Lie Ideals and Generalized Derivations in Semiprime Rings

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\textbf{Abstract.} Let $R$ be a 2-torsion free ring and $L$ a Lie ideal of $R$. An additive mapping $F : R \to R$ is called a generalized derivation on $R$ if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$. In the present paper we describe the action of generalized derivations satisfying several conditions on Lie ideals of semiprime rings.

\textbf{Keywords:} Derivations, Generalized derivations, Semiprime rings, Lie ideals.

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1. Introduction

Let $R$ be an associative ring with center $Z(R)$. A ring $R$ is said to be $n$-torsion free if $nx = 0$ implies $x = 0$ for all $x \in R$. For any $x, y \in R$, the symbol $[x, y]$ will represent the commutator $xy - yx$. Recall that a ring $R$ is prime if $aRb = 0$ implies $a = 0$ or $b = 0$ and $R$ is semiprime if $aRa = 0$ yields $a = 0$. An additive mapping $d : R \to R$ is said to be a derivation of $R$ if

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$d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. In particular, for a fixed $a \in R$ the mapping $I_a : R \to R$ given by $I_a(x) = [x, a]$ is a derivation which is called an inner derivation determined by $a$. In 1991 Bresar [5] introduced the concept of generalized derivation: more precisely an additive mapping $F : R \to R$ is said to be a generalized derivation with associated derivation $d$ if $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. For the sake of convenience, we shall denote by $(F, d)$ a generalized derivation $F$ with associated derivation $d$. A mapping $f : R \to R$ is known to be centralizing on $R$ if $[f(x), x] \in Z(R)$ for all $x \in R$. In particular, if $[f(x), x] = 0$ for all $x \in R$, then $f$ is said to be commuting on $R$. We recall that an additive group $L$ of $R$ is said to be a Lie ideal of $R$ if $[L, R] \subseteq L$.

A well known result of Posner [18] states that a prime ring admitting a nonzero centralizing derivation must be commutative. This theorem indicates that the global structure of a ring $R$ is often tightly connected to the behaviour of additive mappings defined on $R$. Following this line of investigation, several authors studied derivations and generalized derivations acting on appropriate subsets of the ring.

For instance in [19] Quadri et al. prove that if $R$ is a prime ring with a non-zero ideal $I$ and $F$ is a generalized derivation of $R$ such that $F([x, y]) = [x, y]$, for all $x, y \in I$, then $R$ is commutative (Theorem 2.1). Later in [7] Dhara extends all results contained in [19] to semiprime rings.

Further in [10] Gölbasi and Koç investigate the properties of a prime ring $R$ with a generalized derivation $(F, d)$ acting on a Lie ideal $L$ of $R$. They prove that if $[F(u), u] \in Z(R)$, for all $u \in L$, then either $d = 0$ or $L \subseteq Z(R)$ (Theorem 3.3). Moreover if $F([u, v]) = [u, v]$, for all $u, v \in L$, then either $d = 0$ of $L \subseteq Z(R)$ (Theorem 3.6).

In this note we will consider a similar situation and extend the cited results to the case of semiprime rings with a generalized derivation $(F, d)$ acting on a Lie ideal. More precisley we prove the following:

**Theorem 1.** Let $R$ be a 2-torsion free semiprime ring and $L$ be a non-central Lie ideal of $R$. Suppose that $(F, d)$ is a generalized derivation of $R$ such that $F[x, y] \in Z(R)$, for all $x, y \in L$. If $d(L) \neq (0)$, then all the following hold simultaneously:

1. $d(R)[L, R] = (0)$ and $[d(R), L] = (0)$;
2. $a[L, R] = (0)$ and $[a, L] = (0)$;
3. $aI = (0)$ and $d(I) = (0)$ (that is $F(I) = (0)$), where $I$ denotes the ideal of $R$ generated by $[L, L]$.

