Double Integral Characterization for Bergman Spaces

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Abstract. In this paper we characterize Bergman spaces with respect to double integral of the functions $|f(z) - f(w)|/|z - w|$, $|f(z) - f(w)|/\rho(z, w)$ and $|f(z) - f(w)|/\beta(z, w)$, where $\rho$ and $\beta$ are the pseudo-hyperbolic and hyperbolic metrics. We prove some necessary and sufficient conditions that implies a function to be in Bergman spaces.

Keywords: Bergman spaces, Pseudo-hyperbolic metric, Hyperbolic metric, Double integral.


1. Introduction

For $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ in $\mathbb{C}^n$, we define $\langle z, w \rangle = z_1 \overline{w_1} + \cdots + z_n \overline{w_n}$, where $\overline{w_k}$ is the complex conjugate of $w_k$. We also write $|z| = \sqrt{\langle z, z \rangle} = \sqrt{|z_1|^2 + \cdots + |z_n|^2}$. Let $\mathbb{B}_n$ denotes the open unit ball of $\mathbb{C}^n$, that is

$$\mathbb{B}_n = \{ z \in \mathbb{C}^n : |z| < 1 \}.$$

For any $a \in \mathbb{B}_n - \{0\}$, we define

$$\varphi_a(z) = \frac{a - P_a(z) - s_a Q_a(z)}{1 - \langle z, a \rangle} \quad z \in \mathbb{B}_n,$$

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where \( s_a = \sqrt{1-|a|^2} \), \( P_a \) is the orthogonal projection from \( \mathbb{C}^n \) onto the subspace \([a]\) generated by \( a \), and \( Q_a \) is the orthogonal projection from \( \mathbb{C}^n \) onto \( \mathbb{C}^n - [a] \). When \( a = 0 \), we define \( \varphi_a(z) = -z \). These functions are called involutions. (see [9] for more information about these functions)

The hyperbolic metric (Bergman metric) is defined by

\[
\beta(z, w) = \frac{1}{2} \log \frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|}, \quad z, w \in \mathbb{B}_n.
\]

For any \( z \in \mathbb{B}_n \) and \( r > 0 \), we denote Bergman metric ball at \( z \) by \( D(z, r) \). That is

\[
D(z, r) = \{ w \in \mathbb{B}_n : \beta(z, w) < r \}.
\]

Also, pseudo-hyperbolic metric is defined by \( \rho(z, w) = |\varphi_z(w)| \).

For \( \alpha > -1 \) let

\[
dv_{\alpha}(z) = c_\alpha(1-|z|^2)^\alpha \, dv(z)
\]

where \( dv(z) \) is the Lebesgue volume measure on \( \mathbb{B}_n \) and \( c_\alpha \) is a positive constant with \( \nu_\alpha(\mathbb{B}_n) = 1 \). For \( 0 < p < \infty \) and \( \alpha > -1 \), the weighted Bergman space \( A^p_{\alpha} \) consists of all holomorphic functions in \( L^p(\mathbb{B}_n, dv_{\alpha}) \), that is

\[
A^p_{\alpha} = \left\{ f \in H(\mathbb{B}_n) : ||f||_{p, \alpha}^p = \int_{\mathbb{B}_n} |f(z)|^p \, dv_{\alpha}(z) < \infty \right\}
\]

Wulan and Zhu [8], characterized Bergman spaces with standard weights in terms of Lipschitz type conditions in the Euclidean, hyperbolic, and pseudo-hyperbolic metrics. In [4] Li et al. proved that a holomorphic function \( f \) belongs to the \( A^p_{\alpha} \), \( p > n + 1 + \alpha \), if and only if the function \( |f(z) - f(w)|/|1 - \langle z, w \rangle| \) is in \( L^p(\mathbb{B}_n \times \mathbb{B}_n, dv_{\alpha} \times dv_{\alpha}) \), where \( \gamma = (p + \alpha - n - 1)/2 \).

