ for all $x, y \in L$, then either

(1) $g(L) = (0)$;
(2) $d(R)[L, R] = (0)$ and $[d(R), L] = (0)$;
(3) $(a + b)L = (0)$, $[b, L] = (0)$ and $[a, L] = (0)$;
(4) $(a + b)I = (0)$ and $d(I) = (0)$, where $I$ denotes the ideal of $R$ generated by $[L, L]$.

or

(1) $d(L) = (0)$;
(2) $g(R)[L, R] = (0)$ and $[g(R), L] = (0)$;
(3) $[b, L] = (0)$ and $a[L, L] = (0)$;
(4) $aI = (0)$ and $g(I) = (0)$, where $I$ denotes the ideal of $R$ generated by $[L, L]$.

or

(1) $d(R)[L, R] = (0)$ and $[d(R), L] = (0)$;
(2) $g(R)[L, R] = (0)$ and $[g(R), L] = (0)$;
(3) $[a, L] = (0)$, $[b, L] = (0)$, $b[L, R] = a[L, R] = (0)$;
(4) $d(I) = g(I) = (0)$ and $aI = bI = (0)$ (that is $F(I) = G(I) = (0)$), where $I$ denotes the ideal of $R$ generated by $[L, L]$.

In all that follows let $R$ be a non-commutative semiprime ring, $L$ a non-central Lie ideal of $R$, $U$ the right Utumi quotient ring of $R$. We refer the reader to [3] for the definition and the related properties of $U$.

We begin with the following:

**Fact 1.1.** Let $R$ be a semiprime ring. Then every generalized derivation $F$ of $R$ is uniquely extended to its right Utumi quotient ring $U$ and assumes the form $F(x) = ax + d(x)$, where $a \in U$ and $d$ is the derivation of $U$ associated with $F$ (see Theorem 4 in [17]).

**Lemma 1.2.** Let $R$ be a prime ring of characteristic different from 2 and $L$ be a Lie ideal of $R$. Suppose $R$ admits a nonzero generalized derivation $(F, d)$ such that $F(x)[x, y] = 0$ (or $[x, y]F(x) = 0$) for all $x, y \in L$, then $L \subseteq Z(R)$.

Proof. Suppose by contradiction that $L$ is not central in $R$. By [11] (pages 4-5) there exists a non-central ideal $I$ of $R$ such that $0 \neq [I, R] \subseteq L$. By our
assumption it follows that \( F(x)[x, y] = 0 \) (or \( [x, y]F(x) = 0 \)) for all \( x, y \in [I, R] \).

Since \( I \) and \( R \) satisfy the same differential identities (see the main result in [16]), we also have that \( F(x)[x, y] = 0 \) (or \( [x, y]F(x) = 0 \)) for all \( x, y \in [R, R] \). Let \( y_0 \in [R, R] \), be such that \( y_0 \notin Z(R) \) and denote by \( \delta : R \to R \) the non-zero inner derivation of \( R \) induced by the element \( y_0 \). Therefore \( F(x)\delta(x) = 0 \) (or \( \delta(x)F(x) = 0 \)) for all \( x \in [R, R] \). In light of [6], since \( \delta \neq 0 \) and \([R, R]\) is not central in \( R \), one has the contradiction that \( F = 0 \).

**Lemma 1.3.** Let \( R \) be a 2-torsion free semiprime ring and \( L \) be a non-central Lie ideal of \( R \). Suppose \( R \) admits a nonzero generalized derivation \((F, d)\), defined as \( F(x) = ax + d(x) \), for all \( x \in R \) and fixed element \( a \in R \). If \( F(x)[x, y] = 0 \) (or \( [x, y]F(x) = 0 \)) for all \( x, y \in L \), then all the following hold simultaneously:

1. \( d(R)[L, R] = (0) \) and \( [d(R), L] = (0) \);
2. \( a[L, R] = (0) \) and \( [a, L] = (0) \);
3. \((aI = (0) \) and \( d(I) = (0) \) (that is \( F(I) = (0) \)), where \( I \) denotes the ideal of \( R \) generated by \([L, L]\).

