

## A Generalized Singular Value Inequality for Heinz Means

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ABSTRACT. In this paper we will generalize a singular value inequality that was proved before. In particular we obtain an inequality for numerical radius as follows:

$$2\sqrt{t(1-t)}\omega(tA^\nu B^{1-\nu} + (1-t)A^{1-\nu} B^\nu) \leq \omega(tA + (1-t)B),$$

where,  $A$  and  $B$  are positive semidefinite matrices,  $0 \leq t \leq 1$  and  $0 \leq \nu \leq \frac{3}{2}$ .

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### 1. INTRODUCTION

Let  $\mathbb{M}_n$  be the algebra of all  $n \times n$  complex matrices. A norm  $\|\cdot\|$  on  $\mathbb{M}_n$  is said to be unitarily invariant if  $\|UAV\| = \|A\|$  for all  $A \in \mathbb{M}_n$  and all unitary  $U, V \in \mathbb{M}_n$ . Special examples of such norms are the "Ky Fan norms"

$$\|A\|_{(k)} = \sum_{j=1}^k s_j(A), \quad 1 \leq k \leq n.$$

Note that the operator norm, in this notation, is  $\|A\| = \|A\|_{(1)} = s_1(A)$ ; see [4] and [9] for more information.

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If  $\|A\|_{(k)} \leq \|B\|_{(k)}$  for  $1 \leq k \leq n$ , then  $\|A\| \leq \|B\|$  for all unitary invariant norms. This is called the "Fan dominance theorem." If  $A$  is a Hermitian element of  $\mathbb{M}_n$ , then we arrange its eigenvalues in decreasing order as  $\lambda_1(A) \geq \lambda_2(A) \geq \dots \geq \lambda_n(A)$ . If  $A$  is arbitrary, then its singular values are enumerated as  $s_1(A) \geq s_2(A) \geq \dots \geq s_n(A)$ . These are the eigenvalues of the positive semidefinite matrix  $|A| = (A^*A)^{1/2}$ . If  $A$  and  $B$  are Hermitian matrices, and  $A - B$  is positive semidefinite, then we say that  $B \leq A$ .

Weyl's monotonicity theorem [4, p. 63] says that  $B \leq A$  implies  $\lambda_j(A) \leq \lambda_j(B)$ , for all  $j = 1, \dots, n$ . Let  $f$  be a real valued function on an interval  $I$ . Then  $f$  is said to be matrix monotone if  $A, B \in \mathbb{M}_n$  are Hermitian matrices with all their eigenvalues in  $I$  and  $A \geq B$ , then  $f(A) \geq f(B)$  and also,  $f$  is said to be matrix convex if

$$f(tA + (1-t)B) \leq tf(A) + (1-t)f(B), \quad 0 \leq t \leq 1$$

and matrix concave if

$$f(tA + (1-t)B) \geq tf(A) + (1-t)f(B), \quad 0 \leq t \leq 1.$$

In response to a conjecture by Zhan [13], Audenaert [2] has proved that if  $A, B \in \mathbb{M}_n$  are positive semidefinite, then the inequality

$$s_j(A^\nu B^{1-\nu} + A^{1-\nu} B^\nu) \leq s_j(A + B), \quad 1 \leq j \leq n$$

holds, for all  $0 \leq \nu \leq 1$ . In this paper we generalize this inequality as follows: If  $A, B \in \mathbb{M}_n$  are positive semidefinite matrices, then for all  $0 \leq t \leq 1$  and  $0 \leq \nu \leq \frac{3}{2}$

$$2\sqrt{t(1-t)}s_j(tA^\nu B^{1-\nu} + (1-t)A^{1-\nu} B^\nu) \leq s_j(tA + (1-t)B).$$

For more details about inequalities and their generalizations with their history of origin, the reader may refer to [1, 5, 6, 11, 12, 13].

## 2. MAIN RESULTS

**Lemma 2.1.** [14] If  $X = \begin{bmatrix} A & C \\ C^* & B \end{bmatrix}$  is positive, then  $2s_j(C) \leq s_j(X)$  for all  $1 \leq j \leq n$ .