Also, it was shown in [5] that for the case \( 0 < p < n + 1 + \alpha \), \( f \in A^p_{\alpha} \) if and only if the function \( |f(z) - f(w)|/|1 - \langle z, w \rangle| \) is in \( L^p(\mathbb{B}_n \times \mathbb{B}_n, dv_{\alpha} \times dv_{\alpha}) \) if and only if the function \( |f(z) - f(w)|/|z - w| \) is in \( L^p(\mathbb{B}_n \times \mathbb{B}_n, dv_{\alpha} \times dv_{\alpha}) \).

Our aim in this paper is to prove, for \( f \in A^p_{\alpha} \), \( p > n + 1 + \alpha \), the function \( |f(z) - f(w)|/|z - w| \) is in \( L^p(\mathbb{B}_n \times \mathbb{B}_n, dv_{\alpha} \times dv_{\alpha}) \), where \( t = (p + \alpha - n - 1)/2 \) and if \( p = n + 1 + \alpha \), then \( |f(z) - f(w)|/|z - w| \) is in \( L^p(\mathbb{B}_n \times \mathbb{B}_n, dv_{\alpha} \times dv_{\alpha}) \), for any \( \gamma > \alpha \). Our results are applicable for studying the action of symmetric lifting operator on \( A^p_{\alpha} \) in all cases especially for the case \( p = \alpha + 2 \).

Also we replace the Euclidean metric with pseudo-hyperbolic metric \( \rho \) and Bergman metric \( \beta \).

2. Preliminaries

Lemma 2.1. [9] There exists a positive constant \( C \) such that

\[
|f(z)|^p \leq \frac{C}{(1-|z|^2)^{n+1+\alpha}} \int_{D(z,r)} |f(w)|^p \, dv_{\alpha}(w)
\]

for all \( f \in H(\mathbb{B}_n) \) and \( z \in \mathbb{B}_n \).
Lemma 2.2. [9] Suppose $s > -1$, $t$ is real, and
\[ I(z) = \int_{B_n} \frac{(1 - |w|^2)^s}{|1 - \langle z, w \rangle|^{n+1+s+t}} \, dv(w), \quad z \in \mathbb{B}_n. \]

Then $I(z)$ is bounded in $\mathbb{B}_n$ whenever $t < 0$, and $I(z)$ is bounded by $(1 - |z|^2)^{-t}$ whenever $t > 0$.

Theorem 2.3. [8] Suppose that $p > 0$, $\alpha > -1$ and $f$ is analytic in $\mathbb{B}_n$. Then the following conditions are equivalent.

1. $f \in A_p^{\alpha}$.
2. There exists a continuous function $g$ in $L^p(\mathbb{B}_n, dv^\alpha)$ such that
   \[ |f(z) - f(w)| \leq \rho(z, w)(g(z) + g(w)), \quad z, w \in \mathbb{B}_n. \]
3. There exists a continuous function $g$ in $L^p(\mathbb{B}_n, dv^\alpha)$ such that
   \[ |f(z) - f(w)| \leq \beta(z, w)(g(z) + g(w)), \quad z, w \in \mathbb{B}_n. \]
4. There exists a continuous function $g$ in $L^p(\mathbb{B}_n, dv^{p+\alpha})$ such that
   \[ |f(z) - f(w)| \leq |z - w|(g(z) + g(w)), \quad z, w \in \mathbb{B}_n. \]

Lemma 2.4. [4] Let $r > 0$. Then
\[ 1 - |z|^2 \sim 1 - |w|^2 \sim |1 - \langle z, w \rangle| \]
for all $z \in \mathbb{B}_n$ and $w \in D(z, r)$. Furthermore, there exists a positive constant $C$ such that
\[ (1 - |z|^2)^p |\nabla f(z)|^p \leq \frac{C}{(1 - |z|^2)^{n+p+1}} \int_{D(z, r)} |f(w) - f(z)|^p dv(w) \]
for all $z \in \mathbb{B}_n$ and $f \in H(\mathbb{B}_n)$.

