A Generalized Singular Value Inequality for Heinz Means

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Abstract. In this paper we will generalize a singular value inequality that was proved before. In particular we obtain an inequality for numerical radius as follows:

\[ 2\sqrt{t(1-t)}\omega(tA^\nu B^{1-\nu} + (1-t)A^{1-\nu} B^\nu) \leq \omega(tA + (1-t)B), \]

where, \( A \) and \( B \) are positive semidefinite matrices, \( 0 \leq t \leq 1 \) and \( 0 \leq \nu \leq \frac{3}{2} \).

Keywords: Matrix monotone functions, Numerical radius, Singular values, Unitarily invariant norms.


1. Introduction

Let \( \mathbb{M}_n \) be the algebra of all \( n \times n \) complex matrices. A norm \( \| \cdot \| \) on \( \mathbb{M}_n \) is said to be unitarily invariant if \( \| UAV \| = \| A \| \) for all \( A \in \mathbb{M}_n \) and all unitary \( U, V \in \mathbb{M}_n \). Special examples of such norms are the "Ky Fan norms"

\[ \| A \|_{(k)} = \sum_{j=1}^{k} s_j(A), \quad 1 \leq k \leq n. \]

Note that the operator norm, in this notation, is \( \| A \| = \| A \|_{(1)} = s_1(A) \); see [4] and [9] for more information.
If \( \|A\|_k \leq \|B\|_k \) for \( 1 \leq k \leq n \), then \( \|A\| \leq \|B\| \) for all unitary invariant norms. This is called the "Fan dominance theorem." If \( A \) is a Hermitian element of \( M_n \), then we arrange its eigenvalues in decreasing order as \( \lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A) \). If \( A \) is arbitrary, then its singular values are enumerated as \( s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A) \). These are the eigenvalues of the positive semidefinite matrix \( |A| = (A^*A)^{1/2} \). If \( A \) and \( B \) are Hermitian matrices, and \( A - B \) is positive semidefinite, then we say that \( B \leq A \).

Weyl's monotonicity theorem [4, p. 63] says that if \( A \) and \( B \) are Hermitian matrices, then

\[
A \leq B \implies \lambda_j(A) \leq \lambda_j(B), \quad \text{for all } j = 1, \ldots, n.
\]

Let \( f \) be a real valued function on an interval \( I \). Then \( f \) is said to be matrix monotone if \( A, B \in M_n \) are Hermitian matrices with all their eigenvalues in \( I \) and \( A \geq B \), then \( f(A) \geq f(B) \) and also, \( f \) is said to be matrix convex if

\[
f(tA + (1-t)B) \leq tf(A) + (1-t)f(B), \quad 0 \leq t \leq 1
\]

and matrix concave if

\[
f(tA + (1-t)B) \geq tf(A) + (1-t)f(B), \quad 0 \leq t \leq 1.
\]

In response to a conjecture by Zhan [13], Audenaert [2] has proved that if \( A, B \in M_n \) are positive semidefinite, then the inequality

\[
s_j(A^{1/2}B^{1/2} + B^{1/2}A^{1/2}) \leq s_j(A + B), \quad 1 \leq j \leq n
\]

holds, for all \( 0 \leq \nu \leq 1 \). In this paper we generalize this inequality as follows: If \( A, B \in M_n \) are positive semidefinite matrices, then for all \( 0 \leq t \leq 1 \) and \( 0 \leq \nu \leq \frac{3}{2} \)

\[
2\sqrt{t(1-t)}s_j(tA^{1/2}B^{1/2} + (1-t)B^{1/2}A^{1/2}) \leq s_j(tA + (1-t)B).
\]

For more details about inequalities and their generalizations with their history of origin, the reader may refer to [1, 5, 6, 11, 12, 13].

2. MAIN RESULTS

Lemma 2.1. [14] If \( X = \begin{bmatrix} A & C \\ C^* & B \end{bmatrix} \) is positive, then \( 2s_j(C) \leq s_j(X) \) for all \( 1 \leq j \leq n \).

