Lie Ideals and Generalized Derivations in Semiprime Rings

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Abstract. Let \( R \) be a 2-torsion free ring and \( L \) a Lie ideal of \( R \). An additive mapping \( F : R \to R \) is called a generalized derivation on \( R \) if there exists a derivation \( d : R \to R \) such that \( F(xy) = F(x)y + xd(y) \) holds for all \( x, y \in R \). In the present paper we describe the action of generalized derivations satisfying several conditions on Lie ideals of semiprime rings.

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1. Introduction

Let \( R \) be an associative ring with center \( Z(R) \). A ring \( R \) is said to be \( n \)-torsion free if \( nx = 0 \) implies \( x = 0 \) for all \( x \in R \). For any \( x, y \in R \), the symbol \([x, y]\) will represent the commutator \( xy - yx \). Recall that a ring \( R \) is prime if \( aRb = 0 \) implies \( a = 0 \) or \( b = 0 \) and \( R \) is semiprime if \( aRa = 0 \) yields \( a = 0 \). An additive mapping \( d : R \to R \) is said to be a derivation of \( R \) if

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\[ d(xy) = d(x)y + xd(y) \] for all \( x, y \in R \). In particular, for a fixed \( a \in R \) the mapping \( I_a : R \to R \) given by \( I_a(x) = [x, a] \) is a derivation which is called an inner derivation determined by \( a \). In 1991 Bresar [5] introduced the concept of generalized derivation: more precisely an additive mapping \( F : R \to R \) is said to be a generalized derivation with associated derivation \( d \) if

\[ F(xy) = F(x)y + xd(y) \]

for all \( x, y \in R \). For the sake of convenience, we shall denote by \((F, d)\) a generalized derivation \( F \) with associated derivation \( d \). A mapping \( f : R \to R \) is known to be centralizing on \( R \) if \([f(x), x] \in Z(R)\) for all \( x \in R \). In particular, if \([f(x), x] = 0\) for all \( x \in R \), then \( f \) is said to be commuting on \( R \). We recall that an additive group \( L \) of \( R \) is said to be a Lie ideal of \( R \) if \([L, R] \subseteq L\).

A well known result of Posner [18] states that a prime ring admitting a nonzero centralizing derivation must be commutative. This theorem indicates that the global structure of a ring \( R \) is often tightly connected to the behaviour of additive mappings defined on \( R \).

Following this line of investigation, several authors studied derivations and generalized derivations acting on appropriate subsets of the ring.

For instance in [19] Quadri et al. prove that if \( R \) is a prime ring with a non-zero ideal \( I \) and \( F \) is a generalized derivation of \( R \) such that \( F([x, y]) = [x, y] \), for all \( x, y \in I \), then \( R \) is commutative (Theorem 2.1). Later in [7] Dhara extends all results contained in [19] to semiprime rings.

Further in [10] Gölbasi and Koç investigate the properties of a prime ring \( R \) with a generalized derivation \((F, d)\) acting on a Lie ideal \( L \) of \( R \). They prove that if \([F(u), u] \in Z(R)\), for all \( u \in L \), then either \( d = 0 \) or \( L \subseteq Z(R) \) (Theorem 3.3). Moreover if \( F([u, v]) = [u, v] \), for all \( u, v \in L \), then either \( d = 0 \) of \( L \subseteq Z(R) \) (Theorem 3.6).

In this note we will consider a similar situation and extend the cited results to the case of semiprime rings with a generalized derivation \((F, d)\) acting on a Lie ideal. More precisely we prove the following:

**Theorem 1.** Let \( R \) be a 2-torsion free semiprime ring and \( L \) be a non-central Lie ideal of \( R \). Suppose that \((F, d)\) is a generalized derivation of \( R \) such that \( F[x, y] \in Z(R) \), for all \( x, y \in L \). If \( d(L) \neq (0) \), then all the following hold simultaneously:

1. \( d(R)[L, R] = (0) \) and \([d(R), L] = (0)\);
2. \( a[L, R] = (0) \) and \([a, L] = (0)\);
3. \( aI = (0) \) and \( d(I) = (0) \) (that is \( F(I) = (0) \)), where \( I \) denotes the ideal of \( R \) generated by \([L, L]\).