3. PSEUDO-HYPERBOLIC METRIC

Lemma 3.1. Suppose $\alpha > -1$ and $f \in H(\mathbb{B}_n)$. Then there exists a positive constant $C$ such that
\[ \int_{\mathbb{B}_n} |f(z) - f(0)|^p dv_\alpha(z) \leq C \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z) - f(w)|^p}{\rho(z, w)^p} dv_\alpha(z)dv_\alpha(w). \]

Proof. Let
\[ J(f) = \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z) - f(w)|^p}{\rho(z, w)^p} dv_\alpha(z)dv_\alpha(w). \]
By making a change of variable, we have
\[
J(f) = \int_{B_n} \frac{|f(z) - f(\varphi_z(w))|^p}{\rho(z, \varphi_z(w))^p} \frac{(1 - |z|^2)^{n+1+\alpha}}{|1 - \langle z, w \rangle|^{2(\alpha+1+\alpha)}} \, dv_\alpha(w)
\]
\[
= \int_{B_n} dv_\alpha(z) \int_{B_n} |f(z) - f(\varphi_z(w))|^p \frac{(1 - |z|^2)^{n+1+\alpha}}{|1 - \langle z, w \rangle|^{2(\alpha+1+\alpha)}} \, dv_\alpha(w)
\]
\[
\geq \int_{B_n} dv_\alpha(z) \int_{B_n} |f(z) - f(\varphi_z(w))|^p \frac{(1 - |z|^2)^{n+1+\alpha}}{|1 - \langle z, w \rangle|^{2(\alpha+1+\alpha)}} \, dv_\alpha(w)
\]
\[
\geq \int_{B_n} dv_\alpha(z) \int_{D(z,r)} |f(z) - f(\varphi_z(w))|^p \frac{(1 - |z|^2)^{n+1+\alpha}}{|1 - \langle z, w \rangle|^{2(\alpha+1+\alpha)}} \, dv_\alpha(w).
\]

From the first part of Lemma 2.4, there exists a positive constant $C'$ such that
\[
J(f) \geq C' \int_{B_n} dv_\alpha(z) \int_{D(z,r)} |f(z) - f(\varphi_z(w))|^p \frac{(1 - |z|^2)^{n+1+\alpha}}{|1 - \langle z, w \rangle|^{2(\alpha+1+\alpha)}} \, dv_\alpha(w).
\]

Then Lemma 2.1 implies that there exists another positive constant $C$ such that
\[
J(f) \geq C \int_{B_n} |f(z) - f(\varphi_z(z))|^p \, dv_\alpha(z) = C \int_{B_n} |f(z) - f(0)|^p \, dv_\alpha(z).
\]

The proof is complete. \qed

**Lemma 3.2.** Suppose $\alpha > -1$ and $f \in A^p_\alpha$. Then
\[
\int_{B_n} \int_{B_n} \frac{|f(z) - f(w)|^p}{\rho(z, w)^p} \, dv_\alpha(z) dv_\alpha(w) < \infty.
\]

**Proof.** Given $f \in A^p_\alpha$, from Theorem 2.3, there exists a continuous function $g \in L^p(\mathbb{B}_n, dv_\alpha)$ such that for all $z, w \in \mathbb{B}_n$,
\[
|f(z) - f(w)| \leq \rho(z, w)(g(z) + g(w)).
\]

There exists a positive constant $C$ such that
\[
\frac{|f(z) - f(w)|^p}{\rho(z, w)^p} \leq C(g(z)^p + g(w)^p).
\]

So,
\[
\int_{B_n} \int_{B_n} \frac{|f(w) - f(z)|^p}{\rho(z, w)^p} \, dv_\alpha(z) dv_\alpha(w)
\]
\[
\leq C \int_{B_n} \int_{B_n} (g(z)^p + g(w)^p) \, dv_\alpha(z) dv_\alpha(w)
\]
\[
= 2C \int_{B_n} \int_{B_n} g(z)^p \, dv_\alpha(z) dv_\alpha(w) < \infty.
\]

We can combine these two lemmas and obtain the following theorem.
**Theorem 3.3.** Suppose that $\alpha > -1$. Then $f \in A^p_\alpha$ if and only if
\[ \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z) - f(w)|^p}{\rho(z,w)^p} \, dv_\alpha(z)dv_\alpha(w) < \infty. \]

4. **Bergman metric**

Now, we replace metric $\rho$ by Bergman metric $\beta$.