Theorem 2.2. Let \( f \) be a matrix monotone function on \([0, \infty)\) and \( A \) and \( B \) be positive semidefinite matrices. Then

\[
tAf(A) + (1-t)Bf(B) \geq (tA + (1-t)B)^{1/2}(tf(A) + (1-t)f(B))(tA + (1-t)B)^{1/2}
\]

(2.1)

for all \( 0 \leq t \leq 1 \).

Proof. The function \( f \) is also matrix concave, and \( g(x) = xf(x) \) is matrix convex. (See [4]). The matrix convexity of \( g \) implies the inequality

\[
(tA + (1-t)B)f(tA + (1-t)B) \leq tAf(A) + (1-t)Bf(B), \quad 0 \leq t \leq 1.
\]

(2.2)
Since the matrix $tA + (1 - t)B$ is positive semidefinite, in view of the spectral decomposition theorem, it is easy to see that for all $0 \leq t \leq 1$,

$$(tA + (1 - t)B)f((tA + (1 - t)B)^{1/2}f(tA + (1 - t)B)(tA + (1 - t)B)^{1/2}).$$

(2.3)

Also, the matrix concavity of $f$ implies that

$$tf(A) + (1 - tf(B) \leq f(tA + (1 - t)B), \quad 0 \leq t \leq 1.$$ (2.4)

Combining the relations (2.2), (2.3) and (2.4), we get (2.1). □

**Theorem 2.3.** Let $A, B \in \mathbb{M}_n$ be positive semidefinite matrices. Then for all $0 \leq t \leq 1$ and $0 \leq \nu \leq \frac{a}{2}$,

$$2\sqrt{t(1 - t)s_j(tA^\nu B^{1-\nu} + (1 - t)A^{1-\nu} B^\nu)} \leq s_j(tA + (1 - t)B).$$ (2.5)

**Proof.** The proof depends on the fact that the matrices $XY$ and $YX$ have the same eigenvalues. Let $f(x) = x^r, 0 \leq r \leq 1$. This function is matrix monotone on $[0, \infty)$. Hence from (2.1) and Weyl’s monotonicity theorem we have

$$\lambda_j((tA + (1 - t)B)(tA^r + (1 - t)B^r)) \geq \lambda_j(tA^{r+1} + (1 - t)B^{r+1}) \geq \lambda_j((tA + (1 - t)B)(tA^r + (1 - t)B^r)).$$ (2.6)

Except for trivial zeroes the eigenvalues of $(tA + (1 - t)B)(tA^r + (1 - t)B^r)$ are the same as those of the matrix

$$\begin{bmatrix}
    tA + (1 - t)B & 0 & \sqrt{tA^{r/2}} & \sqrt{1 - t}B^{r/2} \\
    0 & 0 & \sqrt{1 - t}A^{r/2} & 0 \\
    \sqrt{tA^{r/2}} & \sqrt{1 - t}A^{r/2} & 0 & 0 \\
    \sqrt{1 - t}B^{r/2} & 0 & 0 & \sqrt{1 - t}B^{r/2}
\end{bmatrix}
$$

and in turn, these are the same as the eigenvalues of

$$\begin{bmatrix}
    \sqrt{tA^{r/2}} & 0 & \sqrt{t(1 - t)A^{r/2}} & \sqrt{t(1 - t)B^{r/2}} \\
    0 & \sqrt{t(1 - t)B^{r/2}} & 0 & 0 \\
    \sqrt{t(1 - t)A^{r/2}} & \sqrt{t(1 - t)B^{r/2}} & 0 & 0 \\
    0 & 0 & \sqrt{t(1 - t)A^{r/2}} & \sqrt{t(1 - t)B^{r/2}}
\end{bmatrix}.
$$