**Theorem 2.** Let \( R \) be a 2-torsion free semiprime ring and \( L \) be a non-central Lie ideal of \( R \). Suppose \( R \) admits a generalized derivation \((F, d)\), defined as \( F(x) = ax + d(x) \), for all \( x \in R \) and fixed element \( a \in R \). If \([F(x), x] \in Z(R)\) for all \( x \in L \) and \( d(L) \neq (0) \), then all the following hold simultaneously:
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(1) \(d(R)[L,R] = (0)\) and \([d(R),L] = (0)\);
(2) \([a,L] = 0\) and \([d(R),L] = (0)\);
(3) \(aI = d(I) = (0)\) (that is \(F(I) = (0)\)), where \(I\) denotes the ideal of \(R\) generated by \([L,L]\).

Theorem 3. Let \(R\) be a 2-torsion free semiprime ring and \(L\) be a non-central Lie ideal of \(R\). Suppose \(R\) admits two generalized derivations \((F,d)\) and \((G,g)\).

Write \(F(x) = ax + d(x)\) and \(G(x) = bx + g(x)\), for some \(a, b \in U\). If \(F([x,y]) = [y,G(x)]\) for all \(x, y \in L\), then either

(1) \(g(L) = (0)\);
(2) \(d(R)[L,R] = (0)\) and \([d(R),L] = (0)\);
(3) \((a+b)[L,R] = (0)\), \([b,L] = (0)\) and \([a,L] = (0)\);
(4) \((a+b)I = (0)\) and \(d(I) = (0)\), where \(I\) denotes the ideal of \(R\) generated by \([L,L]\).

or

(1) \(d(L) = (0)\);
(2) \(g(R)[L,R] = (0)\) and \([g(R),L] = (0)\);
(3) \([b,L] = (0)\) and \([a,L] = (0)\);
(4) \(aI = (0)\) and \(g(I) = (0)\), where \(I\) denotes the ideal of \(R\) generated by \([L,L]\).

or

(1) \(d(R)[L,R] = (0)\) and \([d(R),L] = (0)\);
(2) \(g(R)[L,R] = (0)\) and \([g(R),L] = (0)\);
(3) \([a,L] = (0)\), \([b,L] = (0)\), \(b[L,R] = a[L,R] = (0)\);
(4) \(d(I) = g(I) = (0)\) and \(aI = bI = (0)\) (that is \(F(I) = G(I) = (0)\)), where \(I\) denotes the ideal of \(R\) generated by \([L,L]\).

In all that follows let \(R\) be a non-commutative semiprime ring, \(L\) a non-central Lie ideal of \(R\), \(U\) the right Utumi quotient ring of \(R\). We refer the reader to [3] for the definition and the related properties of \(U\).

We begin with the following:

Fact 1.1. Let \(R\) be a semiprime ring. Then every generalized derivation \(F\) of \(R\) is uniquely extended to its right Utumi quotient ring \(U\) and assumes the form \(F(x) = ax + d(x)\), where \(a \in U\) and \(d\) is the derivation of \(U\) associated with \(F\) (see Theorem 4 in [17]).

Lemma 1.2. Let \(R\) be a prime ring of characteristic different from 2 and \(L\) be a Lie ideal of \(R\). Suppose \(R\) admits a nonzero generalized derivation \((F,d)\) such that \(F(x)[x,y] = 0\) (or \([x,y]F(x) = 0\)) for all \(x, y \in L\), then \(L \subseteq Z(R)\).

Proof. Suppose by contradiction that \(L\) is not central in \(R\). By [11] (pages 4-5) there exists a non-central ideal \(I\) of \(R\) such that \(0 \neq [I,R] \subseteq L\). By our
assumption it follows that $F(x)[x, y] = 0$ (or $[x, y]F(x) = 0$) for all $x, y \in [I, R]$. Since $I$ and $R$ satisfy the same differential identities (see the main result in [16]), we also have that $F(x)[x, y] = 0$ (or $[x, y]F(x) = 0$) for all $x, y \in [R, R]$. Let $y_0 \in [R, R]$ be such that $y_0 \notin Z(R)$ and denote by $\delta : R \to R$ the non-zero inner derivation of $R$ induced by the element $y_0$. Therefore $F(x)\delta(x) = 0$ (or $\delta(x)F(x) = 0$) for all $x \in [R, R]$. In light of [6], since $\delta \neq 0$ and $[R, R]$ is not central in $R$, one has the contradiction that $F = 0$.