**Proof.** Let \( P \) be a prime ideal if \( R \) such that \([L, L] \notin P\).

Assume first that \( d(P) \subseteq P \). Then \( F \) induces a canonical generalized derivation \( \bar{F} \) on \( \bar{R} = \frac{R}{P} \). Therefore \( \bar{F}(\bar{x})[\bar{y}, \bar{y}] = 0 \) for all \( \bar{x}, \bar{y} \in \bar{L} \). Moreover \( \bar{L} \) is a Lie ideal of \( \bar{R} \), such that \([\bar{L}, \bar{L}] \neq 0 \) since \([L, L] \notin P \). By Lemma 1.2 it follows that \( \bar{F}(\bar{R}) = 0 \) that is \( aR \subseteq P \), \( d(R) \subseteq P \) and \( F(R) \subseteq P \). Assume now that \( d(P) \notin P \), then \( \bar{d}(P) \neq 0 \) and \( \bar{d}(P)\bar{R} \neq 0 \). Moreover note that, for any \( p \in P \) and \( r, s \in R \), \( d(pr)s = d(p)s + pd(r)s \) implies that \( d(P)R \subseteq \bar{d}(PR) + P \), in particular \( \bar{d}(P)\bar{R} \) is a non-zero right ideal of \( \bar{R} \).

Starting from our main assumption and linearizing we have that \( F(x)[z, y] + F(z)[x, y] = 0 \), for all \( x, y, z \in L \). For any \( p \in P \), \( r \in R \), \( u \in L \), replace \( x \) by \([pr, u] \). By computation it follows \( [\bar{p}, \bar{u}][\bar{x}, \bar{y}] = 0 \), for all \( \bar{p} \in \bar{d}(P)\bar{R} \) and \( \bar{x}, \bar{y} \in \bar{L} \). By using the same argument of Lemma 1.2, since \( \bar{L} \) is not central in \( \bar{R} \), one has that \( \bar{d}(P)\bar{R} \) is a central right ideal of \( \bar{R} \), which implies that \( \bar{R} \) is commutative, a contradiction.

Therefore, for any prime ideal \( P \) of \( R \), either \( aR \subseteq P \), \( d(R) \subseteq P \) and \( F(R) \subseteq P \) or \([L, L] \subseteq P \). In this last case, by applying Theorem 3 in [15] in the prime ring \( \bar{R} \) since \( char(\bar{R}) \neq 2 \) and \([\bar{L}, \bar{L}] = 0 \), we have that \( \bar{L} \) is central in \( \bar{R} \), which means \([L, R] \subseteq P \).

Hence in any case it follows that \( d(R)[L, R] = (0) \), \( a[R, L] = (0) \) and \( [d(R), L] = (0) \).

By \( a[R, L] = (0) \) we get \( aR[R, L] = (0) \) and so both \( aLR[a, L] = (0) \) and \( LaR[a, L] = (0) \), that is \( [a, L]R[a, L] = (0) \). By the semiprimeness of \( R \) it follows \( [a, L] = (0) \).

Moreover, if \( I = R[L, L]R \) denotes the ideal of \( R \) generated by \([L, L]\), it follows that \( aI = (0) \) and \( d(I) = (0) \), that is \( F(I) = (0) \).

\( \square \)
Corollary 1.4. Let \( R \) be a 2-torsion free semiprime ring and \( L \) be a non-central Lie ideal of \( R \). Suppose \( a \in R \) be such that \( ax, y = 0 \) for all \( x, y \in L \), then \( a[L, R] = 0 \), \( [a, L] = 0 \) and \( aI = 0 \), where \( I \) denotes the ideal of \( R \) generated by \( [L, L] \).

