

## The explicit relation among the edge versions of detour index

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ABSTRACT. The vertex version of detour index was defined during the works on connected graph in chemistry. The edge versions of detour index have been introduced recently. In this paper, the explicit relations among edge versions of detour index have been declared and due to these relations, we compute the edge detour indices for some well-known graphs.

**Keywords:** Vertex detour index, Edge detour indices, Molecular graph.

**2000 Mathematics subject classification:** Primary 05C12, Secondary 92E10.

### 1. INTRODUCTION

The detour matrix is one of the particularly important distance matrices which are based on the topological distance for vertices in a graph. It was introduced in 1969 by Frank Harary [5] and it was discussed in 1990 by Buckley and Harary

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[2]. The detour matrix was introduced in 1994 under the name “*the maximum path matrix of a molecular graph*” [1,7,8,11 and 14] and theoretical graph theory contribution to finding the some interest in chemistry [9,10,15,16,17,18,19 and 21]. During these works, the ordinary (vertex) version of detour index has been defined for a connected graph  $G$  as follows:

$$(1) \quad D(G) = \sum_{\{u,v\} \subseteq V(G)} d_l(u, v | G)$$

where  $d_l(u, v | G)$  denotes the distance between the vertices  $u$  and  $v$  on the longest path, and where the other details are explained below.

In [3,4,6,13 and 20], some work has been done on detour index.

The edge versions of detour index which were based on distance between edges introduced by Iranmanesh et al. in 2008 [12]. These versions have been introduced for a connected graph  $G$  as follow:

The first edge-detour index is:

$$(2) \quad D_{e3}(G) = \sum_{\{e,f\} \subseteq E(G)} d_{l3}(e, f | G)$$

where  $d_{l3}(e, f | G) = \begin{cases} d_{l1}(e, f | G) + 1 & e \neq f \\ 0 & e = f \end{cases}$  and

$d_{l1}(e, f | G) = \min \{d_l(x, u), d_l(x, v), d_l(y, u), d_l(y, v)\}$  such that  $e = xy$  and  $f = uv$ .

The second edge-detour index is:

$$(3) \quad D_{e4}(G) = \sum_{\{e,f\} \subseteq E(G)} d_{l4}(e, f | G)$$

where  $d_{l4}(e, f | G) = \begin{cases} d_{l2}(e, f | G) & e \neq f \\ 0 & e = f \end{cases}$  and

$d_{l2}(e, f | G) = \max \{d_l(x, u), d_l(x, v), d_l(y, u), d_l(y, v)\}$  such that  $e = xy$  and  $f = uv$ .

The third edge detour index is:

$$(4) \quad D_{e0}(G) = \sum_{\{e,f\} \subseteq V(L(G))} d_{l0}(e, f | L(G))$$

where  $d_{l0}(e, f | L(G))$  denotes the distance between the vertices  $e$  and  $f$  on the longest path in line graph  $L(G)$ .

In this paper, the explicit relations between edge versions and vertex version of detour index have been declared and due to these relations, the edge-detour indices of some well-known graphs have been computed.

## 2. EXPLICIT RELATION AMONG THE EDGE-DETOUR INDICES

We recall the conditions of distances.  $d$  is the distance on set  $X$ , if it satisfy in following conditions:

- a-  $\forall u, v \in X; d(u, v) \geq 0$
- b-  $\forall u, v \in X; u = v \Leftrightarrow d(u, v) = 0$
- c-  $\forall u, v \in X; d(u, v) = d(v, u)$
- d-  $\forall u, v, w \in X; d(u, v) + d(v, w) \geq d(u, w)$

At first, we restate the first edge-detour index according to distances between vertices.

**Definition 2-1.** Let  $e = uv, f = xy$  be the edges of connected graph  $G$ . Then, we define:

$$d'_l(e, f) = \frac{d_l(u, x) + d_l(u, y) + d_l(v, x) + d_l(v, y)}{4}.$$

This quantity is not distance since it is not satisfy in distance conditions. We define several classes of graphs which are mentioned following due to the like-distance  $d'_l$ .

**Definition 2-2.** According  $d_l(x, u), d_l(x, v), d_l(y, u)$  and  $d_l(y, v)$  where  $e=uv$  and  $f=xy$ , we define:

$$A_1 = \{\{e, f\} \subseteq E(G) \mid \lceil d'_l(e, f) \rceil = d'_l(e, f)\},$$

$$A_2 = \left\{ \{e, f\} \subseteq E(G) \mid \lceil d'_l(e, f) \rceil = d'_l(e, f) + \frac{2}{4} \right\},$$

$$A_3 = \left\{ \{e, f\} \subseteq E(G) \mid \lceil d'_l(e, f) \rceil = d'_l(e, f) + \frac{2}{4} \right\},$$