\textbf{Lemma 1.3.} Let $R$ be a 2-torsion free semiprime ring and $L$ be a non-central Lie ideal of $R$. Suppose $R$ admits a nonzero generalized derivation $(F, d)$, defined as $F(x) = ax + d(x)$, for all $x \in R$ and fixed element $a \in R$. If $F(x)[x, y] = 0$ (or $[x, y]F(x) = 0$) for all $x, y \in L$, then all the following hold simultaneously:

1. $d(R)[L, R] = (0)$ and $[d(R), L] = (0)$;
2. $a[L, R] = (0)$ and $[a, L] = (0)$;
3. $aI = (0)$ and $d(I) = (0)$ (that is $F(I) = (0)$), where $I$ denotes the ideal of $R$ generated by $[L, L]$.

\textbf{Proof.} Let $P$ be a prime ideal if $R$ such that $[L, L] \notin P$.

Assume first that $d(P) \subseteq P$. Then $F$ induces a canonical generalized derivation $\overline{F}$ on $\overline{R} = \frac{R}{P}$. Therefore $\overline{F}(\overline{\pi}, \overline{\eta}) = 0$ for all $\pi, \eta \in \overline{L}$. Moreover $\overline{L}$ is a Lie ideal of $\overline{R}$, such that $[\overline{L}, \overline{L}] \neq 0$ since $[L, L] \notin P$. By Lemma 1.2 it follows that $\overline{F}(\overline{R}) = 0$, that is $aR \subseteq P$, $d(R) \subseteq P$ and $F(R) \subseteq P$.

Assume now that $d(P) \notin P$, then $\overline{d(P)} \neq 0$ and $\overline{d(P)R} \neq 0$. Moreover note that, for any $p \in P$ and $r, s \in R$, $d(pr)s = d(p)r + pd(r)s$ implies that $d(P)R \subseteq d(P)R + P$, in particular $\overline{d(P)R}$ is a non-zero right ideal of $\overline{R}$.

Starting from our main assumption and linearizing we have that $F(x)[z, y] + F(z)[x, y] = 0$, for all $x, y, z \in L$. For any $p \in P, r \in R, u \in L$, replace $x$ by $[pr, u]$. By computation it follows $[\overline{\pi}, \overline{\eta}] [\overline{\pi}, \overline{\eta}] = 0$, for all $\pi \in \overline{d(P)R}$ and $\pi, \eta \in \overline{L}$. By using the same argument of Lemma 1.2, since $\overline{L}$ is not central in $\overline{R}$, one has that $\overline{d(P)R}$ is a central right ideal of $\overline{R}$, which implies that $\overline{R}$ is commutative, a contradiction.

Therefore, for any prime ideal $P$ of $R$, either $aR \subseteq P$, $d(R) \subseteq P$ and $F(R) \subseteq P$ or $[L, L] \subseteq P$. In this last case, by applying Theorem 3 in [15] in the prime ring $\overline{R}$, since $\text{char}(\overline{R}) \neq 2$ and $[\overline{L}, \overline{L}] = 0$, we have that $\overline{L}$ is central in $\overline{R}$, which means $[L, R] \subseteq P$.

Hence in any case it follows that $d(R)[L, R] = (0)$, $a[R, L] = (0)$ and $[d(R), L] = (0)$.

By $a[R, L] = (0)$ we get $aR[R, L] = (0)$ and so both $aLR[a, L] = (0)$ and $LaR[a, L] = (0)$, that is $[a, L]R[a, L] = (0)$. By the semiprimeness of $R$ it follows $[a, L] = (0)$.