Theorem 1.5. Let \( R \) be a 2-torsion free semiprime ring and \( L \) be a non-central Lie ideal of \( R \). Suppose that \((F, d)\) is a generalized derivation of \( R \) such that \( F[x, y] \in Z(R) \), for all \( x, y \in L \). If \( d(L) \neq (0) \), then all the following hold simultaneously:

1. \( d(R)[L, R] = (0) \) and \( [d(R), L] = (0) \);
2. \( a[L, R] = (0) \) and \( [a, L] = (0) \);
3. \( aI = (0) \) and \( d(I) = (0) \) (that is \( F(I) = (0) \)), where \( I \) denotes the ideal of \( R \) generated by \([L, L]\).

Proof. Assume first that \( R \) is prime and denote \( V = [L, L] \). Hence we have \( F(V) \subseteq Z(R) \). As a consequence of Lemma 2 in \([9]\) we conclude that either \( F = 0 \) or \( V \subseteq Z(R) \). In the first case we have the contradiction \( d = 0 \), and in the other case one has \( L \subseteq Z(R) \) (see Lemma 2 in \([12]\)), a contradiction again. Let \( P \) be a prime ideal of \( R \) such that \([L, L] \not\subseteq P \).

Assume first that \( d(P) \subseteq P \). Then \( F \) induces a canonical generalized derivation \( \overline{F} \) on \( \overline{R} = \frac{R}{P} \). Therefore \( \overline{F}(\overline{x}, \overline{y}) \in Z(\overline{R}) \) for all \( \overline{x}, \overline{y} \in \overline{L} \). Moreover \( \overline{L} \) is a Lie ideal of \( \overline{R} \), such that \([\overline{L}, \overline{L}] \neq 0 \) since \([L, L] \not\subseteq P \). By previous argument it follows that \( \overline{F}(\overline{L}) = 0 \) that \( d(R) \subseteq P \) and \( F(R) \subseteq P \).

Assume now that \( d(P) \not\subseteq P \), then \( d(P) \neq 0 \) and \( d(P)R \neq 0 \). We remark again that \( d(P)R \) is a non-zero right ideal of \( \overline{R} \).

Starting from our main assumption and linearizing we have that

\[
F(x)y + F(x)z + xd(y) + xd(z) + F(y)x + F(z)x - yd(x) - zd(x) \in Z(R), \quad \forall x, y, z \in L.
\]

For any \( p, p', p'' \in P, r, s \in R, u, v \in L \), replace \( y \) by \([pr, u] \) and \( z \) by \([[p's, v], p''] \).

By computation it follows

\[
\overline{x}[\overline{t}, \overline{u}] - [\overline{t}, \overline{u}]\overline{x} \in Z(\overline{R})
\]

that is

\[
[\overline{x}, [\overline{t}, \overline{u}]] \in Z(\overline{R})
\]

for all \( \overline{t} \in d(P)R \) and \( \overline{x}, \overline{u} \in \overline{L} \). As above denote \( \overline{V} = [\overline{L}, \overline{L}] \), which is a Lie ideal for \( \overline{R} \), and \( \delta \) is the derivation of \( \overline{R} \) induced by \( \overline{t} \). Hence we have \( \delta(\overline{V}) \subseteq Z(\overline{R}) \). Again as a consequence of Lemma 2 in \([9]\) it follows that either \( \delta = 0 \) or \( \overline{V} \subseteq Z(\overline{R}) \). Since \( \overline{R} \) is not commutative, then there exists some \( \overline{t} \in \overline{R} \) which is not central. Thus \( \overline{V} \subseteq Z(\overline{R}) \), and \( \overline{L} \subseteq Z(\overline{R}) \) follows from Lemma 2 in \([12]\).

Therefore, for any prime ideal \( P \) of \( R \), either \( d(R) \subseteq P \) and \( F(R) \subseteq P \) or \([L, L] \subseteq P \). In this last case, by applying Theorem 3 in \([15]\) in the prime ring \( \overline{R} \), since \( \text{char}(\overline{R}) \neq 2 \) and \([\overline{L}, \overline{L}] = 0 \), we conclude that \( \overline{L} \) is central in \( \overline{R} \), which
means \([L, R] \subseteq P\).

Hence in any case it follows that \(d(R)[L, R] = (0)\), \(a[R, L] = (0)\) and \([d(R), L] = (0)\). Finally we obtain the required conclusions by following the same argument as in Lemma 1.3.

