A Generalized Singular Value Inequality for Heinz Means

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Abstract. In this paper we will generalize a singular value inequality that was proved before. In particular we obtain an inequality for numerical radius as follows:

$$2\sqrt{t(1-t)}\omega(tA^\nu B^{1-\nu} + (1-t)A^{1-\nu} B^\nu) \leq \omega(tA + (1-t)B),$$

where, $A$ and $B$ are positive semidefinite matrices, $0 \leq t \leq 1$ and $0 \leq \nu \leq \frac{3}{2}$.

Keywords: Matrix monotone functions, Numerical radius, Singular values, Unitarily invariant norms.


1. Introduction

Let $M_n$ be the algebra of all $n \times n$ complex matrices. A norm $\| \cdot \|$ on $M_n$ is said to be unitarily invariant if $\|UAV\| = \|A\|$ for all $A \in M_n$ and all unitary $U, V \in M_n$. Special examples of such norms are the "Ky Fan norms"

$$\|A\|_{(k)} = \sum_{j=1}^{k} s_j(A), \quad 1 \leq k \leq n.$$

Note that the operator norm, in this notation, is $\|A\| = \|A\|_{(1)} = s_1(A)$; see [4] and [9] for more information.

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Received 21 August 2013; Accepted 08 September 2014
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If \(\|A\|_k \leq \|B\|_k\) for \(1 \leq k \leq n\), then \(\|A\| \leq \|B\|\) for all unitary invariant norms. This is called the "Fan dominance theorem." If \(A\) is a Hermitian element of \(M_n\), then we arrange its eigenvalues in decreasing order as \(\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)\). If \(A\) is arbitrary, then its singular values are enumerated as \(s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A)\). These are the eigenvalues of the positive semidefinite matrix \(|A| = (A^* A)^{1/2}\). If \(A\) and \(B\) are Hermitian matrices, and \(A - B\) is positive semidefinite, then we say that \(B \leq A\).

Weyl’s monotonicity theorem [4, p. 63] says that \(B \leq A\) implies \(\lambda_j(A) \leq \lambda_j(B)\), for all \(j = 1, \ldots, n\). Let \(f\) be a real valued function on an interval \(I\). Then \(f\) is said to be matrix monotone if \(A, B \in \mathbb{M}_n\) are Hermitian matrices with all their eigenvalues in \(I\) and \(A \geq B\), then \(f(A) \geq f(B)\) and also, \(f\) is said to be matrix convex if

\[
f(tA + (1 - t)B) \leq tf(A) + (1 - t)f(B), \quad 0 \leq t \leq 1
\]

and matrix concave if

\[
f(tA + (1 - t)B) \geq tf(A) + (1 - t)f(B), \quad 0 \leq t \leq 1.
\]

In response to a conjecture by Zhan [13], Audenaert [2] has proved that if \(A, B \in \mathbb{M}_n\) are positive semidefinite, then the inequality

\[
s_j(A^\nu B^{1-\nu} + A^{1-\nu} B^\nu) \leq s_j(A + B), \quad 1 \leq j \leq n
\]

holds, for all \(0 \leq \nu \leq 1\). In this paper we generalize this inequality as follows: If \(A, B \in \mathbb{M}_n\) are positive semidefinite matrices, then for all \(0 \leq t \leq 1\) and \(0 \leq \nu \leq \frac{3}{2}\)

\[
2\sqrt{t(1-t)}s_j(tA^\nu B^{1-\nu} + (1-t)A^{1-\nu} B^\nu) \leq s_j(tA + (1-t)B).
\]

For more details about inequalities and their generalizations with their history of origin, the reader may refer to [1, 5, 6, 11, 12, 13].

### 2. Main Results

**Lemma 2.1.** [14] If \(X = \begin{bmatrix} A & C \\ C^* & B \end{bmatrix}\) is positive, then \(2s_j(C) \leq s_j(X)\) for all \(1 \leq j \leq n\).

