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Applying Legendre Wavelet Method with Regularization for a Class of Singular Boundary Value Problems

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Abstract. In this paper Legendre wavelet bases have been used for finding approximate solutions to singular boundary value problems arising in physiology. When the number of basis functions are increased the algebraic system of equations would be ill-conditioned (because of the singularity), to overcome this for large M, we use some kind of Tikhonov regularization. Examples from applied sciences are presented to demonstrate the efficiency and accuracy of the method.

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1. Introduction

Many numerical treatments for singular boundary value problems have been emerged in recent years [7, 10, 15, 24, 19, 1]. Recently Kazemi Nasab has applied wavelet analysis method with Chebyshev wavelets for solving some singular differential equations [16]. In their method Chebyshev wavelets operational matrix of derivations is proposed for constructing the linear system. Wavelet Galerkin method has also applied to some nonlinear singular boundary value problems by Nosrati in [17]. They have used the quasilinearization technique to reduce the given nonlinear problem to a sequence of linear problems. SVBP's has also been solved and analyzed by other methods. Some examples for recent works are [18, 20, 21, 14]. In this paper we apply the Galerkin method [11, 9], with Legendre wavelet bases for finding an approximate solution to a special kind of singular boundary value problems. Legendre wavelet bases have been applied to many equations [28, 6, 26], and the results demonstrate the accuracy and efficiency of these bases. The main characteristic of this approach is that it reduces functional equations to a system of algebraic equations, that greatly simplify the problem. This paper is organized as follows: In section 2 we introduce Legendre wavelets and it's properties, in section 3 we present the main BVP and discuss how to apply Legendre wavelet bases to this equation. Section 4 is devoted to examples where we have compared results with other methods.

2. Legendre Wavelet

Wavelet theory is an improvement subject of Fourier analysis. The mathematical properties of wavelets has been studied and organized by Daubechies [8], for the first time. She built an orthogonal wavelet basis with some degree of smoothness. Wavelet bases, especially multiwavelets, have been applied for solving different kinds of functional equations. Some of wavelet bases are constructed based on orthogonal polynomials, among them are Legendre wavelets [6, 26] and Chebyshev wavelets [3, 4]. These wavelet bases are orthogonal and have compact supports, that makes them more appropriate for solving boundary value problems.
Clifford wavelets are another special case of wavelet bases [2]. To construct a wavelet basis one uses dilation and translations of a single function (with some properties), called the mother wavelet. When the dilation parameter a and translation parameter b vary continuously we have the following family of continuous wavelets as [12],

$$\psi_{a,b}(x) = |a|^{-1/2}\psi(\frac{t-a}{b}),\ a, b \in \mathbb{R},\ a \neq 0.$$

If we restrict the parameters a and b to take discrete values $a = a_0^{-k},\ b = nb_0a_0^{-k}$, where $a_0 > 0,\ b_0 > 0$, and n, k positive integers, then one has the following family of discrete wavelets:

$$\psi_{k,n}(x) = |a_0|^{k/2}\psi(a_0^k t - nb_0),$$

where $\psi_{k,n}(x)$ forms a wavelet basis for $L^2(R)$. In particular, when $a_0 = 2$ and $b_0 = 1$, then $\psi_{k,n}(x)$ forms an orthonormal basis [12].
Legendre wavelets $\psi_{k,n}(x) = \psi(k, \hat{n}, m, t)$ have four arguments; $\hat{n} = 2n-1,\ n = 1, 2, ..., 2^{k-1},\ k$ can assume any positive integer, m is the order of Legendre polynomials and t is the normalized time. They are defined on the interval $[0, 1]$ as:

$$\psi_{n,m}(x) = \begin{cases} (m+1/2)^{1/2} 2^{k/2} P_m(2^k t - \hat{n}), & \frac{\hat{n}-1}{2^k} \le t \le \frac{\hat{n}+1}{2^k}, \\ 0, & otherwise, \end{cases} \tag{2.1}$$