In the sequel we will use the following known result:

**Lemma 1.6.** Let \(R\) be a 2-torsion free semiprime ring, \(L\) a Lie ideal of \(R\) such that \(L \nsubseteq Z(R)\). Let \(a \in L\) be such that \(aLa = 0\), then \(a = 0\).

**Remark 1.7.** If \(R\) is a prime ring of characteristic different from 2, \(a \in R\) and \(L\) is a non-central Lie ideal of \(R\) such that \([a, L] \subseteq Z(R)\), then \(a \in Z(R)\).

**Proof.** Denote by \(\delta : R \to R\) the inner derivation of \(R\) induced by the element \(a \in R\). Since \([a, x], r] = 0\) for all \(x \in L\) and \(r \in R\), a fortiori we have \([a, x]_2 = 0\), that is \([\delta(x), x] = 0\), for all \(x \in L\). Thus, by [14] it follows \(\delta = 0\), that is \(a \in Z(R)\). □

**Theorem 1.8.** Let \(R\) be a 2-torsion free semiprime ring and \(L\) be a non-central Lie ideal of \(R\). Suppose \(R\) admits a generalized derivation \((F, d)\), defined as \(F(x) = ax + d(x)\), for all \(x \in R\) and fixed element \(a \in R\). If

\[
[F(x), x] \in Z(R)\text{ for all }x \in L.
\]

and \(d(L) \neq (0)\), then all the following hold simultaneously:

1. \(d(R)[L, R] = (0)\) and \([d(R), L] = (0)\);
2. \([a, L] = a[L, R] = (0)\);
3. \(aI = d(I) = (0)\) (that is \(F(I) = (0)\)), where \(I\) denotes the ideal of \(R\) generated by \([L, L]\).

**Proof.** Let \(P\) be a prime ideal of \(R\) such that \([L, L] \nsubseteq P\).

Assume first that \(d(P) \subseteq P\). Then \(F\) induces a canonical generalized derivation \(\overline{F}\) on \(\overline{R} = \frac{R}{P}\). Therefore \(\overline{F}(\overline{x}, \overline{y}) \in Z(\overline{R})\) for all \(\overline{x}, \overline{y} \in \overline{L}\). Moreover \(\overline{L}\) is a Lie ideal of \(\overline{R}\), such that \([\overline{L}, \overline{L}] = (0)\) since \([L, L] \nsubseteq P\). Since \([L, L] \nsubseteq P\), a fortiori we get \(\overline{L}\) is not central in \(\overline{R}\). Therefore, by Theorem 3.3 in [10], it follows that \(\overline{d}(\overline{R}) = \overline{0}\) that is \(d(R) \subseteq P\).

Assume now that \(d(P) \nsubseteq P\), then \(\overline{d}([P]) \neq \overline{0}\) and \(\overline{d}([P]R) \neq \overline{0}\). By using similar argument as in Lemma 1.3, \(\overline{Rd(P)}\) is a non-zero right ideal of \(\overline{R}\).

Linearizing (1.1) and using (1.1), we obtain

\[
[F(x), y] + [F(y), x] \in Z(R)\text{ for all }x, y \in L.
\]

Now, replace \(y\) by \([rp, u]\), for \(r \in R\), \(p \in P\) and \(u \in L\) and use (1.2) to get

\[
[\overline{F}([rp, u]), \overline{y}] \in Z(\overline{R}).
\]

Moreover, since \(F(r) = ar + d(r)\), for all \(r \in R\), by (1.3) it follows

\[
[\overline{d}([rp, u]), \overline{y}] \subseteq Z(\overline{R}).
\]
By the primeness of $\mathfrak{R}$ and Remark 1.7, one has that $\mathfrak{d}(\mathfrak{r}_p, u) \in Z(\mathfrak{R})$. On the other hand, an easy computation shows that $\mathfrak{d}(\mathfrak{r}_p, u) = [\mathfrak{rd}(p), \mathfrak{r}]$, which implies $[\mathfrak{rd}(P), \mathfrak{L}] \subseteq Z(\mathfrak{R})$. Once again by Remark 1.7, we have $\mathfrak{Rd}(P) \subseteq Z(\mathfrak{R})$. Since $\mathfrak{Rd}(P)$ is a non-zero right ideal of $\mathfrak{R}$, it follows $[\mathfrak{L}, \mathfrak{R}] = (0)$, which contradicts with $[\mathfrak{L}, \mathfrak{L}] \neq (0)$.

