A Generalized Singular Value Inequality for Heinz Means

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Abstract. In this paper we will generalize a singular value inequality that was proved before. In particular we obtain an inequality for numerical radius as follows:

\[ 2\sqrt{t(1-t)}\omega(tA^\nu B^{1-\nu} + (1-t)A^{1-\nu}B^\nu) \leq \omega(tA + (1-t)B), \]

where, \( A \) and \( B \) are positive semidefinite matrices, \( 0 \leq t \leq 1 \) and \( 0 \leq \nu \leq \frac{3}{2} \).

Keywords: Matrix monotone functions, Numerical radius, Singular values, Unitarily invariant norms.


1. Introduction

Let \( \mathbb{M}_n \) be the algebra of all \( n \times n \) complex matrices. A norm \( \| \cdot \| \) on \( \mathbb{M}_n \) is said to be unitarily invariant if \( \| UAV \| = \| A \| \) for all \( A \in \mathbb{M}_n \) and all unitary \( U, V \in \mathbb{M}_n \). Special examples of such norms are the "Ky Fan norms"

\[ \| A \|_{(k)} = \sum_{j=1}^{k} s_j(A), \quad 1 \leq k \leq n. \]

Note that the operator norm, in this notation, is \( \| A \| = \| A \|_{(1)} = s_1(A) \); see [4] and [9] for more information.

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If \( \|A\|_{(k)} \leq \|B\|_{(k)} \) for \( 1 \leq k \leq n \), then \( \|A\| \leq \|B\| \) for all unitary invariant norms. This is called the "Fan dominance theorem." If \( A \) is a Hermitian element of \( \mathbb{M}_n \), then we arrange its eigenvalues in decreasing order as \( \lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A) \). If \( A \) is arbitrary, then its singular values are enumerated as \( s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A) \). These are the eigenvalues of the positive semidefinite matrix \( |A| = (A^*A)^{1/2} \). If \( A \) and \( B \) are Hermitian matrices, and \( A - B \) is positive semidefinite, then we say that \( B \leq A \).

Weyl's monotonicity theorem [4, p. 63] says that \( \lambda_j(A) \leq \lambda_j(B) \), for all \( j = 1, \ldots, n \). Let \( f \) be a real valued function on an interval \( I \). Then \( f \) is said to be matrix monotone if \( A, B \in \mathbb{M}_n \) are Hermitian matrices with all their eigenvalues in \( I \) and \( A \geq B \), then \( f(A) \geq f(B) \) and also, \( f \) is said to be matrix convex if
\[
f(tA + (1 - t)B) \leq tf(A) + (1 - t)f(B), \quad 0 \leq t \leq 1
\]
and matrix concave if
\[
f(tA + (1 - t)B) \geq tf(A) + (1 - t)f(B), \quad 0 \leq t \leq 1.
\]

In response to a conjecture by Zhan [13], Audenaert [2] has proved that if \( A, B \in \mathbb{M}_n \) are positive semidefinite, then the inequality
\[
s_j(A^\nu B^{1-\nu} + A^{1-\nu}B^\nu) \leq s_j(A + B), \quad 1 \leq j \leq n
\]
holds, for all \( 0 \leq \nu \leq 1 \). In this paper we generalize this inequality as follows:

If \( A, B \in \mathbb{M}_n \) are positive semidefinite matrices, then for all \( 0 \leq t \leq 1 \) and \( 0 \leq \nu \leq \frac{3}{2} \)
\[
2\sqrt{t(1-t)s_j(tA^\nu B^{1-\nu} + (1-t)A^{1-\nu}B^\nu)} \leq s_j(tA + (1-t)B).
\]

For more details about inequalities and their generalizations with their history of origin, the reader may refer to [1, 5, 6, 11, 12, 13].

2. Main Results

Lemma 2.1. [14] If \( X = \begin{bmatrix} A & C \\ C^* & B \end{bmatrix} \) is positive, then \( 2s_j(C) \leq s_j(X) \) for all \( 1 \leq j \leq n \).

