Lie Ideals and Generalized Derivations in Semiprime Rings

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Abstract. Let $R$ be a 2-torsion free ring and $L$ a Lie ideal of $R$. An additive mapping $F: R \to R$ is called a generalized derivation on $R$ if there exists a derivation $d: R \to R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$. In the present paper we describe the action of generalized derivations satisfying several conditions on Lie ideals of semiprime rings.

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1. Introduction

Let $R$ be an associative ring with center $Z(R)$. A ring $R$ is said to be $n$-torsion free if $nx = 0$ implies $x = 0$ for all $x \in R$. For any $x, y \in R$, the symbol $[x, y]$ will represent the commutator $xy - yx$. Recall that a ring $R$ is prime if $aRb = 0$ implies $a = 0$ or $b = 0$ and $R$ is semiprime if $aRa = 0$ yields $a = 0$. An additive mapping $d: R \to R$ is said to be a derivation of $R$ if

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Let \( d(xy) = d(x)y + xd(y) \) for all \( x, y \in R \). In particular, for a fixed \( a \in R \) the mapping \( I_a : R \to R \) given by \( I_a(x) = [x, a] \) is a derivation which is called an inner derivation determined by \( a \). In 1991 Bresar [5] introduced the concept of generalized derivation: more precisely an additive mapping \( F : R \to R \) is said to be a generalized derivation with associated derivation \( d \) if \( F(xy) = F(x)y + xd(y) \) for all \( x, y \in R \). For the sake of convenience, we shall denote by \((F, d)\) a generalized derivation \( F \) with associated derivation \( d \). A mapping \( f : R \to R \) is known to be centralizing on \( R \) if \( [f(x), x] \in Z(R) \) for all \( x \in R \). In particular, if \( [f(x), x] = 0 \) for all \( x \in R \), then \( f \) is said to be commuting on \( R \). We recall that an additive group \( L \) of \( R \) is said to be a Lie ideal of \( R \) if \([L, R] \subseteq L\).

A well known result of Posner [18] states that a prime ring admitting a nonzero centralizing derivation must be commutative. This theorem indicates that the global structure of a ring \( R \) is often tightly connected to the behaviour of additive mappings defined on \( R \). Following this line of investigation, several authors studied derivations and generalized derivations acting on appropriate subsets of the ring.

For instance in [19] Quadri et al. prove that if \( R \) is a prime ring with a non-zero ideal \( I \) and \( F \) is a generalized derivation of \( R \) such that \( F([x, y]) = [x, y] \), for all \( x, y \in I \), then \( R \) is commutative (Theorem 2.1). Later in [7] Dhara extends all results contained in [19] to semiprime rings.

Further in [10] Gölbasi and Koç investigate the properties of a prime ring \( R \) with a generalized derivation \((F, d)\) acting on a Lie ideal \( L \) of \( R \). They prove that if \([F(u), u] \in Z(R)\), for all \( u \in L \), then either \( d = 0 \) or \( L \subseteq Z(R) \) (Theorem 3.3). Moreover if \( F([u, v]) = [u, v] \), for all \( u, v \in L \), then either \( d = 0 \) of \( L \subseteq Z(R) \) (Theorem 3.6).

In this note we will consider a similar situation and extend the cited results to the case of semiprime rings with a generalized derivation \((F, d)\) acting on a Lie ideal. More precisley we prove the following:

**Theorem 1.** Let \( R \) be a 2-torsion free semiprime ring and \( L \) be a non-central Lie ideal of \( R \). Suppose that \((F, d)\) is a generalized derivation of \( R \) such that \( F([x, y]) \in Z(R) \), for all \( x, y \in L \). If \( d(L) \neq (0) \), then all the following hold simultaneously:

1. \( d(R)[L, R] = (0) \) and \( d(R), L = (0) \);
2. \( a[L, R] = (0) \) and \( [a, L] = (0) \);
3. \( aI = (0) \) and \( d(I) = (0) \) (that is \( F(I) = (0) \)), where \( I \) denotes the ideal of \( R \) generated by \([L, L]\).