**Lemma 4.1.** Suppose that $\alpha > -1$ and $f \in H(\mathbb{B}_n)$. If
\[ \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z) - f(w)|^p}{\beta(z,w)^p} \, dv_\alpha(z)dt(w) < \infty, \]
then $f \in A^p_\alpha$, where $d\tau(w) = \frac{dv(w)}{(1 - |w|^2)^{n+1}}$ is the Mobius invariant volume measure on $\mathbb{B}_n$.

**Proof.** By Lemma 2.4, there exists a positive constant $C$ such that
\[ (1 - |z|^2)^p |\nabla f(z)|^p \leq \frac{C}{(1 - |z|^2)^{n+1+\alpha}} \int_{D(z,r)} |f(z) - f(w)|^p dv(w) \]
\[ \leq \frac{C}{(1 - |z|^2)^{n+1+\alpha}} \int_{D(z,r)} |f(z) - f(w)|^p dv_\alpha(w). \]

Since $D(z,r)$ is open unit ball in metric $\beta$, we have
\[ (1 - |z|^2)^p |\nabla f(z)|^p \leq \frac{Cr^p}{(1 - |z|^2)^{n+1+\alpha}} \int_{D(z,r)} \frac{|f(z) - f(w)|^p}{\beta(z,w)^p} \, dv_\alpha(w). \]

After integrating
\[ \int_{\mathbb{B}_n} (1 - |z|^2)^p |\nabla f(z)|^p dv_\alpha(z) \leq Cr^p \int_{\mathbb{B}_n} \int_{D(z,r)} \frac{|f(z) - f(w)|^p}{\beta(z,w)^p} \, dv_\alpha(w)dt(z) \]
\[ \leq Cr^p \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z) - f(w)|^p}{\beta(z,w)^p} \, dv_\alpha(z)dv_\alpha(w)dt(z). \]

Therefore $(1 - |z|^2)^p \nabla f(z) \in A^p_\alpha$. It follows from Theorem 2.16 of [9] that $f \in A^p_\alpha$. □

By the same reason as in Lemma 3.2, we can prove the following lemma.

**Lemma 4.2.** Suppose $\alpha > -1$ and $f \in H(\mathbb{B}_n)$. If $f \in A^p_\alpha$, then
\[ \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z) - f(w)|^p}{\beta(z,w)^p} \, dv_\alpha(z)dv_\alpha(w) < \infty. \]
5. Euclidean metric

Theorem 5.1. Suppose $\alpha > -1$, $p = n + 1 + \alpha$ and $f \in A^p_\alpha$, then

$$I(f) = \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z) - f(w)|}{|z - w|^p} \, dv_\gamma(z) \, dv_\gamma(w) < \infty,$$

for any $\gamma > \alpha$.

Proof. Given $f \in A^p_\alpha$, from Theorem 2.3, there exists a continuous function $g \in L^p(\mathbb{B}_n, dv_\alpha)$ such that for all $z, w \in \mathbb{B}_n$,

$$|f(z) - f(w)| \leq \rho(z, w)(g(z) + g(w)) \leq \frac{|z - w|}{1 - \langle z, w \rangle}(g(z) + g(w)).$$

There exists a positive constant $C$ such that

$$I(f) \leq 2C \int_{\mathbb{B}_n} g(z)^p \, dv_\gamma(z) \int_{\mathbb{B}_n} \frac{dv_\gamma(w)}{|1 - \langle z, w \rangle|^p} \leq 2C \int_{\mathbb{B}_n} g(z)^p \, dv_\gamma(z) \int_{\mathbb{B}_n} \frac{dv_\gamma(w)}{|1 - \langle z, w \rangle|^{n+1+\alpha}}.$$