**Theorem 2.2.** Let  $f$  be a matrix monotone function on  $[0, \infty)$  and  $A$  and  $B$  be positive semidefinite matrices. Then

$$tAf(A) + (1-t)Bf(B) \geq (tA + (1-t)B)^{1/2} (tf(A) + (1-t)f(B)) (tA + (1-t)B)^{1/2} \quad (2.1)$$

for all  $0 \leq t \leq 1$ .

*Proof.* The function  $f$  is also matrix concave, and  $g(x) = xf(x)$  is matrix convex. (See [4]). The matrix convexity of  $g$  implies the inequality

$$(tA + (1-t)B)f(tA + (1-t)B) \leq tAf(A) + (1-t)Bf(B), \quad 0 \leq t \leq 1. \quad (2.2)$$

Since the matrix  $tA + (1 - t)B$  is positive semidefinite, in view of the spectral decomposition theorem, it is easy to see that for all  $0 \leq t \leq 1$ ,

$$(tA+(1-t)B)f(tA+(1-t)B) = (tA+(1-t)B)^{1/2}f(tA+(1-t)B)(tA+(1-t)B)^{1/2}. \tag{2.3}$$

Also, the matrix concavity of  $f$  implies that

$$tf(A) + (1 - t)f(B) \leq f(tA + (1 - t)B), \quad 0 \leq t \leq 1. \tag{2.4}$$

Combining the relations (2.2), (2.3) and (2.4), we get (2.1).  $\square$

**Theorem 2.3.** *Let  $A, B \in \mathbb{M}_n$  be positive semidefinite matrices. Then for all  $0 \leq t \leq 1$  and  $0 \leq \nu \leq \frac{3}{2}$*

$$2\sqrt{t(1-t)}s_j(tA^\nu B^{1-\nu} + (1-t)A^{1-\nu}B^\nu) \leq s_j(tA + (1-t)B). \tag{2.5}$$

*Proof.* The proof depends on the fact that the matrices  $XY$  and  $YX$  have the same eigenvalues. Let  $f(x) = x^r, 0 \leq r \leq 1$ . This function is matrix monotone on  $[0, \infty)$ . Hence from (2.1) and Weyl's monotonicity theorem we have

$$\lambda_j(tA^{r+1} + (1-t)B^{r+1}) \geq \lambda_j((tA + (1-t)B)(tA^r + (1-t)B^r)). \tag{2.6}$$

Except for trivial zeroes the eigenvalues of  $(tA + (1-t)B)(tA^r + (1-t)B^r)$  are the same as those of the matrix

$$\begin{aligned} & \begin{bmatrix} tA + (1-t)B & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{t}A^{r/2} & \sqrt{1-t}B^{r/2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{t}A^{r/2} & 0 \\ \sqrt{1-t}B^{r/2} & 0 \end{bmatrix} \\ & \text{and in turn, these are the same as the eigenvalues of} \\ & \begin{bmatrix} \sqrt{t}A^{r/2} & 0 \\ \sqrt{1-t}B^{r/2} & 0 \end{bmatrix} \begin{bmatrix} tA + (1-t)B & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{t}A^{r/2} & \sqrt{1-t}B^{r/2} \\ 0 & 0 \end{bmatrix} \\ & = \begin{bmatrix} tA^{r/2}(tA + (1-t)B)A^{r/2} & \sqrt{t(1-t)}A^{r/2}(tA + (1-t)B)B^{r/2} \\ \sqrt{t(1-t)}B^{r/2}(tA + (1-t)B)A^{r/2} & (1-t)B^{r/2}(tA + (1-t)B)B^{r/2} \end{bmatrix}. \end{aligned}$$