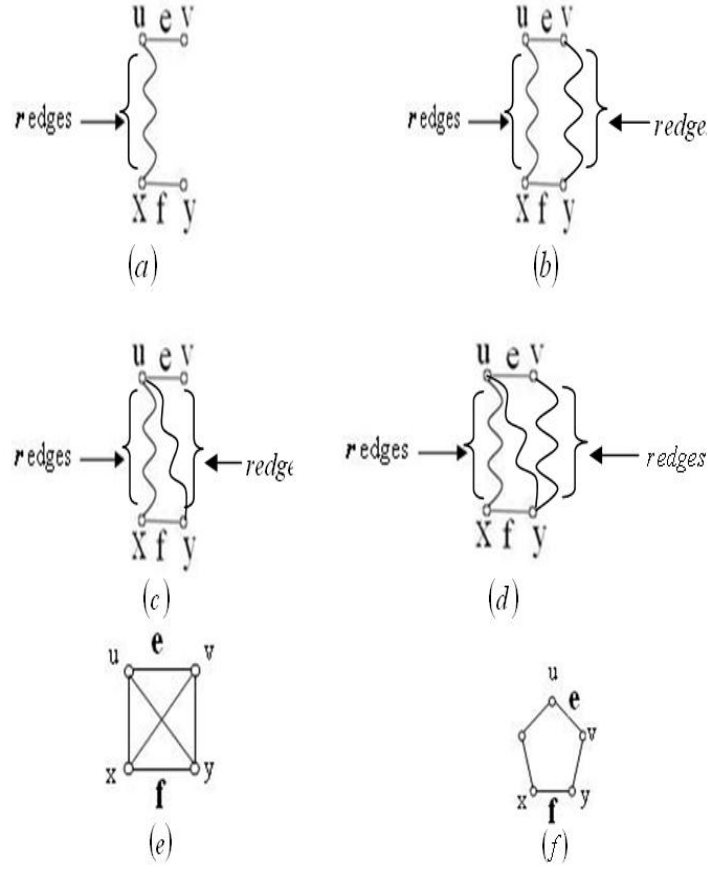
$$A_4 = \left\{ \{e, f\} \subseteq E(G) \mid \lceil d'_l(e, f) \rceil = d'_l(e, f) + \frac{1}{4} \right\}$$

$$A_5 = \left\{ \{e, f\} \subseteq E(G) \mid \lceil d'_l(e, f) \rceil = d'_l(e, f) + \frac{3}{4} \right\}$$

$$C = \{\{e, f\} \subseteq E(G) \mid \lceil d'_l(e, f) \rceil = d'_l(e, f)\}$$

Then, we have:

$$|A_1| + |A_2| + |A_3| + |A_4| + |A_5| + |C| = \binom{|E(G)|}{2}.$$



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Figure 1. Some examples for sets which mention above and  $r$  edges are the longest path between  $u$  and  $x$  in (a).

Now, we find the relation between the like-distance  $d'_i$  and distances  $d_{i3}$ ,  $d_{i4}$  and  $d_{i0}$ .

**Definition 2-3.** The relation between  $d'_i$  and  $d_{i0}$  is

$$d''_i(e, f) = \begin{cases} 4 \lceil d'_i(e, f) \rceil - 7 & , \{e, f\} \in C \\ \lceil d'_i(e, f) \rceil & , \{e, f\} \in A_1 \\ \lceil d'_i(e, f) \rceil - 1 & , \{e, f\} \in A_2 \\ \lceil d'_i(e, f) \rceil & , \{e, f\} \in A_3 \\ \lceil d'_i(e, f) \rceil + 1 & , \{e, f\} \in A_4 \\ \lceil d'_i(e, f) \rceil - 1 & , \{e, f\} \in A_5 \end{cases} ,$$

$$\text{where } d_{l_5}(e, f) = \begin{cases} d_l''(e, f) & e \neq f \\ 0 & e = f \end{cases}.$$

Also,  $d_l''$  do not satisfy the condition (b), and hence they are not a distance and are like-distance.

**Claim.**  $d_{l_0} = d_{l_5}$ .

*Proof.* We have to show for any  $e, f \in E(G)$ ,  $d_{l_0}(e, f) = d_{l_5}(e, f)$ .

i- If  $e = f \in E(G)$ , then  $d_{l_0}(e, f) = d_{l_5}(e, f) = 0$ .

ii- If  $e, f \in E(G)$  are adjacent edges, then,

$d_{l_0}(e, f) = 1$  and  $d_{l_5}(e, f) = d_l''(e, f) = \lceil \frac{1+1+2}{4} \rceil = 1$ . Therefore,  $d_{l_0}(e, f) = d_{l_5}(e, f)$ .

iii- If  $e, f \in E(G)$  are not adjacent such as Figure 1, then:

1) If  $\{e, f\} \in A_1$ , then  $d_{l_0}(e, f) = r + 1$  and

$d_{l_5}(e, f) = d_l''(e, f) = \lceil \frac{d_l(u,x)+d_l(u,y)+d_l(v,x)+d_l(v,y)}{4} \rceil = r + 1$ . Therefore,  $d_{l_0}(e, f) = d_{l_5}(e, f)$ . For example, see the Figure 1 (a).