where $m = 0, 1, ..., M-1,\ n = 1, 2, ..., 2^{k-1}$. In equation (2.1), the coefficient $(m+1)^{1/2}$ is for orthonormality, the dilation parameter is $a = 2^{-k}$, and translation parameter is $b = \hat{n}2^{-k}$. Here, $P_m(x)$ is the well-known Legendre polynomial of order m, which is orthogonal with respect to the weight function $w(x) = 1$ on interval $[-1, 1]$ and satisfy the following recursive formula [5]:

$$P_0(x) = 1,\ P_1(x) = t,$$

$$P_{m+1}(x) = (\frac{2m+1}{m+1})tP_m(x) - (\frac{m}{m+1})P_{m-1}(x),\ m = 1, 2, 3, \ldots.$$

A function $f(x)$ defined over $[0, 1)$ may be expanded as [25]:

$$f(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}(x), \tag{2.2}$$

because of orthonormality we have $c_{n,m} = (f(x), \psi_{n,m}(x))$, where $(,)$ denotes the inner product.
If the infinite series in equation (2.2) is truncated, then it can be written as

$$f(x) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(x) = C^T\Psi(x), \tag{2.3}$$

where C and $\Psi(x)$ are $2^{k-1}M \times 1$ matrices given by

$$C = [c_{10}, ..., c_{1M-1}, c_{20}, ..., c_{2M-1}, ..., c_{2^{k-1}0}, ..., c_{2^{k-1}M-1}]^T, \tag{2.4}$$

$$\Psi(x) = [\psi_{10}(x), ..., \psi_{1M-1}, \psi_{20}, ..., \psi_{2M-1}, ..., \psi_{2^{k-1}0}, ..., \psi_{2^{k-1}M-1}]^T.$$

3. Singular Boundary Value Problem

3.1. **The main equation.** In this paper we consider a nonlinear singular BVP of the form [1],

$$(p(x)y')' = p(x)f(x, y),\ x \in (0, 1], \tag{3.1}$$

with boundary conditions (BCs),

$$y'(0) = 0,\ \alpha y(1) + \beta y'(1) = \gamma, \tag{3.2}$$

or

$$y(0)=A,\ \alpha y(1)+\beta y'(1)=\gamma, \tag{3.3}$$

where $p(x)=x^b g(x),\ x\in[0,1]$. Here $\alpha>0,\ \beta\geq 0$, and A and γ are finite constants. Also, the following resrictions are imposed on $p(x)$ and $f(x,y)$:

(I) $p(x)>0$ on $[0,1],\ p(x)\in C^1(0,1]$, and $\frac{1}{g(x)}$ is analytic in $\{z\ s.t.\ |z|<r\}$ for some $r>1$.

(II) $f(x,y)\in[0,1]\times R$, is continuous, $\frac{\partial f}{\partial y}$ exists, continuous and nonnegative for all $(x,y)\in[0,1]\times R$.

The existence-uniqueness of the solution to equation (3.1) has been established for BCs: $y(0)=A$ and $y(1)=B$, with $0\leq b<1$, and BC's: $y'(0)=0$ and $y(1)=B$ with $b\geq 0$, provided that $\frac{xp'}{p}$ is analytic in $\{z\ s.t.\ |z|<r\}$ for some $r>1$ [22, 23].
Equation (3.1) arises in the study of tumor growth problems, steady-state oxygen diffusion in a cell with Michaleis-Menten uptake kinetics and distribution of heat sources in the human head [1].

3.2. **Applying Legendre Wavelet Method to the BVP.** As we have mentioned in section 2, the support of $\psi_{n,m}(x)$ is $[\frac{n-1}{N},\frac{n}{N}]$, so with $k=1$ we have $N=n=1$, and the support of $\psi_{n,m}(x)$ will be $[0,1]$, that is appropriate for our equation.
Now for selected M, we have basis functions ψ_{1m} for $m=0,...,M-1$ with $supp\ \psi_{1m}\subseteq[0,1]$.
Based on equation (2.3) the approximation of the solution is

$$y(x)=c_0\psi_{10}(x)+...+c_{M-1}\psi_{1,M-1}(x). \tag{3.4}$$

Applying (3.4) to equation (3.1) we have,

$$\begin{aligned} &p(x)(c_0\psi''_{10}(x)+...+c_{M-1}\psi''_{1,M-1}(x))\\ &+p'(x)(c_0\psi'_{10}(x)+...+c_{M-1}\psi'_{1,M-1}(x)) \quad = p(x)f(x,y). \end{aligned} \tag{3.5}$$