So, Lemma 2.1 and inequality (2.6) together give

$$\lambda_j((tA^{r+1} + (1 - t)B^{r+1}) \geq 2\sqrt{t(1 - t)s_j((tA^{r/2} + (1 - t)B^{r/2}))}$$

$$= 2\sqrt{t(1 - t)s_j(tA^{1+\frac{r}{2}} B^{\frac{r}{2}} + (1 - t)A^{1-\frac{r}{2}} B^{1+\frac{r}{2}})}.$$

Replacing $A$ and $B$ by $A^{1/r+1}$ and $B^{1/r+1}$, respectively, we get from this

$$s_j(tA + (1 - t)B) \geq 2\sqrt{t(1 - t)s_j((tA^{r+\frac{2r}{r+2}} + (1 - t)A^{\frac{2r}{r+2}} B^{\frac{2r}{r+2}}))}, \quad 0 \leq r, t \leq 1.$$

Now, if we put $\nu = \frac{r + 2}{2r + 2}$, then trivially, we get

$$s_j(tA + (1 - t)B) \geq 2\sqrt{t(1 - t)s_j(tA^\nu B^{1-\nu} + (1 - t)A^{1-\nu} B^\nu)},$$
for all $0 \leq t \leq 1$ and $\frac{1}{2} \leq \nu \leq \frac{3}{2}$ and we have proved (2.5) for this special range.

Symmetry, if we put $\nu = \frac{r}{2r + 2}$, then it is easy to see that the inequality (2.5) holds for all $0 \leq t \leq 1$ and $0 \leq \nu \leq \frac{1}{2}$. Hence the proof is complete. □

If in Theorem 2.3, we put $t = \frac{1}{2}$, then we have the following corollary, which obtained by Audenaert in [2] and by Bhatia and Kittaneh in [6].

**Corollary 2.4.** Let $A, B \in M_n$ be positive semidefinite matrices. Then for all $0 \leq \nu \leq 1$

$$s_j(A^\nu B^{1-\nu} + A^{1-\nu} B^\nu) \leq s_j(A + B).$$

**Corollary 2.5.** Let $A, B \in M_n$ be positive semidefinite matrices. Then for all $0 \leq t \leq 1$ and $0 \leq \nu \leq \frac{3}{2}$

$$2\sqrt{t(1-t)} \left\| tA^\nu B^{1-\nu} + (1-t)A^{1-\nu} B^\nu \right\| \leq \| tA + (1-t)B \| .$$

For $A \in M_n$, the numerical radius of $A$ is defined and denoted by

$$\omega(A) = \max\{|x^*Ax| : x \in \mathbb{C}^n, x^*x = 1\}.$$

The quantity $\omega(A)$ is useful in studying perturbations, convergence, stability, approximation problems, iterative method, etc. For more information see [3, 7]. It is known that $\omega(.)$ is a vector norm on $M_n$, but is not unitarily invariant. We recall the following results about the numerical radius of matrices which can be found in [8] (see also [10, Chapter 1]).

**Lemma 2.6.** Let $A \in M_n$ and $\omega(.)$ be the numerical radius. Then the following assertions are true:

(i) $\omega(U^*AU) = \omega(A)$, where $U$ is unitary;

(ii) $\frac{1}{2} \| A \| \leq \omega(A) \leq \| A \| ;$

(iii) $\omega(A) = \| A \|$ if (but not only if) $A$ is normal.

Utilizing Lemma 2.6 (parts (ii) and (iii)) and by Corollary 2.5 we obtain the following corollary.

**Corollary 2.7.** Let $A, B \in M_n$ be positive semidefinite matrices. Then for all $0 \leq t \leq 1$ and $0 \leq \nu \leq \frac{3}{2}$

$$2\sqrt{t(1-t)}\omega(tA^\nu B^{1-\nu} + (1-t)A^{1-\nu} B^\nu) \leq \omega(tA + (1-t)B).$$

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