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Quasi-Exact Sequence and Finitely Presented Modules

A. Madanshekaf

Department of Mathematics, Semnan University, Semnan, Iran

E-mail: amadanshekaf@semnan.ac.ir

Abstract. The notion of quasi-exact sequence of modules was introduced by B. Davvaz and coauthors in 1999 as a generalization of the notion of exact sequence. In this paper we investigate further this notion. In particular, some interesting results concerning this concept and torsion functor are given.

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1. Introduction

All rings in this lecture are assumed to be commutative with non-zero identity and all modules are unitary. Exact sequences have been used intensively in many discipline of mathematics such as commutative algebras. Let R be a ring and $A \xrightarrow{f} B \xrightarrow{g} C$ an exact sequence of R-modules. Then $\mathrm{im} f = \ker g (= g^{-1}(\{0\}))$. It is raising a natural question:
What does happen if we substitute a submodule U of C instead of the trivial submodule $\{0\}$ above? In [3], Davvaz and Parniam-Gramaleky introduced the concept of quasi-exact sequences and answered the above question. They generalized some results from the standard case to the modified case. In [2] Davvaz and Shabani-Solt introduced a generalization of some notions in homological algebra. They defined the concepts of chain U-complex, U-homology,

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chain (U, U')-map, chain (U, U')-homotopy and U-functor. They gave a generalization of the Lambek lemma, snake lemma, connecting homomorphism, exact triangle and established new basic properties of the U-homological algebra (See for example [6]). In [1], Anvariyeh and Davvaz studied U-split exact sequences and established several connections between U-split sequences and projective modules.

In this article we investigate further this notion. In particular, some interesting results concerning this concept and torsion functor are given. Now, we recall some basic definitions from [4] and then we present some examples to indicate how quasi-exact sequences occur naturally. (See also [2] and [3].) For unexplained notions we refer the reader to [8], [6] or [7].

Definition 1.1. *Given a ring R and a sequence of R-module homomorphisms $A \xrightarrow{f} B \xrightarrow{g} C$. Moreover assume that U is a submodule of C. It is said to be quasi-exact if* $\mathrm{im} f = g^{-1}(U)$. *In this case, we also say that the sequence is U-exact (at B).*

Definition 1.2. *A sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is said to be a short U-exact sequence if f is injective, g is surjective, and* $\mathrm{im} f = g^{-1}(U)$.

For example, if U and V be two submodules of an R-module A for which $V \subseteq U$ then, the sequence $0 \longrightarrow U \xrightarrow{\subseteq} A \xrightarrow{\pi} A/V \longrightarrow 0$ is a short U/V-exact where π is the natural epimorphism.

In general, given a sequence of R-modules and R-module homomorphisms

$$\cdots \longrightarrow A_{i+2} \xrightarrow{f_{i+2}} A_{i+1} \xrightarrow{f_{i+1}} A_i \xrightarrow{f_i} \cdots,$$

a family of submodules $\{U_i \mid i \in I\}$ of $\{A_i \mid i \in I\}$. We say that it is a quasi-exact sequence if it is a U_n-exact sequence at each of its module components.

2. Finitely Presented Modules and Torsion Functor

Following [7] we say that an R-module M is of *finite presentation* if there exists an exact sequence of the form

$$R^p \longrightarrow R^q \longrightarrow M \longrightarrow 0.$$

This means that M can be generated by q elements $\omega_1, \cdots, \omega_q$ in such a way that the module $N = \{(a_1, \cdots, a_q) \in A^q \mid \sum a_i\omega_i = 0\}$ of linear relations holding between the ω_i can be generated by p elements.
We will use in the next theorem the following result which is known (See Matsumura [7]).

Lemma 2.1. *Let R be a ring, and $0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\varphi} N \longrightarrow 0$ an exact sequence of R-modules. Then*

(i) *If L and N are both of finite presentation then so is M.*

(ii) If L is finitely generated and M is of finite presentation then N is of finite presentation.

Now we are ready to prove

Theorem 2.2 ([7]). *Let R be a ring, and suppose that M is an R-module of finite presentation. If*

$$0 \longrightarrow K \xrightarrow{\psi} N \xrightarrow{\varphi} M \longrightarrow 0$$

is a U-exact sequence where U is a finitely generated submodule of M and N is finitely generated then so is K.