As $f(x,y)$ is usually a nonlinear function, we use the linear approximation instead, i.e. $f(x,y)\simeq f(x,0)+y\frac{\partial f}{\partial y}(x,0)$. So the main equation (3.5) will lead to:

$$\begin{aligned} &p(x)(c_0\psi''_{10}(x)+...+c_{M-1}\psi''_{1,M-1}(x))\\ &+p'(x)(c_0\psi'_{10}(x)+...+c_{M-1}\psi'_{1,M-1}(x))\\ &\quad = p(x)(f(x,0)+(c_0\psi_{10}(x)+...+c_{M-1}\psi_{1,M-1}(x))\frac{\partial f}{\partial y}(x,0)). \end{aligned} \tag{3.6}$$

For calculating c_i's we use the boundary conditions and collocation method at the uniform mesh points in $[0,1]$ i.e. $x_i = \frac{i}{M-1}$ for $i = 1, ..., M-2$.

3.3. **Error analysis.** By the assumption $y(x) = \sum c_m \phi_{1m}(x)$ and because of the orthonormality of the Legendre wavelet bases we have

$$\begin{aligned} Err_M = ||y(x) - \sum_{m=0}^{m=M-1} c_m \phi_{1m}(x)|| &= || \sum_{m=M}^{m=\infty} c_m \phi_{1m}(x)|| \\ &\leq \sum_{m=M}^{m=\infty} |c_m| ||\phi_{1m}(x)|| \leq \sum_{m=M}^{m=\infty} |c_m|. \end{aligned} \tag{3.7}$$

It is clear that $\lim Err_M = 0$ when $M \to \infty$ if $\sum_{m=0}^{m=\infty} |c_m|$ converges. One sufficient condition for the convergence of the series is that for $n \geq k$, $|\frac{c_{n+1}}{c_n}| \leq l < 1$, for some constants k, l.

4. Numerical Examples

In this section we apply the Legendre wavelet bases to some special case of equation (3.1). Comparing the results with other numerical methods verify the efficiency of the approach.

4.1. **Example 1.** Consider the boundary value problem [1]:

$$\begin{gathered} (p(x)y')' = p(x)f(x,y), \qquad x \in [0,1], \\ p(x) = x^b g(x), \quad g(x) = e^x, \quad f(x,y) = \frac{5x^3(5x^5e^y - x - b - 4)}{4 + x^5}, \\ y'(0) = 0, \quad y(1) + 5y'(1) = -5 - ln5. \end{gathered} \tag{4.1}$$

This problem is an application of oxygen diffusion for special values of the parameters. We solve this equation based on Legendre wavelet method for $b = .5$, with $M = 8, 16, 32$, and compare the results with other numerical methods such as, Pandey's method [19], and cubic spline method with economized Chebyshev polynomial [1]. Now by the previous discussion in Section 3, for $M = 8$ the coefficients of the approximate solution $appr_{y(x)} = \sum_{m=0}^{7} c_m \phi_{1m}(x)$ would be:

$c_0 = -1.592645395$, $c_1 = -0.5503774870e - 1$, $c_2 = .2646779215$,
$c_3 = .3803579572$, $c_4 = .2579476901$, $c_5 = .1006784330$,
$c_6 = 0.2210981278e - 1$, $c_7 = 0.2182403182e - 2$.

Here we have $|\frac{c_{n+1}}{c_n}| \leq .5$, for $n \geq 5$ that shows the convergence of the result. TABLE 1. Compares the error of the above solution (LWM) with Pandey's method and cubic spline method (CSM).