**Theorem 2.2.** Let \(f\) be a matrix monotone function on \([0, \infty)\) and \(A\) and \(B\) be positive semidefinite matrices. Then

\[
tAf(A) + (1-t)Bf(B) \geq (tA + (1-t)B)^{1/2}(tf(A) + (1-t)f(B))(tA + (1-t)B)^{1/2}
\]

for all \(0 \leq t \leq 1\).

**Proof.** The function \(f\) is also matrix concave, and \(g(x) = xf(x)\) is matrix convex. (See [4]). The matrix convexity of \(g\) implies the inequality

\[
(tA + (1-t)B)f(tA + (1-t)B) \leq tAf(A) + (1-t)Bf(B), \quad 0 \leq t \leq 1.
\]
Since the matrix \( tA + (1 - t)B \) is positive semidefinite, in view of the spectral decomposition theorem, it is easy to see that for all \( 0 \leq t \leq 1 \),

\[
(tA + (1 - t)B) f(tA+(1-t)B) = (tA+(1-t)B)^{1/2} f(tA+(1-t)B)(tA+(1-t)B)^{1/2}.
\]

(2.3)

Also, the matrix concavity of \( f \) implies that

\[
tf(A) + (1 - t)f(B) \leq f(tA + (1 - t)B), \quad 0 \leq t \leq 1.
\]

(2.4)

Combining the relations (2.2), (2.3) and (2.4), we get (2.1).

**Theorem 2.3.** Let \( A, B \in \mathbb{M}_n \) be positive semidefinite matrices. Then for all \( 0 \leq t \leq 1 \) and \( 0 \leq \nu \leq \frac{3}{2} \)

\[
2\sqrt{t(1-t)} s_j(tA^{\nu}B^{1-\nu} + (1-t)A^{1-\nu}B^{\nu}) \leq s_j(tA + (1-t)B).
\]

(2.5)

**Proof.** The proof depends on the fact that the matrices \( XY \) and \( YX \) have the same eigenvalues. Let \( f(x) = x^r, \ 0 \leq r \leq 1 \). This function is matrix monotone on \([0, \infty)\). Hence from (2.1) and Weyl’s monotonicity theorem we have

\[
\lambda_j(tA^{r+1} + (1-t)B^{r+1}) \geq \lambda_j((tA + (1-t)B)(tA^r + (1-t)B^r)).
\]

(2.6)

Except for trivial zeroes the eigenvalues of \((tA + (1-t)B)(tA^r + (1-t)B^r)\) are the same as those of the matrix

\[
\begin{bmatrix}
 tA + (1 - t)B & 0 \\
 0 & 0
\end{bmatrix}
\begin{bmatrix}
 \sqrt{t}A^{r/2} & \sqrt{1-t}B^{r/2} \\
 \sqrt{1-t}B^{r/2} & 0
\end{bmatrix}
\begin{bmatrix}
 \sqrt{t}A^{r/2} & 0 \\
 0 & \sqrt{1-t}B^{r/2}
\end{bmatrix}
\]

and in turn, these are the same as the eigenvalues of

\[
\begin{bmatrix}
 \sqrt{t}A^{r/2} & (tA + (1 - t)B)A^{r/2} \\
 (tA + (1 - t)B)A^{r/2} & \sqrt{1-t}B^{r/2}
\end{bmatrix}
\begin{bmatrix}
 \sqrt{t}A^{r/2} & 0 \\
 0 & \sqrt{1-t}B^{r/2}
\end{bmatrix}
\begin{bmatrix}
 \sqrt{t}A^{r/2} & \sqrt{1-t}B^{r/2} \\
 \sqrt{1-t}B^{r/2} & 0
\end{bmatrix}
\]

\[
= \begin{bmatrix}
 tA^{r/2}(tA + (1 - t)B)A^{r/2} & \sqrt{t(1-t)}A^{r/2}(tA + (1 - t)B)B^{r/2} \\
 \sqrt{t(1-t)}B^{r/2}(tA + (1 - t)B)A^{r/2} & (1-t)B^{r/2}(tA + (1 - t)B)B^{r/2}
\end{bmatrix}
\]