2) If  $\{e, f\} \in A_2$ , then  $d_{l_0}(e, f) = r + 1$  and

$d_{l_5}(e, f) = d_l''(e, f) = \lceil \frac{d_l(u,x)+d_l(u,y)+d_l(v,x)+d_l(v,y)}{4} \rceil - 1 = r + 1$ . Therefore,  $d_{l_0}(e, f) = d_{l_5}(e, f)$ . For example, see the Figure 1 (b).

3) If  $\{e, f\} \in A_3$ , then  $d_{l_0}(e, f) = r + 2$  and

$d_{l_5}(e, f) = d_l''(e, f) = \lceil \frac{d_l(u,x)+d_l(u,y)+d_l(v,x)+d_l(v,y)}{4} \rceil = r + 2$ . Therefore,  $d_{l_0}(e, f) = d_{l_5}(e, f)$ . For example, see the Figure 1 (c).

4) If  $\{e, f\} \in A_4$ , then  $d_{l_0}(e, f) = r + 3$  and

$d_{l_5}(e, f) = d_l''(e, f) = \lceil \frac{d_l(u,x)+d_l(u,y)+d_l(v,x)+d_l(v,y)}{4} \rceil + 1 = r + 3$ . Therefore,  $d_{l_0}(e, f) = d_{l_5}(e, f)$ . For example, see the Figure 1 (d).

5) If  $\{e, f\} \in A_5$ , then  $d_{l_0}(e, f) = r + 2$  and

$d_{l_5}(e, f) = d_l''(e, f) = \lceil \frac{d_l(u,x)+d_l(u,y)+d_l(v,x)+d_l(v,y)}{4} \rceil - 1 = r + 2$ . Therefore,  $d_{l_0}(e, f) = d_{l_5}(e, f)$ . For example, see the Figure 1 (f).

6) If  $\{e, f\} \in C$ , then  $d_{l_0}(e, f) = 4r + 1$  and

$d_{l_5}(e, f) = d_l''(e, f) = 4 \times \frac{(r+2)+(r+2)+(r+2)+(r+2)}{4} - 7 = 4r + 1$ . Therefore,  $d_{l_0}(e, f) = d_{l_5}(e, f)$ .  $\square$

**Definition 2-4.** The relation between  $d_l'$  and  $d_{l_3}$  is:

$$d_l'''(e, f) = \begin{cases} \lceil d_l'(e, f) \rceil & , \{e, f\} \notin C \\ \lceil d_l'(e, f) \rceil + 1 & , \{e, f\} \in C \end{cases},$$

where  $d_{l_6}(e, f) = \begin{cases} d_l'''(e, f) & e \neq f \\ 0 & e = f \end{cases}$ . Also,  $d_l'''$  do not satisfy the condition

(b), and hence they are not a distance and are like-distance.

**Claim.**  $d_{l_3} = d_{l_6}$ .

*Proof.* We have to show for any  $e, f \in E(G)$ ,  $d_{l_3}(e, f) = d_{l_6}(e, f)$ .

i- If  $e = f \in E(G)$ , then  $d_{l_3}(e, f) = d_{l_6}(e, f) = 0$

ii- If  $e, f \in E(G)$  are adjacent edges, then,

$$d_{l_3}(e, f) = d_{l_1}(e, f) + 1 = 0 + 1 = 1 \text{ and } d_{l_6}(e, f) = d_l'''(e, f) = \lceil \frac{1+1+2}{4} \rceil = 1.$$

Therefore,  $d_{l_3}(e, f) = d_{l_6}(e, f)$ .

iii- If  $e, f \in E(G)$  are not adjacent such as Figure 1, then:

1) If  $\{e, f\} \notin C$ , then

$$(1) \text{ If } \{e, f\} \in A_1, \text{ then } d_{l_3}(e, f) = d_{l_1}(e, f) + 1 = r + 1 \text{ and } d_{l_6}(e, f) = d_l'''(e, f) = \lceil \frac{d_l(u,x)+d_l(u,y)+d_l(v,x)+d_l(v,y)}{4} \rceil = r+1. \text{ Therefore, } d_{l_3}(e, f) = d_{l_6}(e, f).$$

$$(2) \text{ If } \{e, f\} \in A_2, \text{ then } d_{l_3}(e, f) = d_{l_1}(e, f) + 1 = r + 2 \text{ and } d_{l_6}(e, f) = d_l'''(e, f) = \lceil \frac{d_l(u,x)+d_l(u,y)+d_l(v,x)+d_l(v,y)}{4} \rceil = r+2. \text{ Therefore, } d_{l_3}(e, f) = d_{l_6}(e, f).$$