The previous argument shows that, for any prime ideal $P$ of $\mathfrak{R}$, either $[L, L] \subseteq P$ or $d(R) \subseteq P$. Thus $d(R)|L, L| \subseteq \cap P = (0)$. Hence, by Lemma 1.3 and since $L \not\subseteq Z(\mathfrak{R})$, we finally get the required conclusions:

1. $d(R)|L, R| = (0)$ and $[d(R), L] = (0)$;
2. $a[L, R] = (0)$ and $[a, L] = (0)$;
3. $aI = d(I) = (0)$, where $I$ denotes the ideal of $R$ generated by $[L, L]$.

\[\square\]

**Theorem 1.9.** Let $R$ be a 2-torsion free semiprime ring and $L$ be a non-central Lie ideal of $R$. Suppose $R$ admits two generalized derivations $(F, d)$ and $(G, g)$. Write $F(x) = ax + d(x)$ and $G(x) = bx + g(x)$, for some $a, b \in U$. If $F([x, y]) = [y, G(x)]$ for all $x, y \in L$, then either

1. $g(L) = (0)$;
2. $d(R)|L, R| = (0)$ and $[d(R), L] = (0)$;
3. $a[L, R] = (0)$ and $[a, L] = (0)$;
4. $aI = d(I) = (0)$, where $I$ denotes the ideal of $R$ generated by $[L, L]$.

or

1. $d(L) = (0)$;
2. $g(R)|L, R| = (0)$ and $[g(R), L] = (0)$;
3. $a[L, L] = (0)$ and $[a, L] = (0)$;
4. $aI = (0)$ and $g(I) = (0)$, where $I$ denotes the ideal of $R$ generated by $[L, L]$.

or

1. $d(R)|L, R| = (0)$ and $[d(R), L] = (0)$;
2. $g(R)|L, R| = (0)$ and $[g(R), L] = (0)$;
3. $a[L, R] = (0)$, $[a, L] = (0)$, $b[L, R] = a[L, R] = (0)$;
4. $d(I) = g(I) = (0)$ and $aI = bI = (0)$ (that is $F(I) = G(I) = (0)$), where $I$ denotes the ideal of $R$ generated by $[L, L]$.

**Proof.** Assume first $g(L) = (0)$, then $F([x, y]) = [y, bx]$ for all $x, y \in L$. Thus

$$a[x, y] + d([x, y]) = b[y, x]$$

for all $x, y \in L$, that is $(a + b)[x, y] + d([x, y]) = 0$ for all $x, y \in L$. Therefore, applying Theorem 1.5, one has

1. $d(R)|L, R| = (0)$ and $[d(R), L] = (0)$;
2. $(a + b)[L, R] = (0)$ and $[a, b, L] = (0)$.
(3) \((a+b)I = (0)\) and \(d(I) = (0)\), where \(I\) denotes the ideal of \(R\) generated by \([L, L]\).

In particular \(d([L, L]) = (0)\) and \(a[x, y] = -b[x, y]\) for all \(x, y \in L\), so that (1.5) reduces to \((by - yb)x = 0\), for all \(x, y \in L\), that is \([b, L]L = (0)\). Hence by Lemma 1.6, we have \([b, L] = (0)\) and so also \([a, L] = (0)\).

Let now \(d(L) = (0)\), then \([G(x), x] = 0\) for all \(x, y \in L\). In this case, for \(x = y\), we have \([G(y), y] = 0\) and by Theorem 1.8 the following hold:

1. \(g(R)[L, R] = (0)\) and \([g(R), L] = (0)\);
2. \([b, L] = (0)\), \(b[L, R] = (0)\) and \([a, L] = (0)\);
3. \((a+b)I = (0)\) and \(g(I) = (0)\), where \(I\) denotes the ideal of \(R\) generated by \([L, L]\).