**Theorem 2.** Let $R$ be a 2-torsion free semiprime ring and $L$ be a non-central Lie ideal of $R$. Suppose $R$ admits a generalized derivation $(F, d)$, defined as $F(x) = ax + d(x)$, for all $x \in R$ and fixed element $a \in R$. If $[F(x), x] \in Z(R)$ for all $x \in L$ and $d(L) \neq (0)$, then all the following hold simultaneously:
(1) \(d(R)[L,R] = (0)\) and \([d(R),L] = (0)\);
(2) \([a,L] = a[L,R] = (0)\);
(3) \(aI = d(I) = (0)\) (that is \(F(I) = (0)\)), where \(I\) denotes the ideal of \(R\) generated by \([L,L]\).

**Theorem 3.** Let \(R\) be a 2-torsion free semiprime ring and \(L\) be a non-central Lie ideal of \(R\). Suppose \(R\) admits two generalized derivations \((F,d)\) and \((G,g)\).

Write \(F(x) = ax + d(x)\) and \(G(x) = bx + g(x)\), for some \(a,b \in U\). If \(F([x,y]) = [y,G(x)]\) for all \(x,y \in L\), then either

(1) \(g(L) = (0)\);
(2) \(d(R)[L,R] = (0)\) and \([d(R),L] = (0)\);
(3) \((a+b)[L,R] = (0), [b,L] = (0)\) and \([a,L] = (0)\);
(4) \((a+b)I = (0)\) and \(d(I) = (0)\), where \(I\) denotes the ideal of \(R\) generated by \([L,L]\).

or

(1) \(d(L) = (0)\);
(2) \(g(R)[L,R] = (0)\) and \([g(R),L] = (0)\);
(3) \([b,L] = (0)\) and \(a[L,L] = (0)\);
(4) \(aI = (0)\) and \(g(I) = (0)\), where \(I\) denotes the ideal of \(R\) generated by \([L,L]\).

or

(1) \(d(R)[L,R] = (0)\) and \([d(R),L] = (0)\);
(2) \(g(R)[L,R] = (0)\) and \([g(R),L] = (0)\);
(3) \([a,L] = (0), [b,L] = (0), b[L,R] = a[L,R] = (0)\);
(4) \(d(I) = g(I) = (0)\) and \(aI = bI = (0)\) (that is \(F(I) = G(I) = (0)\)), where \(I\) denotes the ideal of \(R\) generated by \([L,L]\).

In all that follows let \(R\) be a non-commutative semiprime ring, \(L\) a non-central Lie ideal of \(R\), \(U\) the right Utumi quotient ring of \(R\). We refer the reader to [3] for the definition and the related properties of \(U\).

We begin with the following:

**Fact 1.1.** Let \(R\) be a semiprime ring. Then every generalized derivation \(F\) of \(R\) is uniquely extended to its right Utumi quotient ring \(U\) and assumes the form \(F(x) = ax + d(x)\), where \(a \in U\) and \(d\) is the derivation of \(U\) associated with \(F\) (see Theorem 4 in [17]).

**Lemma 1.2.** Let \(R\) be a prime ring of characteristic different from 2 and \(L\) be a Lie ideal of \(R\). Suppose \(R\) admits a nonzero generalized derivation \((F,d)\) such that \(F(x)[x,y] = 0\) (or \([x,y]F(x) = 0\) for all \(x,y \in L\), then \(L \subseteq Z(R)\).

**Proof.** Suppose by contradiction that \(L\) is not central in \(R\). By [11] (pages 4-5) there exists a non-central ideal \(I\) of \(R\) such that \(0 \neq [I,R] \subseteq L\). By our
assumption it follows that $F(x)[x, y] = 0$ (or $[x, y]F(x) = 0$) for all $x, y \in [I, R]$. Since $I$ and $R$ satisfy the same differential identities (see the main result in [16]), we also have that $F(x)[x, y] = 0$ (or $[x, y]F(x) = 0$) for all $x, y \in [R, R]$. Let $y_0 \in [R, R]$ be such that $y_0 \notin Z(R)$ and denote by $\delta : R \rightarrow R$ the non-zero inner derivation of $R$ induced by the element $y_0$. Therefore $F(x)\delta(x) = 0$ (or $\delta(x)F(x) = 0$) for all $x \in [R, R]$. In light of [6], since $\delta \neq 0$ and $[R, R]$ is not central in $R$, one has the contradiction that $F = 0$. □