**Theorem 2.** Let \( R \) be a 2-torsion free semiprime ring and \( L \) be a non-central Lie ideal of \( R \). Suppose \( R \) admits a generalized derivation \((F, d)\), defined as \( F(x) = ax + d(x) \), for all \( x \in R \) and fixed element \( a \in R \). If \([F(x), x] \in Z(R) \) for all \( x \in L \) and \( d(L) \neq (0) \), then all the following hold simultaneously:
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(1) \(d(R)[L, R] = (0)\) and \([d(R), L] = (0)\);
(2) \([a, L] = a[L, R] = (0)\);
(3) \(aI = d(I) = (0)\) (that is \(F(I) = (0)\)), where \(I\) denotes the ideal of \(R\) generated by \([L, L]\).

**Theorem 3.** Let \(R\) be a 2-torsion free semiprime ring and \(L\) be a non-central Lie ideal of \(R\). Suppose \(R\) admits two generalized derivations \((F, d)\) and \((G, g)\). Write \(F(x) = ax + d(x)\) and \(G(x) = bx + g(x)\), for some \(a, b \in U\). If \(F([x, y]) = [y, G(x)]\) for all \(x, y \in L\), then either

1. \(g(L) = (0)\);
2. \([d(R)[L, R] = (0)\) and \([d(R), L] = (0)\);
3. \([a + b][L, R] = (0)\), \([b, L] = (0)\) and \([a, L] = (0)\);
4. \((a+b)I = (0)\) and \(d(I) = (0)\), where \(I\) denotes the ideal of \(R\) generated by \([L, L]\).

or

1. \(d(L) = (0)\);
2. \([g(R)[L, R] = (0)\) and \([g(R), L] = (0)\);
3. \([b, L] = (0)\) and \([a, L] = (0)\);
4. \(aI = (0)\) and \(g(I) = (0)\), where \(I\) denotes the ideal of \(R\) generated by \([L, L]\).

or

1. \([d(R)[L, R] = (0)\) and \([d(R), L] = (0)\);
2. \([g(R)[L, R] = (0)\) and \([g(R), L] = (0)\);
3. \([a, L] = (0)\), \([b, L] = (0)\), \([b[L, R] = a[L, R] = (0)\);
4. \(d(I) = g(I) = (0)\) and \(aI = bI = (0)\) (that is \(F(I) = G(I) = (0)\)), where \(I\) denotes the ideal of \(R\) generated by \([L, L]\).

In all that follows let \(R\) be a non-commutative semiprime ring, \(L\) a non-central Lie ideal of \(R\), \(U\) the right Utumi quotient ring of \(R\). We refer the reader to [3] for the definition and the related properties of \(U\).

We begin with the following:

**Fact 1.1.** Let \(R\) be a semiprime ring. Then every generalized derivation \(F\) of \(R\) is uniquely extended to its right Utumi quotient ring \(U\) and assumes the form \(F(x) = ax + d(x)\), where \(a \in U\) and \(d\) is the derivation of \(U\) associated with \(F\) (see Theorem 4 in [17]).

**Lemma 1.2.** Let \(R\) be a prime ring of characteristic different from 2 and \(L\) be a Lie ideal of \(R\). Suppose \(R\) admits a nonzero generalized derivation \((F, d)\) such that \(F(x)[x, y] = 0\) (or \([x, y]F(x) = 0\) for all \(x, y \in L\), then \(L \subseteq Z(R)\).

**Proof.** Suppose by contradiction that \(L\) is not central in \(R\). By [11] (pages 4-5) there exists a non-central ideal \(I\) of \(R\) such that \(0 \neq [I, R] \subseteq L\). By our
assumption it follows that \( F(x)[x, y] = 0 \) (or \([x, y]F(x) = 0\)) for all \( x, y \in [I, R]\).

Since \( I \) and \( R \) satisfy the same differential identities (see the main result in [16]),
we also have that \( F(x)[x, y] = 0 \) (or \([x, y]F(x) = 0\)) for all \( x, y \in [R, R]\).