Moreover, if $I = R[L, L]R$ denotes the ideal of $R$ generated by $[L, L]$, it follows that $aI = (0)$ and $d(I) = (0)$, that is $F(I) = (0)$. □
Corollary 1.4. Let $R$ be a 2-torsion free semiprime ring and $L$ be a non-central Lie ideal of $R$. Suppose $a \in R$ be such that $ax[y, z] = 0$ for all $x, y \in L$, then $a[L, R] = (0)$, $[a, L] = (0)$ and $aI = (0)$, where $I$ denotes the ideal of $R$ generated by $[L, L]$.

Theorem 1.5. Let $R$ be a 2-torsion free semiprime ring and $L$ be a non-central Lie ideal of $R$. Suppose that $(F, d)$ is a generalized derivation of $R$ such that $F[x, y] \in Z(R)$, for all $x, y \in L$. If $d(L) \neq (0)$, then all the following hold simultaneously:

1. $d(R)[L, R] = (0)$ and $[d(R), L] = (0)$;
2. $a[L, R] = (0)$ and $[a, L] = (0)$;
3. $aI = (0)$ and $d(I) = (0)$ (that is $F(I) = (0)$), where $I$ denotes the ideal of $R$ generated by $[L, L]$.

Proof. Assume first that $R$ is prime and denote $V = [L, L]$. Hence we have $F(V) \subseteq Z(R)$. As a consequence of Lemma 2 in [9] we conclude that either $F = 0$ or $V \subseteq Z(R)$. In the first case we have the contradiction $d = 0$, and in the other case one has $L \subseteq Z(R)$ (see Lemma 2 in [12]), a contradiction again. Let now $P$ be a prime ideal of $R$ such that $[L, L] \not\subseteq P$.

Assume first that $d(P) \subseteq P$. Then $F$ induces a canonical generalized derivation $\overline{F}$ on $\overline{R} = \frac{R}{P}$. Therefore $\overline{F}(\overline{x}, \overline{y}) \in Z(\overline{R})$ for all $x, y \in L$. Moreover $\overline{L}$ is a Lie ideal of $\overline{R}$, such that $[\overline{L}, \overline{L}] \neq 0$ since $[L, L] \not\subseteq P$. By previous argument it follows that $\overline{F}(\overline{R}) = \overline{0}$ that $d(R) \subseteq P$ and $F(R) \subseteq P$.

Assume now that $d(P) \not\subseteq P$, then $d(P) \neq 0$ and $d(P)\overline{R} \neq \overline{0}$. We remark again that $d(P)\overline{R}$ is a non-zero right ideal of $\overline{R}$.

Starting from our main assumption and linearizing we have that

$$F(x)y + F(x)z + xd(y) + xd(z) - F(y)x - F(z)x - yd(x) - zd(x) \in Z(R), \forall x, y, z \in L.$$

For any $p, p', p'' \in P, r, s \in R, u, v \in L$, replace $y$ by $[pr, u]$ and $z$ by $[p's, v]$, $p''$ by computation it follows

$$x[\overline{t}, \overline{u}] - [\overline{t}, \overline{u}]x \in Z(\overline{R})$$

that is

$$\left[\overline{x}, [\overline{t}, \overline{u}]\right] \in Z(\overline{R})$$

for all $\overline{t} \in d(P)\overline{R}$ and $\overline{x}, \overline{u} \in \overline{L}$. As above denote $\overline{V} = [\overline{L}, \overline{L}]$, which is a Lie ideal for $\overline{R}$, and $\delta$ is the derivation of $\overline{R}$ induced by $\overline{t}$. Hence we have $\delta(\overline{V}) \subseteq Z(\overline{R})$. Again as a consequence of Lemma 2 in [9] it follows that either $\delta = 0$ or $\overline{V} \subseteq Z(\overline{R})$. Since $\overline{R}$ is not commutative, then there exists some $\overline{t} \in \overline{R}$ which is not central. Thus $\overline{V} \subseteq Z(\overline{R})$, and $\overline{L} \subseteq Z(\overline{R})$ follows from Lemma 2 in [12].