**Lemma 1.3.** Let $R$ be a 2-torsion free semiprime ring and $L$ be a non-central Lie ideal of $R$. Suppose $R$ admits a nonzero generalized derivation $(F, d)$, defined as $F(x) = ax + d(x)$, for all $x \in R$ and fixed element $a \in R$. If $F(x)[x, y] = 0$ (or $[x, y]F(x) = 0$) for all $x, y \in L$, then all the following hold simultaneously:

1. $d(R)[L, R] = (0)$ and $[d(R), L] = (0)$;
2. $a[L, R] = (0)$ and $[a, L] = (0)$;
3. $aI = (0)$ and $d(I) = (0)$ (that is $F(I) = (0)$), where $I$ denotes the ideal of $R$ generated by $[L, L]$.

**Proof.** Let $P$ be a prime ideal if $R$ such that $[L, L] \nsubseteq P$.

Assume first that $d(P) \subseteq P$. Then $F$ induces a canonical generalized derivation $\overline{F}$ on $R = \frac{R}{P}$. Therefore $\overline{F}(\overline{\pi}) = 0$ for all $\overline{\pi}, \overline{\nu} \in \overline{L}$. Moreover $\overline{L}$ is a Lie ideal of $\overline{R}$, such that $[\overline{L}, \overline{L}] \neq 0$ since $[L, L] \nsubseteq P$. By Lemma 1.2 it follows that $\overline{F}(\overline{R}) = 0$ that is $aR \subseteq P$, $d(R) \subseteq P$ and $F(R) \subseteq P$.

Assume now that $d(P) \nsubseteq P$, then $\overline{d(P)} \neq 0$ and $\overline{d(P)}R \neq 0$. Moreover note that, for any $p \in P$ and $r, s \in R$, $d(pr)s = d(p)rs + pd(r)s$ implies that $d(P)R \subseteq d(\overline{P})R + P$, in particular $\overline{d(P)}R$ is a non-zero right ideal of $\overline{R}$.

Starting from our main assumption and linearizing we have that $F(x)[x, y] + F(z)[x, y] = 0$, for all $x, y, z \in L$. For any $p \in P, r \in R, u \in L$, replace $x$ by $[pr, u]$. By computation it follows $[\overline{\pi}, \overline{\nu}][\overline{\pi}, \overline{\nu}] = 0$, for all $\overline{\pi} \in \overline{d(P)}R$ and $\overline{\pi}, \overline{\nu} \in \overline{L}$. By using the same argument of Lemma 1.2, since $\overline{L}$ is not central in $\overline{R}$, one has that $\overline{d(P)}R$ is a central right ideal of $\overline{R}$, which implies that $\overline{R}$ is commutative, a contradiction.

Therefore, for any prime ideal $P$ of $R$, either $aR \subseteq P$, $d(R) \subseteq P$ and $F(R) \subseteq P$ or $[L, L] \subseteq P$. In this last case, by applying Theorem 3 in [15] in the prime ring $\overline{R}$, since $\text{char}(\overline{R}) \neq 2$ and $[L, L] = 0$, we have that $\overline{L}$ is central in $\overline{R}$, which means $[L, R] \subseteq P$.

Hence in any case it follows that $d(R)[L, R] = (0)$, $a[R, L] = (0)$ and $[d(R), L] = (0)$.

By $a[R, L] = (0)$ we get $aR[R, L] = (0)$ and so both $aLR[a, L] = (0)$ and $LaR[a, L] = (0)$, that is $[a, L]R[a, L] = (0)$. By the semiprimeness of $R$ it follows $[a, L] = (0)$.

Moreover, if $I = R[R, L]R$ denotes the ideal of $R$ generated by $[L, L]$, it follows that $aI = (0)$ and $d(I) = (0)$, that is $F(I) = (0)$. □
Corollary 1.4. Let $R$ be a 2-torsion free semiprime ring and $L$ be a non-central Lie ideal of $R$. Suppose $a \in R$ be such that $ax[xy] = 0$ for all $x, y \in L$, then $a[L, R] = (0)$, $[a, L] = (0)$ and $aI = (0)$, where $I$ denotes the ideal of $R$ generated by $[L, L]$.