Let \( y_0 \in [R, R] \) be such that \( y_0 \notin Z(R) \) and denote by \( \delta : R \to R \) the non-zero inner derivation of \( R \) induced by the element \( y_0 \). Therefore \( F(x)\delta(x) = 0 \) (or \( \delta(x)F(x) = 0\)) for all \( x \in [R, R] \). In light of [6], since \( \delta \neq 0 \) and \([R, R]\) is not central in \( R \), one has the contradiction that \( F = 0 \).

\[ \text{Lemma 1.3.} \quad \text{Let} \ R \text{ be a 2-torsion free semiprime ring and} \ L \text{ be a non-central Lie ideal of} \ R. \text{ Suppose} \ R \text{ admits a nonzero generalized derivation} \ (F, d), \text{ defined as} \ F(x) = ax + d(x), \text{ for all} \ x \in R \text{ and fixed element} \ a \in R. \text{ If} \ F(x)[x, y] = 0 \ (\text{or} \ [x, y]F(x) = 0) \text{ for all} \ x, y \in L, \text{ then all the following hold simultaneously:} \]

\[
\begin{align*}
(1) & \quad d(R)[L, R] = (0) \text{ and} \ [d(R), L] = (0); \\
(2) & \quad a[L, R] = (0) \text{ and} \ [a, L] = (0); \\
(3) & \quad aI = (0) \text{ and} \ d(I) = (0) \text{ (that is} \ F(I) = (0)) \text{, where} \ I \text{ denotes the ideal of} \ R \text{ generated by} \ [L, L].
\end{align*}
\]

\[ \text{Proof.} \quad \text{Let} \ P \text{ be a prime ideal if} \ R \text{ such that} \ [L, L] \notin P. \]

Assume first that \( d(P) \subseteq P \). Then \( F \) induces a canonical generalized derivation \( \overline{F} \) on \( \overline{R} = \frac{R}{P} \). Therefore \( \overline{F}([\pi, \gamma]) = 0 \) for all \( \pi, \gamma \in \overline{L} \). Moreover \( \overline{L} \) is a Lie ideal of \( \overline{R} \), such that \([\overline{L}, \overline{L}] \neq 0 \) since \([L, L] \notin P \). By Lemma 1.2 it follows that \( \overline{F}(\overline{R}) = 0 \) that is \( aR \subseteq P \), \( d(R) \subseteq P \) and \( F(R) \subseteq P \).

Assume now that \( d(P) \not\subseteq P \), then \( d(P) \neq 0 \) and \( d(P)R \neq 0 \). Moreover note that, for any \( p \in P \) and \( r, s \in R \), \( d(pr)s = d(p)r + pd(r)s \) implies that \( d(P)R \subseteq d(PR)R + P \), in particular \( d(P)R \) is a non-zero right ideal of \( \overline{R} \).

Starting from our main assumption and linearizing we have that \( F(x)[z, y] + F(z)[x, y] = 0 \), for all \( x, y, z \in L \). For any \( p \in P, r \in R, u \in L \), replace \( x \) by \( [pr, u] \). By computation it follows \([\pi, \overline{u}][\pi, \gamma] = 0 \), for all \( \pi \in d(P) \overline{R} \) and \( \pi, \overline{u}, \gamma \in \overline{L} \). By using the same argument of Lemma 1.2, since \( \overline{L} \) is not central in \( \overline{R} \), one has that \( d(P) \overline{R} \) is a central right ideal of \( \overline{R} \), which implies that \( \overline{R} \) is commutative, a contradiction.

Therefore, for any prime ideal \( P \) of \( R \), either \( aR \subseteq P \), \( d(R) \subseteq P \) and \( F(R) \subseteq P \) or \([L, L] \subseteq P \). In this last case, by applying Theorem 3 in [15] in the prime ring \( \overline{R} \), since \( \text{char}(\overline{R}) \neq 2 \) and \([\overline{L}, \overline{L}] = 0 \), we have that \( \overline{L} \) is central in \( \overline{R} \), which means \([L, R] \subseteq P \).

Hence in any case it follows that \( d(R)[L, R] = (0) \), \( a[R, L] = (0) \) and \([d(R), L] = (0) \).

By \( a[R, L] = (0) \) we get \( aR[R, L] = (0) \) and so both \( aLR[a, L] = (0) \) and \( LaR[a, L] = (0) \), that is \([a, L]R[a, L] = (0) \). By the semiprimeness of \( R \) it follows \([a, L] = (0) \).