So, Lemma 2.1 and inequality (2.6) together give

\[
\lambda_j(tA^{r+1} + (1-t)B^{r+1}) \geq 2\sqrt{t(1-t)} s_j(A^{r/2}(tA + (1-t)B)B^{r/2})
\]

\[
= 2\sqrt{t(1-t)} s_j(tA^{1+\frac{r}{2}}B^{r/2} + (1-t)A^{r/2}B^{1+\frac{r}{2}}).
\]

Replacing \( A \) and \( B \) by \( A^{1/r+1} \) and \( B^{1/r+1} \), respectively, we get from this

\[
s_j(tA + (1-t)B) \geq 2\sqrt{t(1-t)} s_j(tA^{\frac{r+2}{2}}B^{\frac{r}{2r+2}} + (1-t)A^{\frac{r}{2r+2}}B^{\frac{r+2}{2r+2}}), \ 0 \leq r, t \leq 1.
\]

Now, if we put \( \nu = \frac{r+2}{2r+2} \), then trivially, we get

\[
s_j(tA + (1-t)B) \geq 2\sqrt{t(1-t)} s_j(tA^{\nu}B^{1-\nu} + (1-t)A^{1-\nu}B^{\nu}),
\]
for all $0 \leq t \leq 1$ and $\frac{1}{2} \leq \nu \leq \frac{3}{2}$ and we have proved (2.5) for this special range.

Symmetry, if we put $\nu = \frac{r}{2r + 2}$, then it is easy to see that the inequality (2.5) holds for all for all $0 \leq t \leq 1$ and $0 \leq \nu \leq \frac{1}{2}$. Hence the proof is complete. □

If in Theorem 2.3, we put $t = \frac{1}{2}$, then we have the following corollary, which obtained by Audenaert in [2] and by Bhatia and Kittaneh in [6].

**Corollary 2.4.** Let $A, B \in \mathbb{M}_n$ be positive semidefinite matrices. Then for all $0 \leq \nu \leq 1$

$$s_j(A^{\nu}B^{1-\nu} + A^{1-\nu}B^{\nu}) \leq s_j(A + B).$$

**Corollary 2.5.** Let $A, B \in \mathbb{M}_n$ be positive semidefinite matrices. Then for all $0 \leq t \leq 1$ and $0 \leq \nu \leq \frac{3}{2}$

$$2\sqrt{t(1-t)} \left\| tA^{\nu}B^{1-\nu} + (1-t)A^{1-\nu}B^{\nu} \right\| \leq \left\| tA + (1-t)B \right\|.$$

For $A \in \mathbb{M}_n$, the numerical radius of $A$ is defined and denoted by

$$\omega(A) = \max\{|x^*Ax| : x \in \mathbb{C}^n, x^*x = 1\}.$$

The quantity $\omega(A)$ is useful in studying perturbations, convergence, stability, approximation problems, iterative method, etc. For more information see [3, 7]. It is known that $\omega(.)$ is a vector norm on $\mathbb{M}_n$, but is not unitarily invariant. We recall the following results about the numerical radius of matrices which can be found in [8] (see also [10, Chapter 1]).

**Lemma 2.6.** Let $A \in \mathbb{M}_n$ and $\omega(.)$ be the numerical radius. Then the following assertions are true:

(i) $\omega(U^*AU) = \omega(A)$, where $U$ is unitary;
(ii) $\frac{1}{2}\|A\| \leq \omega(A) \leq \|A\|;
(iii) $\omega(A) = \|A\|$ if (but not only if) $A$ is normal.

Utilizing Lemma 2.6 (parts (ii) and (iii)) and by Corollary 2.5 we obtain the following corollary.

**Corollary 2.7.** Let $A, B \in \mathbb{M}_n$ be positive semidefinite matrices. Then for all $0 \leq t \leq 1$ and $0 \leq \nu \leq \frac{3}{2}$

$$2\sqrt{t(1-t)}\omega(tA^{\nu}B^{1-\nu} + (1-t)A^{1-\nu}B^{\nu}) \leq \omega(tA + (1-t)B).$$

**ACKNOWLEDGMENT**

The author would like to thank the anonymous referee for careful reading and the helpful comments improving this paper.
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References