$$(3) \text{ If } \{e, f\} \in A_3, \text{ then } d_{l_3}(e, f) = d_{l_1}(e, f) + 1 = r + 2 \text{ and } d_{l_6}(e, f) = d_l'''(e, f) = \lceil \frac{d_l(u,x)+d_l(u,y)+d_l(v,x)+d_l(v,y)}{4} \rceil = r+2. \text{ Therefore, } d_{l_3}(e, f) = d_{l_6}(e, f).$$

$$(4) \text{ If } \{e, f\} \in A_4, \text{ then } d_{l_3}(e, f) = d_{l_1}(e, f) + 1 = r + 2 \text{ and } d_{l_6}(e, f) = d_l'''(e, f) = \lceil \frac{d_l(u,x)+d_l(u,y)+d_l(v,x)+d_l(v,y)}{4} \rceil = r+2. \text{ Therefore, } d_{l_3}(e, f) = d_{l_6}(e, f).$$

$$(5) \text{ If } \{e, f\} \in A_5, \text{ then } d_{l_3}(e, f) = d_{l_1}(e, f) + 1 = r + 3 \text{ and } d_{l_6}(e, f) = d_l'''(e, f) = \lceil \frac{d_l(u,x)+d_l(u,y)+d_l(v,x)+d_l(v,y)}{4} \rceil = r+3. \text{ Therefore, } d_{l_3}(e, f) = d_{l_6}(e, f).$$

$$2) \text{ If } \{e, f\} \in C, \text{ then } d_{l_3}(e, f) = d_{l_1}(e, f) + 1 = r + 3 \text{ and } d_{l_6}(e, f) = d_l'''(e, f) = \lceil \frac{(r+2)+(r+2)+(r+2)+(r+2)}{4} \rceil + 1 = r + 3. \text{ Therefore, } d_{l_3}(e, f) = d_{l_6}(e, f). \quad \square$$

**Definition 2-5.** The relation between  $d_l'$  and  $d_{l_4}$  is:

$$d_l''''(e, f) = \begin{cases} \lceil d_l'(e, f) \rceil & , \{e, f\} \notin A_1 \\ \lceil d_l'(e, f) \rceil + 1 & , \{e, f\} \in A_1 \end{cases}$$

where  $d_{l_7}(e, f) = \begin{cases} d_l''''(e, f) & e \neq f \\ 0 & e = f \end{cases}$ . Also,  $d_l''''$  do not satisfy the condition

(b), hence, they are not a distance and are like-distance.

**Claim.**  $d_{l_4} = d_{l_7}$ .

*Proof.* We have to show for any  $e, f \in E(G)$ ,  $d_{l_4}(e, f) = d_{l_7}(e, f)$ .

i- If  $e = f \in E(G)$ , then  $d_{l_4}(e, f) = d_{l_7}(e, f) = 0$

ii- If  $e, f \in E(G)$  are adjacent edges, then,







$$d_{l_4}(e, f) = d_{l_2}(e, f) = 2 \text{ and } d_{l_7}(e, f) = d_l''''(e, f) = \lceil \frac{1+1+2}{4} \rceil + 1 = 2. \text{ Therefore, } d_{l_4}(e, f) = d_{l_7}(e, f).$$

iii- If  $e, f \in E(G)$  are not adjacent such as Figure 1, then:

1) If  $\{e, f\} \notin A_1$ , then

- (1) If  $\{e, f\} \in A_2$ , then  $d_{l_4}(e, f) = d_{l_2}(e, f) = r + 2$  and  $d_{l_7}(e, f) = d_l^{''''}(e, f) = \left\lceil \frac{d_l(u, x) + d_l(u, y) + d_l(v, x) + d_l(v, y)}{4} \right\rceil = r + 2$ . Therefore,  $d_{l_4}(e, f) = d_{l_7}(e, f)$ .
- (2) If  $\{e, f\} \in A_3$ , then  $d_{l_4}(e, f) = d_{l_2}(e, f) = r + 2$  and  $d_{l_7}(e, f) = d_l^{''''}(e, f) = \left\lceil \frac{d_l(u, x) + d_l(u, y) + d_l(v, x) + d_l(v, y)}{4} \right\rceil = r + 2$ . Therefore,  $d_{l_4}(e, f) = d_{l_7}(e, f)$ .
- (3) If  $\{e, f\} \in A_4$ , then  $d_{l_4}(e, f) = d_{l_2}(e, f) = r + 2$  and  $d_{l_7}(e, f) = d_l^{''''}(e, f) = \left\lceil \frac{d_l(u, x) + d_l(u, y) + d_l(v, x) + d_l(v, y)}{4} \right\rceil = r + 2$ . Therefore,  $d_{l_4}(e, f) = d_{l_7}(e, f)$ .
- (4) If  $\{e, f\} \in A_5$ , then  $d_{l_4}(e, f) = d_{l_2}(e, f) = r + 3$  and  $d_{l_7}(e, f) = d_l^{''''}(e, f) = \left\lceil \frac{d_l(u, x) + d_l(u, y) + d_l(v, x) + d_l(v, y)}{4} \right\rceil = r + 3$ . Therefore,  $d_{l_4}(e, f) = d_{l_7}(e, f)$ .
- (5) If  $\{e, f\} \in C$ , then  $d_{l_4}(e, f) = d_{l_2}(e, f) = r + 2$  and  $d_{l_7}(e, f) = d_l^{''''}(e, f) = \left\lceil \frac{(r+2) + (r+2) + (r+2) + (r+2)}{4} \right\rceil = r + 2$ . Therefore,  $d_{l_4}(e, f) = d_{l_7}(e, f)$ .
- 2) If  $\{e, f\} \in A_1$ , then  $d_{l_4}(e, f) = d_{l_2}(e, f) = r + 2$  and  $d_{l_7}(e, f) = d_l^{''''}(e, f) = \left\lceil \frac{d_l(u, x) + d_l(u, y) + d_l(v, x) + d_l(v, y)}{4} \right\rceil + 1 = r + 2$ . Therefore,  $d_{l_4}(e, f) = d_{l_7}(e, f)$ .  $\square$

Table 1. Examples for sets which have been defined in above

Set	$c$	$A_1$	$A_2$	$A_3$	$A_4$	$A_5$
Example						
$d_{l_1}(e, f) = \min \left\{ \begin{array}{l} d_l(x, u), d_l(x, v), \\ d_l(y, u), d_l(y, v) \end{array} \right\}$	3	1	2	2	2	3
$d_{l_2}(e, f) = \max \left\{ \begin{array}{l} d_l(x, u), d_l(x, v), \\ d_l(y, u), d_l(y, v) \end{array} \right\}$	3	3	3	3	3	4
$d_{l_3}(e, f   G) = d_{l_6}(e, f   G) = \begin{cases} d_{l_1}(e, f   G) + 1 & , e \neq f \\ 0 & , e = f \end{cases}$	4	2	3	3	3	4
$d_{l_4}(e, f   G) = d_{l_7}(e, f   G) = \begin{cases} d_{l_2}(e, f   G) & , e \neq f \\ 0 & , e = f \end{cases}$	3	3	3	3	3	4
$d_l^{''''}(e, f) = \frac{d_l(u, x) + d_l(u, y) + d_l(v, x) + d_l(v, y)}{4}$	3	2	$\frac{10}{4}$	$\frac{10}{4}$	$\frac{11}{4}$	$\frac{13}{4}$
$d_{l_5}(e, f) = d_{l_0}(e, f) = \begin{cases} d_l^{''''}(e, f) & , e \neq f \\ 0 & , e = f \end{cases}$	5	2	2	3	4	3

**Corollary 2-6.**  $D_{e0}(G) = \sum_{\{e, f\} \subseteq E(G)} d_{l_5}(e, f)$ ,  $D_{e3}(G) = \sum_{\{e, f\} \subseteq E(G)} d_{l_6}(e, f)$  and  $D_{e4}(G) = \sum_{\{e, f\} \subseteq E(G)} d_{l_7}(e, f)$ .

*Proof.* Since  $d_{l_0} = d_{l_5}$ ,  $d_{l_3} = d_{l_6}$  and  $d_{l_4} = d_{l_7}$ , we obtain the desire result.  $\square$

**Theorem 2-7.** The first edge detour index of connected graph with  $m$  edges,  $D_{e3}(G)$ , according to distance between vertices is:

$$D_{e3}(G) = \frac{1}{8} \sum_{x \in V(G)} \sum_{y \in V(G)} \deg(x) \times \deg(y) \times d_l(x, y) - \frac{r_1}{4} - \frac{1}{4} \sum_{i=1}^{r_2} (t_i - 1) + \sum_{\{e,f\} \in A_3} \left(\frac{1}{2}\right) + \sum_{\{e,f\} \in A_2} \left(\frac{1}{2}\right) + \sum_{\{e,f\} \in A_4} \left(\frac{1}{4}\right) + \sum_{\{e,f\} \in A_5} \left(\frac{3}{4}\right) + |C|$$

where  $r_1$  is the pair of adjacent vertices which are not in a cycle,  $r_2$  is the pair of adjacent vertices which are in a cycle and  $t_i$  is the length of longest cycle which pair of vertices occur on it.