Therefore, for any prime ideal $P$ of $R$, either $d(R) \subseteq P$ and $F(R) \subseteq P$ or $[L, L] \subseteq P$. In this last case, by applying Theorem 3 in [15] in the prime ring $\overline{R}$, since $\text{char}(\overline{R}) \neq 2$ and $[\overline{L}, \overline{L}] = \overline{0}$, we conclude that $\overline{L}$ is central in $\overline{R}$, which
means \([L, R] \subseteq P\).

Hence in any case it follows that \(d(R)[L, R] = (0)\), \(a[L, L] = (0)\) and \([d(R), L] = (0)\). Finally we obtain the required conclusions by following the same argument as in Lemma 1.3.

In the sequel we will use the following known result:

**Lemma 1.6.** Let \(R\) be a 2-torsion free semiprime ring, \(L\) a Lie ideal of \(R\) such that \(L \not\subseteq Z(R)\). Let \(a \in L\) be such that \(aL = 0\), then \(a = 0\).

**Remark 1.7.** If \(R\) is a prime ring of characteristic different from 2, \(a \in R\) and \(L\) is a non-central Lie ideal of \(R\) such that \([a, L] \subseteq Z(R)\), then \(a \in Z(R)\).

**Proof.** Denote by \(\delta : R \to R\) the inner derivation of \(R\) induced by the element \(a \in R\). Since \([a, x], r] = 0\) for all \(x \in L\) and \(r \in R\), a fortiori we have \([a, x]_{L} = 0\), that is \([\delta(x), x] = 0\), for all \(x \in L\). Thus, by [14] it follows \(\delta = 0\), that is \(a \in Z(R)\).

**Theorem 1.8.** Let \(R\) be a 2-torsion free semiprime ring and \(L\) be a non-central Lie ideal of \(R\). Suppose \(R\) admits a generalized derivation \((F, d)\), defined as \(F(x) = ax + d(x)\), for all \(x \in R\) and fixed element \(a \in R\). If

\[
[F(x), x] \in Z(R) \quad \text{for all } x \in L.
\]

and \(d(L) \neq (0)\), then all the following hold simultaneously:

1. \(d(R)[L, R] = (0)\) and \([d(R), L] = (0)\);
2. \([a, L] = a[L, R] = (0)\);
3. \(aI = d(I) = (0)\) (that is \(F(I) = (0)\)), where \(I\) denotes the ideal of \(R\) generated by \([L, L]\).

**Proof.** Let \(P\) be a prime ideal of \(R\) such that \([L, L] \not\subseteq P\).

Assume first that \(d(P) \subseteq P\). Then \(F\) induces a canonical generalized derivation \(\mathcal{F}\) on \(\overline{R} = \frac{R}{P}\). Therefore \(\mathcal{F}([\overline{x}, \overline{y}]) \in Z(\overline{R})\) for all \(\overline{x} \in \overline{L}\). Moreover \(\overline{L}\) is a Lie ideal of \(\overline{R}\), such that \([\overline{L}, \overline{L}] \neq 0\) since \([L, L] \not\subseteq P\). Since \([L, L] \not\subseteq P\), a fortiori we get \(\overline{L}\) is not central in \(\overline{R}\). Therefore, by Theorem 3.3 in [10], it follows that \(\overline{\mathcal{F}}(\overline{R}) = 0\) that is \(d(R) \subseteq P\).

Assume now that \(d(P) \not\subseteq P\), then \(\overline{d(P)} \neq 0\) and \(\overline{d}(P)\overline{R} \neq 0\). By using similar argument as in Lemma 1.3, \(\overline{Rd(P)}\) is a non-zero right ideal of \(\overline{R}\).

Linearizing (1.1) and using (1.1), we obtain

\[
[F(x), y] + [F(y), x] \in Z(R) \quad \text{for all } x, y \in L.
\]

Now, replace \(y\) by \([rp, u]\), for \(r \in R\), \(p \in P\) and \(u \in L\) and use (1.2) to get

\[
[\mathcal{F}([rp, u]), \overline{\overline{x}}] \in Z(\overline{R}).
\]

Moreover, since \(F(r) = ar + d(r)\), for all \(r \in R\), by (1.3) it follows

\[
[\overline{d([rp, u])}, \overline{L}] \subseteq Z(\overline{R}).
\]
By the primeness of $R$ and Remark 1.7, one has that $\overline{d([rp,u])} \in Z(R)$. On the other hand, an easy computation shows that $\overline{d([rp,u])} = \overline{rd(p),u}$, which implies $[Rd(P),L] \subseteq Z(R)$. Once again by Remark 1.7, we have $Rd(P) \subseteq Z(R)$. Since $Rd(P)$ is a non-zero right ideal of $R$, it follows $[R,R] = (0)$, which contradicts with $[L,L] \neq (0)$.