Proof. Let $\pi : M \longrightarrow M/U$ be the natural projection, i.e., $\pi(x) = x + U$, for all $x \in M$. By assumption, since the sequence,

$$0 \longrightarrow K \xrightarrow{\psi} N \xrightarrow{\varphi} M \longrightarrow 0$$

is a U-exact sequence we deduce that the sequence

$$0 \longrightarrow K \xrightarrow{\psi} N \xrightarrow{\pi\varphi} M/U \longrightarrow 0$$

is an exact sequence. Now applying the above lemma, since M is of finite presentation and U is finitely generated, it implies that M/U is of finite presentation. On the other hand, by theorem 2.6 of [7], K is finitely generated and one gets the result.

Throughout the following, for a fixed ring R we denote the category of all R-modules and R-homomorphisms by $\mathbf{Mod}(R)$.
We recall the following from [2].

Definition 2.3. *Let R and S be two commutative rings, and F a covariant additive functor from the category $\mathbf{Mod}(R)$ to the category $\mathbf{Mod}(S)$. We say that F is a left U-exact functor whenever*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C$$

is a U-exact sequence of R-modules, then the induced sequence

$$0 \longrightarrow F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$$

of S-modules and S-homomorphisms is $F(U)$-exact.

If A is a module over the ring R and $a \in A$, then it is easy to see that the set $O_a = \{r \in R \mid \ ra = 0\}$ is an ideal of R. If $O_a \neq 0$ we say that a is a torsion element of A. Whenever R is an integral domain, the set $T(A)$ of all torsion elements of A is a submodule of A which is called the torsion submodule of A.

Suppose that R is an integral domain, A and B two R-modules and $f : A \longrightarrow B$ is a homomorphism, then $f(T(A)) \subseteq T(B)$; for let $a \in T(A)$, then there exists $0 \neq r \in R$ such that $ra = 0$. We have $rf(a) = f(ra) = 0$, hence $O_{f(a)} \neq 0$ and so $f(a) \in T(B)$. It follows that f induces by restriction to $T(A)$

and $T(B)$ an R-homomorphism $f_T : T(A) \longrightarrow T(B)$. It is straightforward that the mappings

$$A \mapsto T(A), \quad f \mapsto f_T,$$

where A runs over all objects and f over all morphisms of $\mathbf{Mod}(R)$, define an additive functor

$$T : \mathbf{Mod}(R) \longrightarrow \mathbf{Mod}(R),$$

which will be refereed to as the *torsion functor*.
In the next proposition we shall show that the torsion functor is left U-exact whenever the ground ring is an integral domain.

Proposition 2.4. *If R is an integral domain, then the torsion functor T : $\mathbf{Mod}(R) \longrightarrow \mathbf{Mod}(R)$, is a left U-exact functor.*

Proof. Let

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \tag{1}$$

be a U-exact sequence of R-modules, where R is an integral domain. We show that the sequence

$$0 \longrightarrow T(A) \xrightarrow{f_T} T(B) \xrightarrow{g_T} T(C)$$

is a $T(U)$-exact sequence; but, firstly, f_T is injective, because it acts in the same way as f; next, let $a \in T(A)$. Since $g_T(f_T(a)) = g(f(a))$ and since the sequence (1) is U-exact, so $g(f(a)) \in U$. But $a \in T(A)$ implies that $g(f(a)) \in T(C)$ and therefore $g(f(a)) \in T(U)$. Thus, $\mathrm{im} f_T \subseteq g_T^{-1}(T(U))$.
If $b \in g_T^{-1}(T(U))$, so $b \in g^{-1}(U)$. Again by U-exactness of (1), there exists $a \in A$ such that $f(a) = b$; but $a \in T(A)$, since if $rb = 0$ for some $r \neq 0$, we have $f(ra) = rb = 0$, and as f is injective, $ra = 0$. Therefore, $b = f_T(a) \in \mathrm{im} f_T$. Hence, $g_T^{-1}(T(U)) \subseteq \mathrm{im} f_T$ and one has the equality.

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