Theorem 1.5. Let $R$ be a 2-torsion free semiprime ring and $L$ be a non-central Lie ideal of $R$. Suppose that $(F, d)$ is a generalized derivation of $R$ such that $F[xy] \in Z(R)$, for all $x, y \in L$. If $d(L) \neq (0)$, then all the following hold simultaneously:

1. $d(R)[L, R] = (0)$ and $[d(R), L] = (0)$;
2. $a[L, R] = (0)$ and $[a, L] = (0)$;
3. $aI = (0)$ and $d(I) = (0)$ (that is $F(I) = (0)$), where $I$ denotes the ideal of $R$ generated by $[L, L]$.

Proof. Assume first that $R$ is prime and denote $V = [L, L]$. Hence we have $F(V) \subseteq Z(R)$. As a consequence of Lemma 2 in [9] we conclude that either $F = 0$ or $V \subseteq Z(R)$. In the first case we have the contradiction $d = 0$, and in the other case one has $L \subseteq Z(R)$ (see Lemma 2 in [12]), a contradiction again. Let now $P$ be a prime ideal of $R$ such that $[L, L] \not\subseteq P$.

Assume first that $d(P) \subseteq P$. Then $F$ induces a canonical generalized derivation $\overline{F}$ on $\overline{R} = \frac{R}{P}$. Therefore $\overline{F}([\pi, \eta]) \in Z(\overline{R})$ for all $\pi, \eta \in L$. Moreover $\overline{L}$ is a Lie ideal of $\overline{R}$, such that $[\overline{L}, \overline{L}] \neq 0$ since $[L, L] \not\subseteq P$. By previous argument it follows that $\overline{F}(\overline{R}) = \overline{0}$ that $d(R) \subseteq P$ and $F(R) \subseteq P$.

Assume now that $d(P) \not\subseteq P$, then $\overline{d}(P) \neq \overline{0}$ and $\overline{d}(P)R \neq \overline{0}$. We remark again that $\overline{d}(P)R$ is a non-zero right ideal of $\overline{R}$.

Starting from our main assumption and linearizing we have that

\[ F(x)y + F(x)z + xd(y) + xd(z) - F(y)x - F(z)x - yd(x) - zd(x) \in Z(R), \forall x, y, z \in L. \]

For any $p, p', p'' \in P, r, s \in R, u, v \in L$, replace $y$ by $[pr, u]$ and $z$ by $[[p's, v], p'']$. By computation it follows

\[ \pi[\overline{\pi}] - [\overline{\pi}, \overline{\pi}] \pi \in Z(\overline{R}) \]

that is

\[ [\pi, [\pi, \pi]] \in Z(\overline{R}) \]

for all $\overline{\pi} \in \overline{d}(P)R$ and $\pi, \pi \in \overline{L}$. As above denote $\overline{V} = [\overline{L}, \overline{L}]$, which is a Lie ideal for $\overline{R}$, and $\delta$ is the derivation of $\overline{R}$ induced by $\overline{\pi}$. Hence we have $\delta(\overline{V}) \subseteq Z(\overline{R})$. Again as a consequence of Lemma 2 in [9] it follows that either $\delta = 0$ or $\overline{V} \subseteq Z(\overline{R})$. Since $\overline{R}$ is not commutative, then there exists some $\overline{\pi} \in \overline{R}$ which is not central. Thus $\overline{V} \subseteq Z(\overline{R})$, and $\overline{L} \subseteq Z(\overline{R})$ follows from Lemma 2 in [12].

Therefore, for any prime ideal $P$ of $R$, either $d(R) \subseteq P$ and $F(R) \subseteq P$ or $[L, L] \subseteq P$. In this last case, by applying Theorem 3 in [15] in the prime ring $\overline{R}$, since $char(\overline{R}) \neq 2$ and $[L, L] = \overline{0}$, we conclude that $\overline{L}$ is central in $\overline{R}$, which
means \([L, R] \subseteq P\).