So, Lemma 2.1 and inequality (2.6) together give

$$\begin{aligned} \lambda_j(tA^{r+1} + (1-t)B^{r+1}) & \geq 2\sqrt{t(1-t)}s_j(A^{r/2}(tA + (1-t)B)B^{r/2}) \\ & = 2\sqrt{t(1-t)}s_j(tA^{1+\frac{r}{2}}B^{r/2} + (1-t)A^{r/2}B^{1+\frac{r}{2}}). \end{aligned}$$

Replacing  $A$  and  $B$  by  $A^{1/r+1}$  and  $B^{1/r+1}$ , respectively, we get from this

$$s_j(tA+(1-t)B) \geq 2\sqrt{t(1-t)}s_j(tA^{\frac{r+2}{2r+2}}B^{\frac{r}{2r+2}}+(1-t)A^{\frac{r}{2r+2}}B^{\frac{2+r}{2r+2}}), \quad 0 \leq r, t \leq 1.$$

Now, if we put  $\nu = \frac{r+2}{2r+2}$ , then trivially, we get

$$s_j(tA + (1-t)B) \geq 2\sqrt{t(1-t)}s_j(tA^\nu B^{1-\nu} + (1-t)A^{1-\nu}B^\nu),$$

for all  $0 \leq t \leq 1$  and  $\frac{1}{2} \leq \nu \leq \frac{3}{2}$  and we have proved (2.5) for this special range.

Symmetry, if we put  $\nu = \frac{r}{2r+2}$ , then it is easy to see that the inequality (2.5) holds for all for all  $0 \leq t \leq 1$  and  $0 \leq \nu \leq \frac{1}{2}$ . Hence the proof is complete.  $\square$

If in Theorem 2.3, we put  $t = \frac{1}{2}$ , then we have the following corollary, which obtained by Audenaert in [2] and by Bhatia and Kittaneh in [6].

**Corollary 2.4.** *Let  $A, B \in \mathbb{M}_n$  be positive semidefinite matrices. Then for all  $0 \leq \nu \leq 1$*

$$s_j(A^\nu B^{1-\nu} + A^{1-\nu} B^\nu) \leq s_j(A + B).$$

**Corollary 2.5.** *Let  $A, B \in \mathbb{M}_n$  be positive semidefinite matrices. Then for all  $0 \leq t \leq 1$  and  $0 \leq \nu \leq \frac{3}{2}$*

$$2\sqrt{t(1-t)} \left\| \|tA^\nu B^{1-\nu} + (1-t)A^{1-\nu} B^\nu\| \right\| \leq \|tA + (1-t)B\|.$$

For  $A \in \mathbb{M}_n$ , the numerical radius of  $A$  is defined and denoted by

$$\omega(A) = \max\{|x^* Ax| : x \in \mathbb{C}^n, x^* x = 1\}.$$

The quantity  $\omega(A)$  is useful in studying perturbations, convergence, stability, approximation problems, iterative method, etc. For more information see [3, 7]. It is known that  $\omega(\cdot)$  is a vector norm on  $\mathbb{M}_n$ , but is not unitarily invariant. We recall the following results about the numerical radius of matrices which can be found in [8] (see also [10, Chapter 1]).

**Lemma 2.6.** *Let  $A \in \mathbb{M}_n$  and  $\omega(\cdot)$  be the numerical radius. Then the following assertions are true:*

- (i)  $\omega(U^*AU) = \omega(A)$ , where  $U$  is unitary;
- (ii)  $\frac{1}{2}\|A\| \leq \omega(A) \leq \|A\|$ ;
- (iii)  $\omega(A) = \|A\|$  if (but not only if)  $A$  is normal.

Utilizing Lemma 2.6 (parts (ii) and (iii)) and by Corollary 2.5 we obtain the following corollary.

**Corollary 2.7.** *Let  $A, B \in \mathbb{M}_n$  be positive semidefinite matrices. Then for all  $0 \leq t \leq 1$  and  $0 \leq \nu \leq \frac{3}{2}$*

$$2\sqrt{t(1-t)} \omega(tA^\nu B^{1-\nu} + (1-t)A^{1-\nu} B^\nu) \leq \omega(tA + (1-t)B).$$

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