*Proof.* By the Definitions 2-1 and 2-4 and Corollary 2-6, we have:

$$\begin{aligned} D_{e3}(G) &= \sum_{\{e,f\} \subseteq E(G)} d_{l6}(e, f) = \\ &= \sum_{\substack{\{e,f\} \in A_1 \\ ife = uv, f = xy}} \frac{d_l(u,x)+d_l(u,y)+d_l(v,x)+d_l(v,y)}{4} + \\ &\sum_{\substack{\{e,f\} \in A_2 \\ ife = uv, f = xy}} \left( \frac{d_l(u,x)+d_l(u,y)+d_l(v,x)+d_l(v,y)}{4} + \frac{1}{2} \right) + \\ &\sum_{\substack{\{e,f\} \in A_3 \\ ife = uv, f = xy}} \left( \frac{d_l(u,x)+d_l(u,y)+d_l(v,x)+d_l(v,y)}{4} + \frac{1}{2} \right) + \\ &\sum_{\substack{\{e,f\} \in A_4 \\ ife = uv, f = xy}} \left( \frac{d_l(u,x)+d_l(u,y)+d_l(v,x)+d_l(v,y)}{4} + \frac{1}{4} \right) + \\ &\sum_{\substack{\{e,f\} \in A_5 \\ ife = uv, f = xy}} \left( \frac{d_l(u,x)+d_l(u,y)+d_l(v,x)+d_l(v,y)}{4} + \frac{3}{4} \right) + \\ &\sum_{\substack{\{e,f\} \in C \\ ife = uv, f = xy}} \left( \frac{d_l(u,x)+d_l(u,y)+d_l(v,x)+d_l(v,y)}{4} + 1 \right) = \\ &\sum_{\substack{\{e,f\} \subseteq E(G) \\ ife = uv, f = xy}} \frac{d_l(u,x)+d_l(u,y)+d_l(v,x)+d_l(v,y)}{4} + \\ &\sum_{\{e,f\} \in A_2} \left(\frac{1}{2}\right) + \sum_{\{e,f\} \in A_3} \left(\frac{1}{2}\right) + \sum_{\{e,f\} \in A_4} \left(\frac{1}{4}\right) + \sum_{\{e,f\} \in A_5} \left(\frac{3}{4}\right) + |C| \end{aligned}$$

For each pair of vertices  $u, x \in V(G)$  such that  $u \neq x$  which is not adjacent, the distance  $d_l(u, x)$  in like distance  $d'_l$  is repeated  $\deg(u) \times \deg(x)$  times. And if every pair of vertices  $u, x \in V(G)$ ,  $u \neq x$ , is adjacent, distance  $d_l(u, x)$  is repeated  $\deg(u) \times \deg(x) - 1$  times. Therefore,

$$D_{e3}(G) = \frac{1}{8} \sum_{x \in V(G)} \sum_{y \in V(G)} \deg(x) \times \deg(y) \times d_l(x, y) - \frac{r_1}{2} - \frac{1}{4} \sum_{i=1}^{r_2} (t_i - 2) + \sum_{\{e,f\} \in A_3} \left(\frac{1}{2}\right) + \sum_{\{e,f\} \in A_2} \left(\frac{1}{2}\right) + \sum_{\{e,f\} \in A_4} \left(\frac{1}{4}\right) + \sum_{\{e,f\} \in A_5} \left(\frac{3}{4}\right) + |C|$$

□

**Theorem 2-8.** The second edge detour index of connected graph with  $m$  edges,  $D_{e4}(G)$ , according to distance between vertices is:



$$D_{e4}(G) = \frac{1}{8} \sum_{x \in V(G)} \sum_{y \in V(G)} \deg(x) \times \deg(y) \times d_l(x, y) - \frac{r_1}{4} - \frac{1}{4} \sum_{i=1}^{r_2} (t_i - 1) \\ \sum_{\{e,f\} \in A_3} \left(\frac{1}{2}\right) + \sum_{\{e,f\} \in A_2} \left(\frac{1}{2}\right) + \sum_{\{e,f\} \in A_4} \left(\frac{1}{4}\right) + \sum_{\{e,f\} \in A_5} \left(\frac{3}{4}\right) + |A_1|$$

where  $r_1$  is the pair of adjacent vertices which are not in a cycle,  $r_2$  is the pair of adjacent vertices which are in a cycle and  $t_i$  is the length of longest cycle which pair of vertices occur on it.