Hence in any case it follows that \(d(R)[L, R] = (0), a[R, L] = (0)\) and \([d(R), L] = (0)\). Finally we obtain the required conclusions by following the same argument as in Lemma 1.3. \(\square\)

In the sequel we will use the following known result:

**Lemma 1.6.** Let \(R\) be a 2-torsion free semiprime ring, \(L\) a Lie ideal of \(R\) such that \(L \not\subseteq Z(R)\). Let \(a \in L\) be such that \(aLa = 0\), then \(a = 0\).

**Remark 1.7.** If \(R\) is a prime ring of characteristic different from 2, \(a \in R\) and \(L\) is a non-central Lie ideal of \(R\) such that \([a, L] \subseteq Z(R)\), then \(a \in Z(R)\).

**Proof.** Denote by \(\delta : R \to R\) the inner derivation of \(R\) induced by the element \(a \in R\). Since \([[a, x], r] = 0\) for all \(x \in L\) and \(r \in R\), a fortiori we have \([a, x]_2 = 0\), that is \([\delta(x), x] = 0\) for all \(x \in L\). Thus, by [14] it follows \(\delta = 0\), that is \(a \in Z(R)\). \(\square\)

**Theorem 1.8.** Let \(R\) be a 2-torsion free semiprime ring and \(L\) be a non-central Lie ideal of \(R\). Suppose \(R\) admits a generalized derivation \((F, d)\), defined as \(F(x) = ax + d(x)\), for all \(x \in R\) and fixed element \(a \in R\). If

\[
[F(x), x] \in Z(R)\quad \text{for all } x \in L.\tag{1.1}
\]

and \(d(L) \neq (0)\), then all the following hold simultaneously:

1. \(d(R)[L, R] = (0)\) and \([d(R), L] = (0)\);
2. \([a, L] = a[L, R] = (0)\);
3. \(aI = d(I) = (0)\) (that is \(F(I) = (0)\)), where \(I\) denotes the ideal of \(R\) generated by \([L, L]\).

**Proof.** Let \(P\) be a prime ideal of \(R\) such that \([L, L] \not\subseteq P\).

Assume first that \(d(P) \subseteq P\). Then \(F\) induces a canonical generalized derivation \(\mathcal{F}\) on \(\overline{R} = \overline{\mathcal{F}}\). Therefore \([\mathcal{F}(\overline{x}), \overline{x}] \in Z(\overline{R})\) for all \(\overline{x} \in \overline{L}\). Moreover \(\overline{L}\) is a Lie ideal of \(\overline{R}\), such that \(\overline{[L, L]} \neq 0\) since \([L, L] \not\subseteq P\). Since \([L, L] \not\subseteq P\), a fortiori we get \(\overline{L}\) is not central in \(\overline{R}\). Therefore, by Theorem 3.3 in [10], it follows that \(\overline{d(\overline{R})} = 0\) that is \(d(R) \subseteq P\).

Assume now that \(d(P) \not\subseteq P\), then \(\overline{d(P)} \neq 0\) and \(\overline{d(P)\overline{R}} \neq 0\). By using similar argument as in Lemma 1.3, \(\overline{Rd(P)}\) is a non-zero right ideal of \(\overline{R}\).

Linearizing (1.1) and using (1.1), we obtain

\[
[F(x), y] + [F(y), x] \in Z(R)\quad \text{for all } x, y \in L.\tag{1.2}
\]

Now, replace \(y\) by \([rp, u]\), for \(r \in R\), \(p \in P\) and \(u \in L\) and use (1.2) to get

\[
[\mathcal{F}([rp, u]), \overline{x}] \in Z(\overline{R}).\tag{1.3}
\]

Moreover, since \(F(r) = ar + d(r)\), for all \(r \in R\), by (1.3) it follows

\[
[\overline{d([rp, u])}, \overline{L}] \subseteq Z(\overline{R}).\tag{1.4}
\]
By the primeness of $R$ and Remark 1.7, one has that $d([rp,u]) \in Z(R)$. On the other hand, an easy computation shows that $d([rp,u]) = [rd(p),\pi]$, which implies $[Rd(P),L] \subseteq Z(R)$. Once again by Remark 1.7, we have $Rd(P) \subseteq Z(R)$. Since $Rd(P)$ is a non-zero right ideal of $R$, it follows $[Rd(P),L] = (0)$, which contradicts with $[L,L] \neq (0)$.