For $M = 16$ the corresponding coefficients are:
$c_0 = -1.2028100292365663886$, $c_1 = .62395284352695186896$,
$c_2 = 1.1603186798256836839$, $c_3 = 1.3764189093866981764$,

Table 1. Error Comparison of Example 1 for M=8

x	LWM	Pandey's method	CSM
0	0.0143	0.01	0.01
0.1	0.0143	0.01	0.01
0.2	0.0143	0.01	0.01
0.3	0.0143	0.01	0.01
0.4	0.0144	0.01	0.01
0.5	0.0144	0.01	0.01
0.6	0.0144	0.01	0.01
0.7	0.0143	0.01	0.01
0.8	0.0142	0.01	0.01
0.9	0.0140	0.01	0.01
1	0.0138	0.01	0.01

$c_4 = 1.19254462210253 41289, \quad c_5 = .83323308639869402496,$
$c_6 = .49506950020581057962, \quad c_7 = .24763588752320647322,$
$c_8 = 0.97328977716939060262e-1, \quad c_9 = 0.25678152930182439947e-1,$
$c_{10} = 0.17416008269448818104e-2, \quad c_{11} = -0.21579300142439125728e-2,$
$c_{12} = -0.12139357853083133079e-2, \quad c_{13} = -0.34670540047873153539e-3,$
$c_{14} = -0.57156327436667180150e-4, \quad c_{15} = -0.44131194744956478780e-5.$

Here we have $|\frac{c_{n+1}}{c_n}| \leq .6$ for $n \geq 11$, that shows the convergence of the result. TABLE 2. Compares the error of the above solution with Pandey's method and cubic spline method.

Table 2. Error Comparison of Example 1 for M=16

x	LWM	Pandey's method	CSM
0	0.001	0.004	0.0005
0.1	0.001	0.004	0.0006
0.2	0.001	0.004	0.0007
0.3	0.001	0.004	0.0008
0.4	0.001	0.004	0.0009
0.5	0.001	0.004	0.001
0.6	0.001	0.004	0.001
0.7	0.001	0.004	0.001
0.8	0.0009	0.004	0.001
0.9	0.0006	0.004	0.001
1	0.0003	0.004	0.001

For the case $M = 32$ after solving the obtained system of equations the coefficients will read:

TABLE 3. Error Comparison of Example 1 for M=32

x	LWM	CSM	Pandey's method
0	0.001	0.0002	0.001
0.1	0.001	0.0002	0.001
0.2	0.001	0.0002	0.001
0.3	0.001	0.0002	0.001
0.4	0.001	0.0003	0.001
0.5	0.001	0.0003	0.001
0.6	0.001	0.0003	0.001
0.7	0.001	0.0004	0.001
0.8	0.0009	0.0004	0.001
0.9	0.0006	0.0004	0.001
1	0.0003	0.0004	0.001

$c_0 = -1.5668260410297211678, \quad c_1 = 0.31637315031420237439e^{-1},$
$c_2 = .47387878260649028773, \quad c_3 = .65878260230042063800,$
$c_4 = .46548259990476093538, \quad c_5 = .10511439460261074765,$
$c_6 = -.21095600158051484798, \quad c_7 = -.38786801631568180130,$
$c_8 = -.40746450342068380586, \quad c_9 = -.30303162031950710379,$
$c_{10} = -.14142051146184570842, \quad c_{11} = 0.10525780886752447026e^{-1},$
$c_{12} = .11245184223067013243, \quad c_{13} = .15541004211045685499,$
$c_{14} = .15192765061995161302, \quad c_{15} = .12255589530900139502,$
$c_{16} = 0.85755699399196535512e^{-1}, \quad c_{17} = 0.53187349489469040464e^{-1},$
$c_{18} = 0.29537830272285305238e^{-1}, \quad c_{19} = 0.14749458491914828458e^{-1},$
$c_{20} = 0.66240114851239284737e^{-2}, \quad c_{21} = 0.26684115548987174765e^{-2},$
$c_{22} = 0.95917608050199466452e^{-3}, \quad c_{23} = 0.30527693739622463659e^{-3},$
$c_{24} = 0.85125989085420864322e^{-4}, \quad c_{25} = 0.20506845231683404921e^{-4},$
$c_{26} = 0.41876336045955541507e^{-5}, \quad c_{27} = 7.0595970148133046232e^{-7},$
$c_{28} = 9.4491021347606179554e^{-8}, \quad c_{29} = 9.4321810986874658009e^{-9},$
$c_{30} = 6.2544537267760213078e^{-10}, \quad c_{31} = 2.0707985493565530533e^{-11}.$