*Proof.* Due to the definition of  $D_{e4}(G)$  and Definitions 2-1 and 2-5 and Corollary 2-6, we have:

$$D_{e4}(G) = \sum_{\{e,f\} \subseteq E(G)} d_{l7}(e, f) = \\ \sum_{\{e,f\} \subseteq E(G)} d'_l(e, f) + \sum_{\{e,f\} \in A_2} \frac{2}{4} + \sum_{\{e,f\} \in A_3} \frac{2}{4} + \sum_{\{e,f\} \in A_4} \frac{1}{4} + \\ \sum_{\{e,f\} \in A_5} \frac{3}{4} + |A_1|$$

For each pair of vertices  $u$  and  $x$  such that  $u \neq x$  which is not adjacent, the distance  $d_l(u, x)$  in like-distance  $d'_l$  is repeated  $\deg(u) \times \deg(x)$  times. And if every pair of vertices  $u$  and  $x$ ,  $u \neq x$ , which is adjacent, distance  $d_l(u, x)$  is repeated  $\deg(u) \times \deg(x) - 1$  times. Therefore,

$$D_{e4}(G) = \frac{1}{8} \sum_{x \in V(G)} \sum_{y \in V(G)} \deg(x) \times \deg(y) \times d_l(x, y) - \frac{r_1}{4} - \frac{1}{4} \sum_{i=1}^{r_2} (t_i - 1) \\ \sum_{\{e,f\} \in A_3} \left(\frac{1}{2}\right) + \sum_{\{e,f\} \in A_2} \left(\frac{1}{2}\right) + \sum_{\{e,f\} \in A_4} \left(\frac{1}{4}\right) + \sum_{\{e,f\} \in A_5} \left(\frac{3}{4}\right) + |A_1|.$$

□

**Theorem 2-9.** The third edge detour index of connected graph with  $m$  edges,  $D_{e0}(G)$ , according to distance between vertices is:

$$D_{e0}(G) = \frac{1}{8} \sum_{x \in V(G)} \sum_{y \in V(G)} \deg(x) \times \deg(y) \times d_l(x, y) - \frac{r_1}{4} - \frac{1}{4} \sum_{i=1}^{r_2} (t_i - 1) + \\ \sum_{\substack{\{e,f\} \in C \\ ife = uv, f = xy}} \left( 3 \left\lceil \frac{d_l(u,x) + d_l(u,y) + d_l(v,x) + d_l(v,y)}{4} \right\rceil \right) + \\ \sum_{\{e,f\} \in A_3} \left(\frac{1}{2}\right) + \sum_{\{e,f\} \in A_2} \left(\frac{1}{2}\right) + \sum_{\{e,f\} \in A_4} \left(\frac{1}{4}\right) + \\ \sum_{\{e,f\} \in A_5} \left(\frac{3}{4}\right) - 7|C| - |A_2| + |A_4| - |A_5|$$

where  $r_1$  is the pair of adjacent vertices which are not in a cycle,  $r_2$  is the pair of adjacent vertices which are in a cycle and  $t_i$  is the length of longest cycle which pair of vertices occur on it.

*Proof.* Due to the definition of  $D_{e0}(G)$  and Definitions (2-1 and 2-3) and Corollary (2-6), we have:

$$\begin{aligned}
D_{e0}(G) &= \sum_{\{e,f\} \subseteq E(G)} d_{l5}(e, f) = \\
&= \sum_{\substack{\{e,f\} \in A_1 \\ ife = uv, f = xy}} \frac{d_l(u,x)+d_l(u,y)+d_l(v,x)+d_l(v,y)}{4} + \\
&\sum_{\substack{\{e,f\} \in A_2 \\ ife = uv, f = xy}} \left( \left\lceil \frac{d_l(u,x)+d_l(u,y)+d_l(v,x)+d_l(v,y)}{4} \right\rceil - 1 \right) + \\
&\sum_{\substack{\{e,f\} \in A_3 \\ ife = uv, f = xy}} \left( \left\lceil \frac{d_l(u,x)+d_l(u,y)+d_l(v,x)+d_l(v,y)}{4} \right\rceil \right) + \\
&\sum_{\substack{\{e,f\} \in A_4 \\ ife = uv, f = xy}} \left( \left\lceil \frac{d_l(u,x)+d_l(u,y)+d_l(v,x)+d_l(v,y)}{4} \right\rceil + 1 \right) + \\
&\sum_{\substack{\{e,f\} \in A_5 \\ ife = uv, f = xy}} \left( \left\lceil \frac{d_l(u,x)+d_l(u,y)+d_l(v,x)+d_l(v,y)}{4} \right\rceil - 1 \right) + \\
&\sum_{\substack{\{e,f\} \in C \\ ife = uv, f = xy}} \left( 4 \left\lceil \frac{d_l(u,x)+d_l(u,y)+d_l(v,x)+d_l(v,y)}{4} \right\rceil - 7 \right) = \\
&\sum_{\substack{\{e,f\} \subseteq E(G) \\ ife = uv, f = xy}} \frac{d_l(u,x)+d_l(u,y)+d_l(v,x)+d_l(v,y)}{4} + \\
&\sum_{\substack{\{e,f\} \in C \\ ife = uv, f = xy}} \left( 3 \left\lceil \frac{d_l(u,x)+d_l(u,y)+d_l(v,x)+d_l(v,y)}{4} \right\rceil \right) + \\
&\sum_{\{e,f\} \in A_2} \left( \frac{1}{2} \right) + \sum_{\{e,f\} \in A_3} \left( \frac{1}{2} \right) + \sum_{\{e,f\} \in A_4} \left( \frac{1}{4} \right) + \\
&\sum_{\{e,f\} \in A_5} \left( \frac{3}{4} \right) - 7|C| - |A_2| + |A_4| - |A_5|
\end{aligned}$$