The previous argument shows that, for any prime ideal $P$ of $R$, either $[L,L] \subseteq P$ or $d(R) \subseteq P$. Thus $d(R)[L,L] \subseteq \cap P_i = (0)$. Hence, by Lemma 1.3 and since $L \nsubseteq Z(R)$, we finally get the required conclusions:

1. $d(R)[L,R] = (0)$ and $[d(R),L] = (0)$;
2. $(a+b)[L,R] = (0)$ and $[a+b,L] = (0)$;
3. $aI = d(I) = (0)$, where $I$ denotes the ideal of $R$ generated by $[L,L]$.

□

**Theorem 1.9.** Let $R$ be a 2-torsion free semiprime ring and $L$ be a non-central Lie ideal of $R$. Suppose $R$ admits two generalized derivations $(F,d)$ and $(G,g)$. Write $F(x) = ax + d(x)$ and $G(x) = bx + g(x)$, for some $a,b \in U$. If $F([x,y]) = [y,G(x)]$ for all $x,y \in L$, then either

1. $g(L) = (0)$;
2. $d(R)[L,R] = (0)$ and $[d(R),L] = (0)$;
3. $(a+b)[L,R] = (0)$, $[b,L] = (0)$ and $[a,L] = (0)$;
4. $(a+b)I = d(I) = (0)$, where $I$ denotes the ideal of $R$ generated by $[L,L]$.

or

1. $d(L) = (0)$;
2. $g(R)[L,R] = (0)$ and $[g(R),L] = (0)$;
3. $[b,L] = (0)$ and $a[L,L] = (0)$;
4. $aI = (0)$ and $g(I) = (0)$, where $I$ denotes the ideal of $R$ generated by $[L,L]$.

or

1. $d(R)[L,R] = (0)$ and $[d(R),L] = (0)$;
2. $g(R)[L,R] = (0)$ and $[g(R),L] = (0)$;
3. $[a,L] = (0)$, $[b,L] = (0)$, $b[L,R] = a[L,R] = (0)$;
4. $d(I) = g(I) = (0)$ and $aI = bI = (0)$ (that is $F(I) = G(I) = (0)$), where $I$ denotes the ideal of $R$ generated by $[L,L]$.

**Proof.** Assume first $g(L) = (0)$, then $F([x,y]) = [y,bx]$ for all $x,y \in L$. Thus

$$a[x,y] + d([x,y]) = b[y,x]$$

(1.5)

for all $x,y \in L$, that is $(a+b)[x,y] + d([x,y]) = 0$ for all $x,y \in L$. Therefore, applying Theorem 1.5, one has

1. $d(R)[L,R] = (0)$ and $[d(R),L] = (0)$;
2. $(a+b)[L,R] = (0)$ and $[a+b,L] = (0)$;
(3) \((a+b)I = (0)\) and \(d(I) = (0)\), where \(I\) denotes the ideal of \(R\) generated by \([L,L]\).

In particular \(d([L,L]) = (0)\) and \(a[x,y] = -b[x,y]\) for all \(x, y \in L\), so that (1.5) reduces to \((by - yb)x = 0\), for all \(x, y \in L\), that is \([b,L]L = (0)\). Hence by Lemma 1.6, we have \([b,L] = (0)\) and so also \([a,L] = (0)\).