Theorem 2.2. Let \( f \) be a matrix monotone function on \([0, \infty)\) and \( A \) and \( B \) be positive semidefinite matrices. Then
\[
tAf(A) + (1-t)Bf(B) \geq (tA + (1-t)B)^{1/2}(tf(A) + (1-t)f(B))(tA + (1-t)B)^{1/2}
\]
(2.1)
for all \( 0 \leq t \leq 1 \).

Proof. The function \( f \) is also matrix concave, and \( g(x) = xf(x) \) is matrix convex. (See [4]). The matrix convexity of \( g \) implies the inequality
\[
(tA + (1-t)B)f(tA + (1-t)B) \leq tAf(A) + (1-t)Bf(B), \quad 0 \leq t \leq 1.
\]
(2.2)
Since the matrix \( tA + (1 - t)B \) is positive semidefinite, in view of the spectral decomposition theorem, it is easy to see that for all \( 0 \leq t \leq 1 \),
\[
(tA + (1 - t)B)f(tA + (1 - t)B) = (tA + (1 - t)B)^{1/2}f(tA + (1 - t)B)(tA + (1 - t)B)^{1/2}.
\]
(2.3)
Also, the matrix concavity of \( f \) implies that
\[
 tf(A) + (1 - t)f(B) \leq f(tA + (1 - t)B), \quad 0 \leq t \leq 1. \tag{2.4}
\]
Combining the relations (2.2), (2.3) and (2.4), we get (2.1).
\[\square\]

**Theorem 2.3.** Let \( A, B \in \mathbb{M}_n \) be positive semidefinite matrices. Then for all \( 0 \leq t \leq 1 \) and \( 0 \leq \nu \leq \frac{r}{2} \)
\[
2\sqrt{t(1-t)}s_j(tA^\nu B^{1-\nu} + (1-t)A^{1-\nu}B^\nu) \leq s_j(tA + (1 - t)B). \tag{2.5}
\]

**Proof.** The proof depends on the fact that the matrices \( XY \) and \( YX \) have the same eigenvalues. Let \( f(x) = x^r, 0 \leq r \leq 1 \). This function is matrix monotone on \([0, \infty)\). Hence from (2.1) and Weyl’s monotonicity theorem we have
\[
\lambda_j(tA^{r+1} + (1-t)B^{r+1}) \geq \lambda_j ((tA + (1-t)B)(tA^r + (1-t)B^r)). \tag{2.6}
\]
Except for trivial zeroes the eigenvalues of \((tA + (1-t)B)(tA^r + (1-t)B^r)\) are the same as those of the matrix
\[
\begin{bmatrix}
tA + (1-t)B & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\sqrt{t}A^{r/2} & \sqrt{1-t}B^{r/2} \\
\sqrt{1-t}B^{r/2} & \sqrt{1-t}B^{r/2}
\end{bmatrix}
\begin{bmatrix}
\frac{\sqrt{t}}{\sqrt{1-t}}A^{r/2} & 0 \\
0 & \frac{\sqrt{1-t}}{\sqrt{1-t}}B^{r/2}
\end{bmatrix}
\]
n and in turn, these are the same as the eigenvalues of
\[
\begin{bmatrix}
\sqrt{t}A^{r/2} & 0 \\
0 & \sqrt{1-t}B^{r/2}
\end{bmatrix}
\begin{bmatrix}
tA + (1-t)B & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\sqrt{t}A^{r/2} & \sqrt{1-t}B^{r/2} \\
\sqrt{1-t}B^{r/2} & \sqrt{1-t}B^{r/2}
\end{bmatrix}
\]
\[
= \begin{bmatrix}
tA^{r/2}(tA + (1-t)B)A^{r/2} & \sqrt{t(1-t)}A^{r/2}(tA + (1-t)B)B^{r/2} \\
\sqrt{t(1-t)}B^{r/2}(tA + (1-t)B)A^{r/2} & (1-t)B^{r/2}(tA + (1-t)B)B^{r/2}
\end{bmatrix}.
\]
So, Lemma 2.1 and inequality (2.6) together give
\[
\lambda_j(tA^{r+1} + (1-t)B^{r+1}) \geq 2\sqrt{t(1-t)}s_j(A^{r/2}(tA + (1-t)B)B^{r/2})
\]
\[
= 2\sqrt{t(1-t)}s_j(tA^{1+\frac{r}{2}}B^{r/2} + (1-t)A^{r/2}B^{1+\frac{r}{2}}).
\]
Replacing \( A \) and \( B \) by \( A^{1/r+1} \) and \( B^{1/r+1} \), respectively, we get from this
\[
s_j(tA + (1-t)B) \geq 2\sqrt{t(1-t)}s_j(tA^{\frac{r+2}{r+2}}B^{\frac{r}{r+2}} + (1-t)A^{\frac{r}{r+2}}B^{\frac{r+2}{r+2}}), 0 \leq r, t \leq 1.
\]
Now, if we put \( \nu = \frac{r + 2}{2r + 2} \), then trivially, we get
\[
s_j(tA + (1-t)B) \geq 2\sqrt{t(1-t)}s_j(tA^\nu B^{1-\nu} + (1-t)A^{1-\nu}B^\nu),
\]
for all \( 0 \leq t \leq 1 \) and \( \frac{1}{2} \leq \nu \leq \frac{3}{2} \) and we have proved (2.5) for this special range.