The previous argument shows that, for any prime ideal $P$ of $R$, either $[L,L] \subseteq P$ or $d(R) \subseteq P$. Thus $d(R)[L,L] \subseteq P_1 = (0)$. Hence, by Lemma 1.3 and since $L \nsubseteq Z(R)$, we finally get the required conclusions:

1. $d(R)[L,R] = (0)$ and $[d(R),L] = (0)$;
2. $(a+b)[L,R] = (0)$ and $[a+b,L] = (0)$;
3. $aI = d(I) = (0)$, where $I$ denotes the ideal of $R$ generated by $[L,L]$.

\[\square\]

**Theorem 1.9.** Let $R$ be a 2-torsion free semiprime ring and $L$ be a non-central Lie ideal of $R$. Suppose $R$ admits two generalized derivations $(F,d)$ and $(G,g)$. Write $F(x) = ax + d(x)$ and $G(x) = bx + g(x)$, for some $a,b \in U$. If $F([x,y]) = [y,G(x)]$ for all $x,y \in L$, then either

1. $g(L) = (0)$;
2. $d(R)[L,R] = (0)$ and $[d(R),L] = (0)$;
3. $(a+b)[L,R] = (0)$, $[b,L] = (0)$ and $[a,L] = (0)$;
4. $(a+b)I = (0)$ and $d(I) = (0)$, where $I$ denotes the ideal of $R$ generated by $[L,L]$.

or

1. $d(L) = (0)$;
2. $g(R)[L,R] = (0)$ and $[g(R),L] = (0)$;
3. $[b,L] = (0)$ and $a[L,L] = (0)$;
4. $aI = (0)$ and $g(I) = (0)$, where $I$ denotes the ideal of $R$ generated by $[L,L]$.

or

1. $d(R)[L,R] = (0)$ and $[d(R),L] = (0)$;
2. $g(R)[L,R] = (0)$ and $[g(R),L] = (0)$;
3. $[a,L] = (0)$, $[b,L] = (0)$, $b[L,R] = a[L,R] = (0)$;
4. $d(I) = g(I) = (0)$ and $aI = bI = (0)$ (that is $F(I) = G(I) = (0)$), where $I$ denotes the ideal of $R$ generated by $[L,L]$.

**Proof.** Assume first $g(L) = (0)$, then $F([x,y]) = [y,bx]$ for all $x,y \in L$. Thus

$$a[x,y] + d([x,y]) = b[y,x]$$

(1.5)

for all $x,y \in L$, that is $(a+b)[x,y] + d([x,y]) = 0$ for all $x,y \in L$. Therefore, applying Theorem 1.5, one has

1. $d(R)[L,R] = (0)$ and $[d(R),L] = (0)$;
2. $(a+b)[L,R] = (0)$ and $[a+b,L] = (0)$;
(3) \((a+b)I = (0)\) and \(d(I) = (0)\), where \(I\) denotes the ideal of \(R\) generated by \([L,L]\).

In particular \(d([L,L]) = (0)\) and \(a[x,y] = -b[x,y]\) for all \(x, y \in L\), so that (1.5) reduces to \((by - yb)x = 0\), for all \(x, y \in L\), that is \([b, L]L = (0)\). Hence by Lemma 1.6, we have \([b, L] = (0)\) and so also \([a, L] = (0)\).