Here we have $|\frac{c_{n+1}}{c_n}| \leq .5$, for $n \geq 18$, that shows the convergence of the result. TABLE 3. compares the error of the above solution with Pandey's method and cubic spline method.

4.2. **Example 2.** Consider the boundary value problem with different boundary conditions [1]

$$\begin{gathered} (p(x)y')' = p(x)f(x,y), \qquad x \in [0,1], \\ p(x) = x^b g(x), \quad g(x) = e^x, \quad f(x,y) = \frac{5x^3(5x^5e^y - x - b - 4)}{4 + x^5}, \\ y(0) = -ln(4), \quad y(1) + 5y'(1) = -5 - ln5. \end{gathered} \tag{4.2}$$

Table 4. Error Comparison of Example 2 for M=8

x	LWM	Pandey's method	CSM
0	0.002	0.01	0.01
0.1	0.001	0.01	0.01
0.2	0.003	0.01	0.01
0.3	0.005	0.01	0.01
0.4	0.006	0.01	0.01
0.5	0.006	0.01	0.01
0.6	0.006	0.01	0.01
0.7	0.006	0.01	0.01
0.8	0.007	0.01	0.01
0.9	0.006	0.01	0.01
1	0.007	0.01	0.01

Like Example 1. we solve this equation based on Legendre wavelet method for $b = .5$, with $M = 8, 16, 32$, and compare the results with other numerical methods i.e. Pandey's method [19], and cubic spline method (CSM) with economized Chebyshev polynomial [1]. Now by the previous discussion in Section 3, for $M = 8$ we have,

$c_0 = -1.6323890648023684584$, $c_1 = -.12650608370150377892$, $c_2 = .20029627168678081152$,

$c_3 = 0.33317488451180765903$, $c_4 = 0.23186579146191620765$, $c_5 = 0.089467617979900008625$,

$c_6 = 0.018840642930622664031$, $c_7 = 0.0015998548484048946861$.

Here we have $|\frac{c_{n+1}}{c_n}| \leq .5$, for $n \geq 4$, that shows the convergence of the result. TABLE 4. compares the error of the above solution with Pandey's method and cubic spline method (CSM).

For the case $M = 16$ we have,
$c_0 = -37.085253912476237102$, $c_1 = -56.382387992419169590$,
$c_2 = -60.703824875283847864$, $c_3 = -54.921822452560235841$,
$c_4 = -43.612294016803071709$, $c_5 = -30.776654094799875849$,
$c_6 = -19.343588818066637877$, $c_7 = -10.808394161573297029$,
$c_8 = -5.3420120863121277913$, $c_9 = -2.3135269148713642482$,
$c_{10} = -.86462951838697314197$, $c_{11} = -.27246910604372655571$,
$c_{12} = -0.069894533975678137 57$, $c_{13} = -0.013785817113691259 20$,
$c_{14} = -0.0018788553980270614 8$, $c_{15} = -0.00013554317589891528$.

Here we have $|\frac{c_{n+1}}{c_n}| \leq .5$ for $n \geq 7$, that shows the convergence of the result. TABLE 5. compares the error of the our method with cubic spline method.