Symmetry, if we put \( \nu = \frac{r}{2r + 2} \), then it is easy to see that the inequality (2.5) holds for all \( 0 \leq t \leq 1 \) and \( 0 \leq \nu \leq \frac{1}{2} \). Hence the proof is complete. \( \square \)

If in Theorem 2.3, we put \( t = \frac{1}{2} \), then we have the following corollary, which obtained by Audenaert in [2] and by Bhatia and Kittaneh in [6].

**Corollary 2.4.** Let \( A, B \in M_n \) be positive semidefinite matrices. Then for all \( 0 \leq \nu \leq 1 \)

\[ s_j(A^{\nu}B^{1-\nu} + A^{1-\nu}B^{\nu}) \leq s_j(A + B). \]

**Corollary 2.5.** Let \( A, B \in M_n \) be positive semidefinite matrices. Then for all \( 0 \leq t \leq 1 \) and \( 0 \leq \nu \leq \frac{3}{2} \)

\[ 2\sqrt{t(1-t)} \left\| tA^{\nu}B^{1-\nu} + (1-t)A^{1-\nu}B^{\nu} \right\| \leq \left\| tA + (1-t)B \right\|. \]

For \( A \in M_n \), the numerical radius of \( A \) is defined and denoted by

\[ \omega(A) = \max\{ |x^*Ax| : x \in \mathbb{C}^n, x^*x = 1 \}. \]

The quantity \( \omega(A) \) is useful in studying perturbations, convergence, stability, approximation problems, iterative method, etc. For more information see [3, 7]. It is known that \( \omega(.) \) is a vector norm on \( M_n \), but is not unitarily invariant. We recall the following results about the numerical radius of matrices which can be found in [8] (see also [10, Chapter 1]).

**Lemma 2.6.** Let \( A \in M_n \) and \( \omega(.) \) be the numerical radius. Then the following assertions are true:

(i) \( \omega(U^*AU) = \omega(A) \), where \( U \) is unitary;
(ii) \( \frac{1}{2}\|A\| \leq \omega(A) \leq \|A\|\);
(iii) \( \omega(A) = \|A\| \) if (but not only if) \( A \) is normal.

Utilizing Lemma 2.6 (parts (ii) and (iii)) and by Corollary 2.5 we obtain the following corollary.

**Corollary 2.7.** Let \( A, B \in M_n \) be positive semidefinite matrices. Then for all \( 0 \leq t \leq 1 \) and \( 0 \leq \nu \leq \frac{3}{2} \)

\[ 2\sqrt{t(1-t)}\omega(tA^{\nu}B^{1-\nu} + (1-t)A^{1-\nu}B^{\nu}) \leq \omega(tA + (1-t)B). \]

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