Let now \(d(L) = (0)\), then \(a[x,y] = [y, G(x)]\) for all \(x, y \in L\). In this case, for \(x = y\), we have \([G(y), y] = 0\) and by Theorem 1.8 the following hold:

1. \(g(R)[L,L] = (0)\) and \([g(R), L] = (0)\);
2. \([b, L] = (0)\), \(b[L,R] = (0)\) and \([a, L] = (0)\);
3. \(bI = (0)\) and \(g(I) = (0)\), where \(I\) denotes the ideal of \(R\) generated by \([L,L]\).

Moreover, since \([[L,L], R] \subseteq [L,L]\), we also have \(0 = a[[L,L], R] = aR[L,L]\), which implies \(aI = (0)\).

Assume finally that both \(g(L) \neq (0)\) and \(d(L) \neq (0)\). Once again for \(x = y \in L\) we have \([G(x), x] = 0\) for any \(x \in L\). Thus by Theorem 1.8, we have that all the following hold:

1. \(g(R)[L,L] = (0)\) and \([g(R), L] = (0)\);
2. \([b, L] = (0)\) and \(b[L,R] = (0)\);
3. \(bI = (0)\) and \(g(I) = (0)\), where \(I\) denotes the ideal of \(R\) generated by \([L,L]\).

Hence by the main assumption it follows that \((a+b)[x, y] + d([x,y]) = 0\), for all \(x, y \in L\). Denote \(H(x) = (a-b)x + d(x)\), then \(H(u) = 0\) for all \(u \in [L,L]\).

Finally, by applying Theorem 1.5, one has

1. \(d(R)[L,L] = (0)\) and \([d(R), L] = (0)\);
2. \((a+b)[L,R] = (0)\) and \([a, L] = (0)\);
3. \((a+b)I = (0)\) and \(d(I) = (0)\), where \(I\) denotes the ideal of \(R\) generated by \([L,L]\).

Note that, since both \(bI = (0)\) and \((a+b)I = (0)\), we are done. \(\square\)

We conclude our paper with some applications to generalized derivations acting on ideals of semiprime rings:

**Theorem 1.10.** Let \(R\) be a \(2\)-torsion free semiprime ring and \(I\) be a non-central ideal of \(R\). Suppose \(R\) admits a generalized derivation \((F,d)\), defined as \(F(x) = ax + d(x)\), for all \(x \in R\) and fixed element \(a \in R\). If \([F(x), x] = 0\) for all \(x \in I\), then either \(d(I) = 0\) or \(R\) contains a non-zero central ideal.

**Proof.** By Theorem 1.8, we have that if \(d(I) \neq (0)\) then \([d(R), I] = (0)\). Hence, by applying Main Theorem in [13], it follows that \(R\) must contain a non-zero central ideal. \(\square\)

**Corollary 1.11.** Let \(R\) be a \(2\)-torsion free semiprime ring \(F\) a generalized derivation of \(R\). If \([F(x), x] = 0\) for all \(x \in R\), then either \(R\) contains a...
non-zero central ideal or there exists \( \lambda \in Z(R) \) such that \( F(x) = \lambda x \), for all \( x \in R \).

**Theorem 1.12.** Let \( R \) be a 2-torsion free semiprime ring and \( I \) be a non-central ideal of \( R \). Suppose \( R \) admits two generalized derivations \((F,d)\) and \((G,g)\). Write \( F(x) = ax + d(x) \) and \( G(x) = bx + g(x) \), for some \( a,b \in U \). If \( F([x,y]) = [y,G(x)] \) for all \( x,y \in L \), then either \( d(I) = g(I) = (0) \) or \( R \) contains a non-zero central ideal.

**Proof.** Assume either \( d(I) \neq 0 \) or \( g(I) \neq 0 \). Thus, by Theorem 1.9 respectively we have that either \([d(R),I] = (0)\) or \([g(R),I] = (0)\). In any case, again by [13], \( R \) must contain some non-zero central ideals. \( \square \)

**Corollary 1.13.** Let \( R \) be a 2-torsion free semiprime ring and \( F,G \) two generalized derivations of \( R \). If \( F([x,y]) = [y,G(x)] \) for all \( x,y \in R \), then either \( R \) contains a non-zero central ideal or there exist \( \lambda \in Z(R) \) such that \( F(x) = G(x) = \lambda x \), for all \( x \in R \).

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**References**