TABLE 5. Error Comparison of Example 2 for M=16

x	L. W. M.	C. S. M
0	0.0002	0.0008
0.1	0.0003	0.0009
0.2	0.0004	0.001
0.3	0.0005	0.001
0.4	0.0005	0.001
0.5	0.0005	0.001
0.6	0.0005	0.001
0.7	0.0005	0.001
0.8	0.0004	0.001
0.9	0.0001	0.001
1	0.0001	0.001

Similarly, for $M = 32$ we come to the following coefficients:
$c_0 = -4.6358693076092977194 * 10^6$, $c_1 = -6.7582264983241763273 * 10^6$,
$c_2 = -5.9099142767810083574 * 10^6$, $c_3 = -3.1770021176532618076 * 10^6$,
$c_4 = 2.1188760718345176772 * 10^5$, $c_5 = 3.0419453143786254232 * 10^6$,
$c_6 = 4.4978182063968003491 * 10^6$, $c_7 = 4.3729982946561054292 * 10^6$,
$c_8 = 3.0178295836150515531 * 10^6$, $c_9 = 1.0918111655220634394 * 10^6$,
$c_{10} = -7.3825057261862819434$, $c_{11} = -2.0216877298043717294 * 10^6$,
$c_{12} = -2.6041990921200130978 * 10^6$, $c_{13} = -2.5783384180672618095 * 10^6$,
$c_{14} = -2.1664432264550479477 * 10^6$, $c_{15} = -1.6038311314732019976 * 10^6$,
$c_{16} = -1.0646835980362944770 * 10^6$, $c_{17} = -6.3943235427383833427 * 10^5$,
$c_{18} = -3.4890121013572833342 * 10^5$, $c_{19} = -1.7316778052680677547 * 10^5$,
$c_{20} = -78099.00000139694805$, $c_{21} = -31912.487693990200350$,
$c_{22} = -11757.36500885325238$, $c_{23} = -3878.575888516679465$,
$c_{24} = -1134.7567159384624654$, $c_{25} = -290.64966939574863$,
$c_{26} = -64.022020715699201532$, $c_{27} = -11.825148046192491$,
$c_{28} = -1.763743121252254$, $c_{29} = -.199859762088155$,
$c_{30} = -0.15358571058327e - 1$, $c_{31} = -0.6035246206578086981e - 3$.

As the coefficient matrix is ill-conditioned for $M = 32$, the coefficients are very large (and so far from the main solution). Here we use Tikhonov regularization method (TRM) [27] to stabilize the solution. In regularization we change the ill-posed main equation with another equation that is less ill-posed. Tikhonov regularization is a well-known method of regularization. One important part in Tikhonov method is to find the optimum value of regularization parameter (μ). For more details one can see [13]. The results after applying TRM (with $\mu = 5 \times 10^{-13}$) are as follows:

TABLE 6. Error Comparison of Example 2 for M=32

x	LWM	CSM
0	0.00000001	0.0001
0.1	0.0002	0.0001
0.2	0.0004	0.0001
0.3	0.0005	0.0001
0.4	0.0005	0.0001
0.5	0.0006	0.0001
0.6	0.0006	0.0001
0.7	0.0006	0.0001
0.8	0.0005	0.0001
0.9	0.0002	0.0001
1	0.00001	0.0001

$$
\begin{aligned}
&c_0 = -.89023366514234935536, \quad c_1 = .54799351644467802312,\\
&c_2 = .22987869633682692909, \quad c_3 = -.14654224228351210740,\\
&c_4 = -.19880287660064923473, \quad c_5 = 0.023855558285009152128,\\
&c_6 = 0.15688511320033616955, \quad c_7 = 0.042604455456950835228,\\
&c_8 = -.10699218169413489446, \quad c_9 = -0.078671951067718633939,\\
&c_{10} = 0.054008137231122075279, \quad c_{11} = 0.090551316510355457336,\\
&c_{12} = -0.00424055919158032434, \quad c_{13} = -0.080176365603922724 55,\\
&c_{14} = -0.036818277057667280 97, \quad c_{15} = 0.052365148810533068 66,\\
&c_{16} = 0.061149049442388247 72, \quad c_{17} = -0.012451210708302498 77,\\
&c_{18} = -0.063727709020852189 76, \quad c_{19} = -0.028067636992952869 68,\\
&c_{20} = 0.040481079250968621 55, \quad c_{21} = 0.055272959122776167 50,\\
&c_{22} = 0.003083559406557017 00, \quad c_{23} = -0.050261578505449440 20,\\
&c_{24} = -0.048309119199519198 30, \quad c_{25} = 0.001712515405426443 26,\\
&c_{26} = 0.050266417806977757 11, \quad c_{27} = 0.064217526786791245 89,\\
&c_{28} = 0.047466652246753417 27, \quad c_{29} = 0.023347956100351489 280,\\
&c_{30} = 0.007338760448545312 3500, \quad c_{31} = 0.001233847929898666 33.
\end{aligned}
$$