Moreover, if \( I = R[L, L]R \) denotes the ideal of \( R \) generated by \([L, L] \), it follows that \( aI = (0) \) and \( d(I) = (0) \), that is \( F(I) = (0) \). \( \square \)
Corollary 1.4. Let $R$ be a 2-torsion free semiprime ring and $L$ be a non-central Lie ideal of $R$. Suppose $a \in R$ be such that $ax[x, y] = 0$ for all $x, y \in L$, then $a[L, R] = (0)$, $[a, L] = (0)$ and $aI = (0)$, where $I$ denotes the ideal of $R$ generated by $[L, L]$.

Theorem 1.5. Let $R$ be a 2-torsion free semiprime ring and $L$ be a non-central Lie ideal of $R$. Suppose that $(F, d)$ is a generalized derivation of $R$ such that $F[x, y] \in Z(R)$, for all $x, y \in L$. If $d(L) \neq (0)$, then all the following hold simultaneously:

1. $d(R)[L, R] = (0)$ and $[d(R), L] = (0)$;
2. $a[L, R] = (0)$ and $[a, L] = (0)$;
3. $aI = (0)$ and $d(I) = (0)$ (that is $F(I) = (0)$), where $I$ denotes the ideal of $R$ generated by $[L, L]$.

Proof. Assume first that $R$ is prime and denote $V = [L, L]$. Hence we have $F(V) \subseteq Z(R)$. As a consequence of Lemma 2 in [9] we conclude that either $F = 0$ or $V \subseteq Z(R)$. In the first case we have the contradiction $d = 0$, and in the other case one has $L \subseteq Z(R)$ (see Lemma 2 in [12]), a contradiction again. Let now $P$ be a prime ideal of $R$ such that $[L, L] \nsubseteq P$.

Assume first that $d(P) \subseteq P$. Then $F$ induces a canonical generalized derivation $\overline{F}$ on $\overline{R} = \frac{R}{P}$. Therefore $\overline{F}([\overline{x}, \overline{y}]) \in Z(\overline{R})$ for all $\overline{x}, \overline{y} \in \overline{L}$. Moreover $\overline{L}$ is a Lie ideal of $\overline{R}$, such that $[\overline{L}, \overline{L}] \neq 0$ since $[L, L] \nsubseteq P$. By previous argument it follows that $\overline{F}(\overline{R}) = \overline{0}$ that $d(\overline{R}) \subseteq P$ and $F(\overline{R}) \subseteq P$.

Assume now that $d(P) \nsubseteq P$, then $\overline{d}(P) \neq \overline{0}$ and $\overline{d}(P)\overline{R} \neq \overline{0}$. We remark again that $\overline{d}(P)\overline{R}$ is a non-zero right ideal of $\overline{R}$.

Starting from our main assumption and linearizing we have that

$$F(x)y + F(x)z + xd(y) + xd(z) - F(y)x - F(z)x - yd(x) - zd(x) \in Z(R), \forall x, y, z \in L.$$ 

For any $p, p', p'' \in P, r, s \in R, u, v \in L$, replace $y$ by $[pr, u]$ and $z$ by $[[p's, v], p'']$. By computation it follows

$$\overline{\xi}[\overline{\tau}, \overline{\nu}] - [\overline{\tau}, \overline{\nu}][\overline{\xi}] \in Z(\overline{R})$$

that is

$$\left[\overline{\xi}, [\overline{\tau}, \overline{\nu}]\right] \in Z(\overline{R})$$

for all $\overline{\tau} \in \overline{d}(P)\overline{R}$ and $\overline{\xi}, \overline{\nu} \in \overline{L}$. As above denote $\overline{V} = [\overline{L}, \overline{L}]$, which is a Lie ideal for $\overline{R}$, and $\delta$ is the derivation of $\overline{R}$ induced by $\overline{V}$. Hence we have $\delta(\overline{V}) \subseteq Z(\overline{R})$. Again as a consequence of Lemma 2 in [9] it follows that either $\delta = 0$ or $\overline{V} \subseteq Z(\overline{R})$. Since $\overline{R}$ is not commutative, then there exists some $\overline{\tau} \in \overline{R}$ which is not central. Thus $\overline{V} \subseteq Z(\overline{R})$, and $\overline{L} \subseteq Z(\overline{R})$ follows from Lemma 2 in [12].

