A Generalized Singular Value Inequality for Heinz Means

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Abstract. In this paper we will generalize a singular value inequality that was proved before. In particular we obtain an inequality for numerical radius as follows:

$$2\sqrt{t(1-t)}\omega(tA\nu B^{1-\nu} + (1-t)A^{1-\nu}B^\nu) \leq \omega(tA + (1-t)B),$$

where, $A$ and $B$ are positive semidefinite matrices, $0 \leq t \leq 1$ and $0 \leq \nu \leq \frac{3}{2}$.

Keywords: Matrix monotone functions, Numerical radius, Singular values, Unitarily invariant norms.


1. Introduction

Let $M_n$ be the algebra of all $n \times n$ complex matrices. A norm $\| \cdot \|$ on $M_n$ is said to be unitarily invariant if $\|UAV\| = \|A\|$ for all $A \in M_n$ and all unitary $U, V \in M_n$. Special examples of such norms are the "Ky Fan norms"

$$\|A\|_k = \sum_{j=1}^{k} s_j(A), \quad 1 \leq k \leq n.$$ 

Note that the operator norm, in this notation, is $\|A\| = \|A\|_1 = s_1(A)$; see [4] and [9] for more information.
If $\|A\|_{(k)} \leq \|B\|_{(k)}$ for $1 \leq k \leq n$, then $\|A\| \leq \|B\|$ for all unitary invariant norms. This is called the "Fan dominance theorem." If $A$ is a Hermitian element of $M_n$, then we arrange its eigenvalues in decreasing order as $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$. If $A$ is arbitrary, then its singular values are enumerated as $s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A)$. These are the eigenvalues of the positive semidefinite matrix $|A| = (A^*A)^{1/2}$. If $A$ and $B$ are Hermitian matrices, and $A - B$ is positive semidefinite, then we say that $B \leq A$.

Weyl’s monotonicity theorem [4, p. 63] says that $\lambda_j(A) \leq \lambda_j(B)$, for all $j = 1, \ldots, n$. Let $f$ be a real valued function on an interval $I$. Then $f$ is said to be matrix monotone if $A, B \in M_n$ are Hermitian matrices with all their eigenvalues in $I$ and $A \geq B$, then $f(A) \geq f(B)$ and also, $f$ is said to be matrix convex if

$$f(tA + (1-t)B) \leq tf(A) + (1-t)f(B), \ 0 \leq t \leq 1$$

and matrix concave if

$$f(tA + (1-t)B) \geq tf(A) + (1-t)f(B), \ 0 \leq t \leq 1.$$ 

In response to a conjecture by Zhan [13], Audenaert [2] has proved that if $A, B \in M_n$ are positive semidefinite, then the inequality

$$s_j(A'^\nu B^{1-\nu} + A^{1-\nu}B'^\nu) \leq s_j(A + B), \ 1 \leq j \leq n$$

holds, for all $0 \leq \nu \leq 1$. In this paper we generalize this inequality as follows: If $A, B \in M_n$ are positive semidefinite matrices, then for all $0 \leq t \leq 1$ and $0 \leq \nu \leq \frac{3}{2}$

$$2\sqrt{t(1-t)}s_j(tA'^\nu B^{1-\nu} + (1-t)A^{1-\nu}B'^\nu) \leq s_j(tA + (1-t)B).$$

For more details about inequalities and their generalizations with their history of origin, the reader may refer to [1, 5, 6, 11, 12, 13].

2. MAIN RESULTS

**Lemma 2.1.** [14] If $X = \begin{bmatrix} A & C \\ C^* & B \end{bmatrix}$ is positive, then $2s_j(C) \leq s_j(X)$ for all $1 \leq j \leq n$.

**Theorem 2.2.** Let $f$ be a matrix monotone function on $[0, \infty)$ and $A$ and $B$ be positive semidefinite matrices. Then

$$tAf(A)+(1-t)Bf(B) \geq (tA+(1-t)B)^{1/2}(tf(A)+(1-t)f(B))(tA+(1-t)B)^{1/2}$$

(2.1)

for all $0 \leq t \leq 1$.

**Proof.** The function $f$ is also matrix concave, and $g(x) = xf(x)$ is matrix convex. (See [4]). The matrix convexity of $g$ implies the inequality

$$(tA+(1-t)B)f(tA+(1-t)B) \leq tAf(A)+(1-t)Bf(B), \ 0 \leq t \leq 1.$$ (2.2)
Since the matrix $tA + (1-t)B$ is positive semidefinite, in view of the spectral decomposition theorem, it is easy to see that for all $0 \leq t \leq 1$,
\[(tA+(1-t)B)f(tA+(1-t)B) = (tA+(1-t)B)^{1/2} f(tA+(1-t)B)(tA+(1-t)B)^{1/2}.\]  
(2.3)

Also, the matrix concavity of $f$ implies that
\[tf(A) + (1-t)f(B) \leq f(tA + (1-t)B), \quad 0 \leq t \leq 1.\]  
(2.4)

Combining the relations (2.2), (2.3) and (2.4), we get (2.1).