For each pair of vertices  $u$  and  $x$  such that  $u \neq x$  which is not adjacent, the distance  $d_l(u, x)$  in like-distance  $d'_l$  is repeated  $\deg(u) \times \deg(x)$  times. And if every pair of vertices  $u$  and  $x$ ,  $u \neq x$ , which is adjacent, distance  $d_l(u, x)$  is repeated  $\deg(u) \times \deg(x) - 1$  times. Therefore,

$$\begin{aligned}
D_{e0}(G) &= \frac{1}{8} \sum_{x \in V(G)} \sum_{y \in V(G)} \deg(x) \times \deg(y) \times d_l(x, y) - \frac{r_1}{4} - \frac{1}{4} \sum_{i=1}^{r_2} (t_i - 1) + \\
&\sum_{\substack{\{e,f\} \in C \\ ife = uv, f = xy}} \left( 3 \left\lceil \frac{d_l(u,x)+d_l(u,y)+d_l(v,x)+d_l(v,y)}{4} \right\rceil \right) + \\
&\sum_{\{e,f\} \in A_3} \left( \frac{1}{2} \right) + \sum_{\{e,f\} \in A_2} \left( \frac{1}{2} \right) + \sum_{\{e,f\} \in A_4} \left( \frac{1}{4} \right) + \sum_{\{e,f\} \in A_5} \left( \frac{3}{4} \right) - 7|C| - \\
&|A_2| + |A_4| - |A_5|
\end{aligned}$$

□

**Corollary 2-10.** The explicit relations between edge versions of detour index are

$$D_{e4}(G) = D_{e3}(G) + |A_1| - |C|$$

$$D_{e0}(G) = D_{e4}(G) + \sum_{\substack{\{e, f\} \subseteq E(G) \\ ife = uv, f = xy}} \left( 3 \left\lceil \frac{d_I(u,x) + d_I(u,y) + d_I(v,x) + d_I(v,y)}{4} \right\rceil \right) - |A_1| - 7|C| - |A_2| + |A_4| - |A_5|$$

$$D_{e0}(G) = D_{e3}(G) + \sum_{\substack{\{e, f\} \subseteq E(G) \\ ife = uv, f = xy}} \left( 3 \left\lceil \frac{d_I(u,x) + d_I(u,y) + d_I(v,x) + d_I(v,y)}{4} \right\rceil \right) - 8|C| - |A_2| + |A_4| - |A_5|$$

*Proof.* Due to the Theorems 2-7, 2-8 and 2-9, we can get the desire results.  $\square$

Now, we state the edge detour indices of some well-known graphs.

**Table 2.**

Graph ( $G$ )	$D_{e3}(G)$	$ A_1 $	$ A_2 $	$ A_4 $	$ A_5 $	$ C $
$P_n$	$\frac{1}{6}n(n-1)(n-2)$	$\binom{n-1}{2}$	0	0	0	0
$S_n$	$\frac{1}{2}(n-1)(n-2)$	$\binom{n-1}{2}$	0	0	0	0
$C_n,$ <i>nisodd</i>	$\frac{3}{8}n^3 - \frac{3}{2}n^2 + \frac{17}{8}n$	$\binom{n}{2} - n$	0	0	$n$	0
$C_n,$ <i>niseven</i>	$\frac{3}{8}n^3 - \frac{3}{2}n^2 + 2n$	$\binom{n}{2} - \frac{n}{2}$	$\frac{n}{2}$	0	0	0

Graph ( $G$ )	$D_{e4}(G)$	$D_{e0}(G)$
$P_n$	$\frac{1}{6}(n-1)(n-2)(n+3)$	$\frac{1}{6}n(n-1)(n-2)$
$S_n$	$(n-1)(n-2)$	$\frac{1}{2}(n-1)(n-2)$
$C_n,$ <i>nisodd</i>	$\frac{3}{8}n^3 - \frac{27}{8}n$	$\frac{3}{8}n^3 - \frac{3}{2}n^2 + \frac{9}{8}n$
$C_n,$ <i>niseven</i>	$\frac{3}{8}n^3 - 2n$	$\frac{3}{8}n^3 - \frac{3}{2}n^2 + \frac{3}{2}n$

**Acknowledgement.** We are grateful to the referees for their valuable suggestions, which have improved this paper.

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