Therefore, for any prime ideal $P$ of $R$, either $d(R) \subseteq P$ and $F(R) \subseteq P$ or $[L, L] \subseteq P$. In this last case, by applying Theorem 3 in [15] in the prime ring $\overline{R}$, since $\text{char}(\overline{R}) \neq 2$ and $[\overline{L}, \overline{L}] = \overline{0}$, we conclude that $\overline{L}$ is central in $\overline{R}$, which
means \([L, R] \subseteq P\).

Hence in any case it follows that \(d(R)[L, R] = (0)\), \(a[R, L] = (0)\) and \([d(R), L] = (0)\). Finally we obtain the required conclusions by following the same argument as in Lemma 1.3.

In the sequel we will use the following known result:

**Lemma 1.6.** Let \(R\) be a 2-torsion free semiprime ring, \(L\) a Lie ideal of \(R\) such that \(L \not\subseteq Z(R)\). Let \(a \in L\) be such that \(aLa = 0\), then \(a = 0\).

**Remark 1.7.** If \(R\) is a prime ring of characteristic different from 2, \(a \in R\) and \(L\) is a non-central Lie ideal of \(R\) such that \([a, L] \subseteq Z(R)\), then \(a \in Z(R)\).

**Proof.** Denote by \(\delta : R \rightarrow R\) the inner derivation of \(R\) induced by the element \(a \in R\). Since \([a, x, r] = 0\) for all \(x \in L\) and \(r \in R\), a fortiori we have \([a, x]_2 = 0\), that is \([\delta(x), x] = 0\), for all \(x \in L\). Thus, by [14] it follows \(\delta = 0\), that is \(a \in Z(R)\). \(\square\)

**Theorem 1.8.** Let \(R\) be a 2-torsion free semiprime ring and \(L\) be a non-central Lie ideal of \(R\). Suppose \(R\) admits a generalized derivation \((F, d)\), defined as \(F(x) = ax + d(x)\), for all \(x \in R\) and fixed element \(a \in R\). If

\[
[F(x), x] \in Z(R) \text{ for all } x \in L.
\]

and \(d(L) \neq (0)\), then all the following hold simultaneously:

1. \(d(R)[L, R] = (0)\) and \([d(R), L] = (0)\);
2. \([a, L] = a[L, R] = (0)\);
3. \(aI = d(I) = (0)\) (that is \(F(I) = (0)\)), where \(I\) denotes the ideal of \(R\) generated by \([L, L]\).

**Proof.** Let \(P\) be a prime ideal of \(R\) such that \([L, L] \not\subseteq P\).

Assume first that \((P) \subseteq P\). Then \(F\) induces a canonical generalized derivation \(\overline{F}\) on \(\overline{R} = \frac{R}{P}\). Therefore \([\overline{F}(\overline{x}), \overline{x}] \in Z(\overline{R})\) for all \(\overline{x} \in \overline{L}\). Moreover \(\overline{L}\) is a Lie ideal of \(\overline{R}\), such that \([\overline{L}, \overline{L}] \neq 0\) since \([L, L] \not\subseteq P\). Since \([L, L] \not\subseteq P\), a fortiori we get \(\overline{L}\) is not central in \(\overline{R}\). Therefore, by Theorem 3.3 in [10], it follows that \(\overline{d}(\overline{R}) = 0\) that is \(d(R) \subseteq P\).

Assume now that \((P) \not\subseteq P\), then \(\overline{d}(\overline{F}) \neq 0\) and \(\overline{d}(\overline{F}) \overline{R} \neq 0\). By using similar argument as in Lemma 1.3, \(\overline{Rd}(\overline{F})\) is a non-zero right ideal of \(\overline{R}\). Linearizing (1.1) and using (1.1), we obtain

\[
[F(x), y] + [F(y), x] \in Z(R) \text{ for all } x, y \in L.
\]

Now, replace \(y\) by \([rp, u]\), for \(r \in R\), \(p \in P\) and \(u \in L\) and use (1.2) to get

\[
[\overline{F}([rp, u]), \overline{x}] \in Z(\overline{R}).
\]