Let now \(d(L) = (0)\), then \(a[x,y] = [y,G(x)]\) for all \(x, y \in L\). In this case, for \(x = y\), we have \([G(y),y] = 0\) and by Theorem 1.8 the following hold:

1. \(g(R)[L,R] = (0)\) and \([g(R),L] = (0)\);
2. \([b,L] = (0)\), \(b[L,R] = (0)\) and \(a[L,L] = (0)\);
3. \(bI = (0)\) and \(g(I) = (0)\), where \(I\) denotes the ideal of \(R\) generated by \([L,L]\).

Moreover, since \([[L,L],R] \subseteq [L,L]\), we also have \(0 = a[[L,L],R] = aR[L,L]\), which implies \(aI = (0)\).

Assume finally that both \(g(L) \neq (0)\) and \(d(L) \neq (0)\). Once again for \(x = y \in L\) we have \([G(x),x] = 0\) for any \(x \in L\). Thus by Theorem 1.8, we have that all the following hold:

1. \(g(R)[L,R] = (0)\) and \([g(R),L] = (0)\);
2. \([b,L] = (0)\) and \(b[L,R] = (0)\);
3. \(bI = (0)\) and \(g(I) = (0)\), where \(I\) denotes the ideal of \(R\) generated by \([L,L]\).

Hence by the main assumption it follows that \((a+b)[x,y] + d([x,y]) = 0\), for all \(x, y \in L\). Denote \(H(x) = (a-b)x + d(x)\), then \(H(u) = 0\) for all \(u \in [L,L]\).

Finally, by applying Theorem 1.5, one has

1. \(d(R)[L,R] = (0)\) and \([d(R),L] = (0)\);
2. \((a+b)[L,R] = (0)\) and \([a,L] = (0)\);
3. \((a+b)I = (0)\) and \(d(I) = (0)\), where \(I\) denotes the ideal of \(R\) generated by \([L,L]\).

Note that, since both \(bI = (0)\) and \((a+b)I = (0)\), we are done. \(\Box\)

We conclude our paper with some applications to generalized derivations acting on ideals of semiprime rings:

**Theorem 1.10.** Let \(R\) be a 2-torsion free semiprime ring and \(I\) be a non-central ideal of \(R\). Suppose \(R\) admits a generalized derivation \((F,d)\), defined as \(F(x) = ax + d(x)\), for all \(x \in R\) and fixed element \(a \in R\). If \([F(x),x] = 0\) for all \(x \in I\), then either \(d(I) = 0\) or \(R\) contains a non-zero central ideal.

**Proof.** By Theorem 1.8, we have that if \(d(I) \neq (0)\) then \([d(R),I] = (0)\). Hence, by applying Main Theorem in [13], it follows that \(R\) must contain a non-zero central ideal. \(\Box\)

**Corollary 1.11.** Let \(R\) be a 2-torsion free semiprime ring \(F\) a generalized derivation of \(R\). If \([F(x),x] = 0\) for all \(x \in R\), then either \(R\) contains a
non-zero central ideal or there exists $\lambda \in Z(R)$ such that $F(x) = \lambda x$, for all $x \in R$.

**Theorem 1.12.** Let $R$ be a 2-torsion free semiprime ring and $I$ be a non-central ideal of $R$. Suppose $R$ admits two generalized derivations $(F,d)$ and $(G,g)$. Write $F(x) = ax + d(x)$ and $G(x) = bx + g(x)$, for some $a,b \in U$. If $F([x,y]) = [y,G(x)]$ for all $x,y \in L$, then either $d(I) = g(I) = (0)$ or $R$ contains a non-zero central ideal.

**Proof.** Assume either $d(I) \neq 0$ or $g(I) \neq 0$. Thus, by Theorem 1.9 respectively we have that either $[d(R),I] = (0)$ or $[g(R),I] = (0)$. In any case, again by [13], $R$ must contain some non-zero central ideals. \hfill $\square$

**Corollary 1.13.** Let $R$ be a 2-torsion free semiprime ring and $F,G$ two generalized derivations of $R$. If $F([x,y]) = [y,G(x)]$ for all $x,y \in R$, then either $R$ contains a non-zero central ideal or there exist $\lambda \in Z(R)$ such that $F(x) = G(x) = \lambda x$, for all $x \in R$.

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**References**