Since $\alpha - \gamma < 0$, by Lemma 2.2, the last integral is bounded. Then there exists another positive constant $M$ such that

$$I(f) \leq M \int_{\mathbb{B}_n} g(z)^p \, dv_\gamma(z) = M c_\alpha \int_{\mathbb{B}_n} g(z)^p(1 - |z|^2)^\gamma(1 - |z|^2)^\gamma \, dv_\gamma(z) < M c_\alpha \int_{\mathbb{B}_n} g(z)^p \, dv_\alpha(z) < \infty.$$

\[\square\]

Lemma 5.2. Suppose $\alpha > -1$, $f \in H(\mathbb{B}_n)$ and $\delta$ and $\gamma$ are real parameters such that

$$\delta + \gamma = p + \alpha - (n + 1), \quad -1 < \gamma < p - (n + 1).$$

If $f \in A^p_\alpha$, then

$$I(f) = \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z) - f(w)|}{|z - w|^p} \, dv_\delta(z) \, dv_\gamma(w) < \infty.$$

Proof. By the proof of the previous lemma, there exists a positive constant $C$ such that

$$I(f) \leq 2C \int_{\mathbb{B}_n} g(z)^p \, dv_\delta(z) \int_{\mathbb{B}_n} \frac{dv_\gamma(w)}{|1 - \langle z, w \rangle|^p} \leq 2C \int_{\mathbb{B}_n} g(z)^p \, dv_\delta(z) \int_{\mathbb{B}_n} \frac{dv_\gamma(w)}{|1 - \langle z, w \rangle|^{n+1+\gamma+\delta}}.$$
Since $\delta - \alpha > 0$, by Lemma 2.2, there exists another positive constant $M$ such that

$$I(f) \leq M \int_{B_n} \frac{g(z)^p}{(1 - |z|^2)^{\delta - \alpha}} \, dv_\delta(z) = M \int_{B_n} g(z)^p \, dv_\alpha(z) < \infty.$$ 

□

**Corollary 5.3.** Suppose that $\alpha > -1$, $p > n + 1 + \alpha$ and $f \in A^p_\alpha$, then

$$\int_{B_n} \int_{B_n} \frac{|f(z) - f(w)|^p}{|z - w|^p} \, dv_t(z) dv_t(w) < \infty,$n

where $t = \frac{p+\alpha-(n+1)}{2}$.

If $n = 1$, then we obtain the following corollary.

**Corollary 5.4.** Suppose that $\alpha > -1$, $p > \alpha + 2$ and $f \in A^p_\alpha(D)$, then

$$\int_D \int_D \frac{|f(z) - f(w)|^p}{|z - w|^p} \, dA_t(z) dA_t(w) < \infty,$n

where $t = \frac{p+\alpha-2}{2}$.

The symmetric lifting operator $L : H(D) \rightarrow H(D \times D)$ is defined by

$$L(f)(z, w) = \frac{f(z) - f(w)}{z - w}.$n

The action of symmetric lifting operator on $A^p_\alpha(D)$ in the cases $p > \alpha + 2$ and $p < \alpha + 2$ was studied in [8]. In the case $p = \alpha + 2$, we have the following result.

**Corollary 5.5.** Suppose that $\alpha > -1$, $p = \alpha + 2$. Then the symmetric lifting operator maps $A^p_\alpha(D)$ into $A^p_\gamma(D^2)$, for any $\gamma > \alpha$.

**Proof.** The result follows by letting $n = 1$ in Theorem 5.1. □

If $\alpha > -1$, $p > \alpha + 2$ and $f \in A^p_\alpha(D)$, then by Corollary 5.4, $L(f) \in A^t_\gamma(D^2)$, which means that the symmetric lifting operator maps $f \in A^p_\alpha(D)$ into $A^t_\gamma(D^2)$, for $t = \frac{p+\alpha-2}{2}$. This is the Theorem 4.4 in [8].

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**References**