Moreover, since \(F(r) = ar + d(r)\), for all \(r \in R\), by (1.3) it follows

\[
[\overline{d}([rp, u]), \overline{L}] \subseteq Z(\overline{R}).
\]
By the primeness of $R$ and Remark 1.7, one has that $\bar{d}([rp,u]) \in Z(R)$. On the other hand, an easy computation shows that $\bar{d}([rp,u]) = \bar{rd}(p),\bar{u}$, which implies $[\bar{rd}(P),L] \subseteq Z(R)$. Once again by Remark 1.7, we have $[\bar{rd}(P)] \subseteq Z(R)$. Since $\bar{rd}(P)$ is a non-zero right ideal of $R$, it follows $[R,R] = (0)$, which contradicts with $[L,L] \neq (0)$.

The previous argument shows that, for any prime ideal $P$ of $R$, either $[L,L] \subseteq P$ or $d(R) \subseteq P$. Thus $d(R)[L,L] \subseteq \cap P_i = (0)$. Hence, by Lemma 1.3 and since $L \nsubseteq Z(R)$, we finally get the required conclusions:

1. $d(R)[L,R] = (0)$ and $[d(R),L] = (0)$;
2. $(a+b)[L,R] = (0)$ and $[a+ b,L] = (0)$;
3. $aI = d(I) = (0)$, where $I$ denotes the ideal of $R$ generated by $[L,L]$.

□

**Theorem 1.9.** Let $R$ be a 2-torsion free semiprime ring and $L$ be a non-central Lie ideal of $R$. Suppose $R$ admits two generalized derivations $(F,d)$ and $(G,g)$. Write $F(x) = ax + d(x)$ and $G(x) = bx + g(x)$, for some $a, b \in U$. If $F([x,y]) = [y,G(x)]$ for all $x, y \in L$, then either

1. $g(L) = (0)$;
2. $d(R)[L,R] = (0)$ and $[d(R),L] = (0)$;
3. $(a+b)[L,R] = (0)$, $[b,L] = (0)$ and $[a,L] = (0)$; or
4. $(a+b)I = d(I) = (0)$, where $I$ denotes the ideal of $R$ generated by $[L,L]$.

or

1. $d(L) = (0)$;
2. $g(R)[L,R] = (0)$ and $[g(R),L] = (0)$;
3. $[b,L] = (0)$ and $a[L,L] = (0)$;
4. $aI = (0)$ and $g(I) = (0)$, where $I$ denotes the ideal of $R$ generated by $[L,L]$. or

1. $d(R)[L,R] = (0)$ and $[d(R),L] = (0)$;
2. $g(R)[L,R] = (0)$ and $[g(R),L] = (0)$;
3. $[a,L] = (0)$, $[b,L] = (0)$, $b[L,R] = a[L,R] = (0)$;
4. $d(I) = g(I) = (0)$ and $aI = bI = (0)$ (that is $F(I) = G(I) = (0)$), where $I$ denotes the ideal of $R$ generated by $[L,L]$.

Proof. Assume first $g(L) = (0)$, then $F([x,y]) = [y,bx]$ for all $x, y \in L$. Thus

$$a[x,y] + d([x,y]) = b[y,x]$$ (1.5)

for all $x, y \in L$, that is $(a+b)[x,y] + d([x,y]) = 0$ for all $x, y \in L$. Therefore, applying Theorem 1.5, one has

1. $d(R)[L,R] = (0)$ and $[d(R),L] = (0)$;
2. $(a+b)[L,R] = (0)$ and $[a+b,L] = (0)$;
(3) \((a+b)I = (0)\) and \(d(I) = (0)\), where \(I\) denotes the ideal of \(R\) generated by \([L, L]\).