Moreover, since \([[L, L], R] \subseteq [L, L]\), we also have 0 = \(a[[L, L], R] = aR[L, L]\), which implies \(aI = (0)\).

Assume finally that both \(g(L) \neq (0)\) and \(d(L) \neq (0)\). Once again for \(x = y \in L\) we have \([G(x), x] = 0\) for any \(x \in L\). Thus by Theorem 1.8, we have that all the following hold:

1. \(g(R)[L, R] = (0)\) and \([g(R), L] = (0)\);
2. \([b, L] = (0)\) and \([b, L] = (0)\);
3. \((a+b)I = (0)\) and \(g(I) = (0)\), where \(I\) denotes the ideal of \(R\) generated by \([L, L]\).

Hence by the main assumption it follows that \((a+b)[x, y] + d([x, y]) = 0\), for all \(x, y \in L\). Denote \(H(x) = (a-b)x + d(x)\), then \(H(u) = 0\) for all \(u \in [L, L]\).

Finally, by applying Theorem 1.5, one has

1. \(d(R)[L, R] = (0)\) and \([d(R), L] = (0)\);
2. \((a+b)[L, R] = (0)\) and \([a, L] = (0)\);
3. \((a+b)I = (0)\) and \(d(I) = (0)\), where \(I\) denotes the ideal of \(R\) generated by \([L, L]\).

Note that, since both \(bI = (0)\) and \((a+b)I = (0)\), we are done. \(\square\)

We conclude our paper with some applications to generalized derivations acting on ideals of semiprime rings:

**Theorem 1.10.** Let \(R\) be a \(2\)-torsion free semiprime ring and \(I\) be a non-central ideal of \(R\). Suppose \(R\) admits a generalized derivation \((F, d)\), defined as \(F(x) = ax + dx\), for all \(x \in R\) and fixed element \(a \in R\). If \([F(x), x] = 0\) for all \(x \in I\), then either \(d(I) = 0\) or \(R\) contains a non-zero central ideal.

**Proof.** By Theorem 1.8, we have that if \(d(I) \neq (0)\) then \([d(R), I] = (0)\). Hence, by applying Main Theorem in [13], it follows that \(R\) must contain a non-zero central ideal. \(\square\)

**Corollary 1.11.** Let \(R\) be a \(2\)-torsion free semiprime ring \(F\) a generalized derivation of \(R\). If \([F(x), x] = 0\) for all \(x \in R\), then either \(R\) contains a
non-zero central ideal or there exists \( \lambda \in Z(R) \) such that \( F(x) = \lambda x \), for all \( x \in R \).

**Theorem 1.12.** Let \( R \) be a 2-torsion free semiprime ring and \( I \) be a non-central ideal of \( R \). Suppose \( R \) admits two generalized derivations \( (F,d) \) and \( (G,g) \). Write \( F(x) = ax + d(x) \) and \( G(x) = bx + g(x) \), for some \( a,b \in U \). If \( F([x,y]) = [y,G(x)] \) for all \( x,y \in L \), then either \( d(I) = g(I) = (0) \) or \( R \) contains a non-zero central ideal.

**Proof.** Assume either \( d(I) \neq 0 \) or \( g(I) \neq 0 \). Thus, by Theorem 1.9 respectively we have that either \( [d(R),I] = (0) \) or \( [g(R),I] = (0) \). In any case, again by [13], \( R \) must contain some non-zero central ideals. \( \Box \)

**Corollary 1.13.** Let \( R \) be a 2-torsion free semiprime ring and \( F,G \) two generalized derivations of \( R \). If \( F([x,y]) = [y,G(x)] \) for all \( x,y \in R \), then either \( R \) contains a non-zero central ideal or there exist \( \lambda \in Z(R) \) such that \( F(x) = G(x) = \lambda x \), for all \( x \in R \).

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**REFERENCES**