**Theorem 2.3.** Let $A, B \in \mathbb{M}_n$ be positive semidefinite matrices. Then for all $0 \leq t \leq 1$ and $0 \leq \nu \leq \frac{2}{3}$
\[2\sqrt{t(1-t)}s_j(tA^\nu B^{1-\nu} + (1-t)A^{1-\nu}B^\nu) \leq s_j(tA + (1-t)B).\]  
(2.5)

**Proof.** The proof depends on the fact that the matrices $XY$ and $YX$ have the same eigenvalues. Let $f(x) = x^r$, $0 \leq r \leq 1$. This function is matrix monotone on $[0, \infty)$. Hence from (2.1) and Weyl’s monotonocity theorem we have
\[\lambda_j(tA^{r+1} + (1-t)B^{r+1}) \geq \lambda_j((tA + (1-t)B)(tA^r + (1-t)B^r)).\]  
(2.6)

Except for trivial zeroes the eigenvalues of $(tA + (1-t)B)(tA^r + (1-t)B^r)$ are the same as those of the matrix
\[
\begin{bmatrix}
 tA + (1-t)B & 0 \\
 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
 \sqrt{t}A^{r/2} & \sqrt{1-t}B^{r/2} \\
 \sqrt{1-t}B^{r/2} & \sqrt{1-t}B^{r/2} \\
\end{bmatrix}
\begin{bmatrix}
 \sqrt{t}A^{r/2} & 0 \\
 0 & 0 \\
\end{bmatrix}
\]
and in turn, these are the same as the eigenvalues of
\[
\begin{bmatrix}
 \sqrt{t}A^{r/2} & 0 \\
 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
 tA + (1-t)B & 0 \\
 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
 \sqrt{t}A^{r/2} & \sqrt{1-t}B^{r/2} \\
 \sqrt{1-t}B^{r/2} & \sqrt{1-t}B^{r/2} \\
\end{bmatrix}
\begin{bmatrix}
 0 & 0 \\
 0 & 0 \\
\end{bmatrix}
\]

So, Lemma 2.1 and inequality (2.6) together give
\[\lambda_j(tA^{r+1} + (1-t)B^{r+1}) \geq 2\sqrt{t(1-t)}s_j(A^{r/2}(tA + (1-t)B)B^{r/2})\]
\[= 2\sqrt{t(1-t)}s_j(tA^{1+\frac{r}{2}}B^{r/2} + (1-t)A^{1/2}B^{1+\frac{r}{2}}).\]

Replacing $A$ and $B$ by $A^{1/r+1}$ and $B^{1/r+1}$, respectively, we get from this
\[s_j(tA+(1-t)B) \geq 2\sqrt{(1-t)}s_j(tA^{1+r/2}B^{r/2} + (1-t)A^{1/r+1}B^{1+r/2}), \quad 0 \leq r, t \leq 1.\]

Now, if we put $\nu = \frac{r+2}{2r+2}$, then trivially, we get
\[s_j(tA + (1-t)B) \geq 2\sqrt{(1-t)}s_j(tA^\nu B^{1-\nu} + (1-t)A^{1-\nu}B^\nu),\]
for all $0 \leq t \leq 1$ and $\frac{1}{2} \leq \nu \leq \frac{3}{2}$ and we have proved (2.5) for this special range.

Symmetry, if we put $\nu = \frac{r}{2r + 2}$, then it is easy to see that the inequality (2.5) holds for all $0 \leq t \leq 1$ and $0 \leq \nu \leq \frac{1}{2}$. Hence the proof is complete. □

If in Theorem 2.3, we put $t = \frac{1}{2}$, then we have the following corollary, which obtained by Audenaert in [2] and by Bhatia and Kittaneh in [6].

**Corollary 2.4.** Let $A, B \in \mathbb{M}_n$ be positive semidefinite matrices. Then for all $0 \leq \nu \leq 1$

$$s_j(A^\nu B^{1-\nu} + A^{1-\nu} B^\nu) \leq s_j(A + B).$$

**Corollary 2.5.** Let $A, B \in \mathbb{M}_n$ be positive semidefinite matrices. Then for all $0 \leq t \leq 1$ and $0 \leq \nu \leq \frac{3}{2}$

$$2\sqrt{t(1-t)} \| tA^\nu B^{1-\nu} + (1-t)A^{1-\nu} B^\nu \| \leq \| tA + (1-t)B \|.$$  

For $A \in \mathbb{M}_n$, the numerical radius of $A$ is defined and denoted by

$$\omega(A) = \max \{|x^*Ax| : x \in \mathbb{C}^n, x^*x = 1\}.$$  

The quantity $\omega(A)$ is useful in studying perturbations, convergence, stability, approximation problems, iterative method, etc. For more information see [3, 7]. It is known that $\omega(.)$ is a vector norm on $\mathbb{M}_n$, but is not unitarily invariant. We recall the following results about the numerical radius of matrices which can be found in [8] (see also [10, Chapter 1]).

**Lemma 2.6.** Let $A \in \mathbb{M}_n$ and $\omega(.)$ be the numerical radius. Then the following assertions are true:

(i) $\omega(U^*AU) = \omega(A)$, where $U$ is unitary;
(ii) $\frac{1}{2}||A|| \leq \omega(A) \leq ||A||$;
(iii) $\omega(A) = ||A||$ if (but not only if) $A$ is normal.

Utilizing Lemma 2.6 (parts (ii) and (iii)) and by Corollary 2.5 we obtain the following corollary.

**Corollary 2.7.** Let $A, B \in \mathbb{M}_n$ be positive semidefinite matrices. Then for all $0 \leq t \leq 1$ and $0 \leq \nu \leq \frac{3}{2}$

$$2\sqrt{t(1-t)}\omega(tA^\nu B^{1-\nu} + (1-t)A^{1-\nu} B^\nu) \leq \omega(tA + (1-t)B).$$

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REFERENCES