In particular \(d([L, L]) = (0)\) and \([a[x, y] = -b[x, y]\) for all \(x, y \in L\), so that (1.5) reduces to \((by - yb)x = 0\), for all \(x, y \in L\), that is \([b, L]L = (0)\). Hence by Lemma 1.6, we have \([b, L] = (0)\) and so also \([a, L] = (0)\)

Let now \(d(L) = (0)\), then \([a, y] = [y, G(x)]\) for all \(x, y \in L\). In this case, for \(x = y\), we have \([G(y), y] = (0)\) and by Theorem 1.8 the following hold:

1. \([g(R)[L, R] = (0)\) and \([g(R), L] = (0)\);
2. \([b, L] = (0)\), \(b[L, R] = (0)\) and \(a[L, L] = (0)\);
3. \(bI = (0)\) and \(g(I) = (0)\), where \(I\) denotes the ideal of \(R\) generated by \([L, L]\).

Moreover, since \([([L, L], R) \subseteq [L, L]\), we also have \(0 = a[[L, L], R] = aR[L, L],\) which implies \(aI = (0)\).

Assume finally that both \(g(L) \neq (0)\) and \(d(L) \neq (0)\). Once again for \(x = y \in L\) we have \([G(x), x] = 0\) for any \(x \in L\). Thus by Theorem 1.8, we have that all the following hold:

1. \([g(R), L] = (0)\) and \([g(R), L] = (0)\);
2. \([b, L] = (0)\) and \(b[L, R] = (0)\);
3. \(bI = (0)\) and \(g(I) = (0)\), where \(I\) denotes the ideal of \(R\) generated by \([L, L]\).

Hence by the main assumption it follows that \((a+b)[x, y] + d([x, y]) = 0\), for all \(x, y \in L\). Denote \(H(x) = (a - b)x + d(x)\), then \(H(u) = 0\) for all \(u \in [L, L]\).

Finally, by applying Theorem 1.5, one has

1. \([d(R)[L, R] = (0)\) and \([d(R), L] = (0)\);
2. \((a+b)[L, R] = (0)\) and \([a, L] = (0)\);
3. \((a+b)I = (0)\) and \(d(I) = (0)\), where \(I\) denotes the ideal of \(R\) generated by \([L, L]\).

Note that, since both \(bI = (0)\) and \((a+b)I = (0)\), we are done. \(\square\)

We conclude our paper with some applications to generalized derivations acting on ideals of semiprime rings:

**Theorem 1.10.** Let \(R\) be a 2-torsion free semiprime ring and \(I\) be a non-central ideal of \(R\). Suppose \(R\) admits a generalized derivation \((F, d)\), defined as \(F(x) = ax + d(x)\), for all \(x \in R\) and fixed element \(a \in R\). If \([F(x), x] = 0\) for all \(x \in I\), then either \(d(I) = 0\) or \(R\) contains a non-zero central ideal.

**Proof.** By Theorem 1.8, we have that if \(d(I) \neq (0)\) then \([d(R), I] = (0)\). Hence, by applying Main Theorem in [13], it follows that \(R\) must contain a non-zero central ideal. \(\square\)

**Corollary 1.11.** Let \(R\) be a 2-torsion free semiprime ring \(F\) a generalized derivation of \(R\). If \([F(x), x] = 0\) for all \(x \in R\), then either \(R\) contains a
non-zero central ideal or there exists $\lambda \in Z(R)$ such that $F(x) = \lambda x$, for all $x \in R$.

**Theorem 1.12.** Let $R$ be a 2-torsion free semiprime ring and $I$ be a non-central ideal of $R$. Suppose $R$ admits two generalized derivations $(F,d)$ and $(G,g)$. Write $F(x) = ax + d(x)$ and $G(x) = bx + g(x)$, for some $a,b \in U$. If $F([x,y]) = [y,G(x)]$ for all $x,y \in L$, then either $d(I) = g(I) = (0)$ or $R$ contains a non-zero central ideal.

**Proof.** Assume either $d(I) \neq 0$ or $g(I) \neq 0$. Thus, by Theorem 1.9 respectively we have that either $[d(R),I] = (0)$ or $[g(R),I] = (0)$. In any case, again by [13], $R$ must contain some non-zero central ideals. \qed

**Corollary 1.13.** Let $R$ be a 2-torsion free semiprime ring and $F,G$ two generalized derivations of $R$. If $F([x,y]) = [y,G(x)]$ for all $x,y \in R$, then either $R$ contains a non-zero central ideal or there exist $\lambda \in Z(R)$ such that $F(x) = G(x) = \lambda x$, for all $x \in R$.

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**References**