Here we have $|\frac{c_{n+1}}{c_n}| \leq .6$ for $n \geq 27$, that shows the convergence of the result (see TABLE 6).

4.3. **Example 3.** Consider the oxygen diffusion equation of the form [24]:

$$
\begin{aligned}
y'' + \frac{m}{x}y' &= f(x,y), \qquad x \in [0,1],\\
y'(0) = 0, &\quad \alpha y(1) + \beta y'(1) = \gamma. \qquad (4.3)
\end{aligned}
$$

Where $f(x,y) = \frac{ny}{y+k}$, and $m = 2$, $n = .76129$, $k = .03119$, $\alpha = \gamma = 5$, $\beta = 1$. Applying Legendre wavelet method with $k = 1$ and $M = 32$ we have :

TABLE 7. Solution Value Comparison of Example 3 for M=32

x	LWM	PSM
0	0.828476	0.828483
0.1	0.829700	0.829706
0.2	0.833373	0.833374
0.3	0.839495	0.839489
0.4	0.848067	0.848052
0.5	0.859088	0.859064
0.6	0.872560	0.872528
0.7	0.888484	0.888445
0.8	0.906860	0.906818
0.9	0.927690	0.927650
1	0.950974	0.950945

$$y \approx \sum_{n=0}^{n=31} c_i \psi_{1,m}(x).$$

The coefficients are as follows:
$c_0 = .18637357115746482205$, $c_1 = -1.1141233030658516145$,
$c_2 = -1.2623851202435653868$, $c_3 = -1.0644847689670253742$,
$c_4 = -.66554436091794497683$, $c_5 = -.22290530480950376606$,
$c_6 = .16043912855554241194$, $c_7 = .42100765080111951420$,
$c_8 = .54145133504248797753$, $c_9 = .54218085557660869357$
$c_{10} = .46430169915258098816$, $c_{11} = .35187677758224507944$,
$c_{12} = .23947228060133984272$, $c_{13} = .14713182630357617675$,
$c_{14} = 0.81545096255112911761e-1$, $c_{15} = 0.40494645804697287309e-1$,
$c_{16} = 0.17744240927783100144e-1$, $c_{17} = 0.66407774085453815 66e-2$,
$c_{18} = 0.19553596541082371574e-2$, $c_{19} = 0.32134055776911934200e-3$,
$c_{20} = -0.8994773725108570e-4$, $c_{21} = -0.11516214722833478323e-3$,
$c_{22} = -0.66376937395763533374e-4$, $c_{23} = -0.28523642688714613922e-4$,
$c_{24} = -0.99867189381073832 40e-5$, $c_{25} = -0.29164113544247215 27e-5$,
$c_{26} = -7.100299022121911 86e-7$, $c_{27} = -1.4185969891799602 21e-7$,
$c_{28} = -2.2525886046482555e-8$, $c_{29} = -2.683910447798546e-9$,
$c_{30} = -2.14635754835878e-10$, $c_{31} = -8.695564890323e-12$.
TABLE 7. Reports the comparison of the resulted solution with that of Pandey and Singh method (PSM) [19]. It should be mentioned that the errors are about 10^{-5}.

5. Conclusion

In this paper Legendre wavelet method has been applied to some singular boundary value problems. The results are comparable with other numerical approaches. In the proposed method, the results are efficient near the singular point $x = 0$. Moreover the estimation error obtained in Section 3, are verified with the results.

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