Double Integral Characterization for Bergman Spaces

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Abstract. In this paper we characterize Bergman spaces with respect to double integral of the functions $|f(z) - f(w)|/|z - w|$, $|f(z) - f(w)|/\rho(z, w)$, and $|f(z) - f(w)|/\beta(z, w)$, where $\rho$ and $\beta$ are the pseudo-hyperbolic and hyperbolic metrics. We prove some necessary and sufficient conditions that implies a function to be in Bergman spaces.

Keywords: Bergman spaces, Pseudo-hyperbolic metric, Hyperbolic metric, Double integral.


1. Introduction

For $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ in $\mathbb{C}^n$, we define $\langle z, w \rangle = z_1 \overline{w_1} + \cdots + z_n \overline{w_n}$, where $\overline{w_k}$ is the complex conjugate of $w_k$. We also write $|z| = \sqrt{\langle z, z \rangle} = \sqrt{|z_1|^2 + \cdots + |z_n|^2}$. Let $\mathbb{B}_n$ denotes the open unit ball of $\mathbb{C}^n$, that is

$$\mathbb{B}_n = \{ z \in \mathbb{C}^n : |z| < 1 \}.$$

For any $a \in \mathbb{B}_n - \{0\}$, we define

$$\varphi_a(z) = \frac{a - P_a(z) - s_a Q_a(z)}{1 - \langle z, a \rangle} \quad z \in \mathbb{B}_n,$$

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Received 13 July 2013; Accepted 22 November 2015

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where \( s_n = \sqrt{1 - |a|^2} \), \( P_a \) is the orthogonal projection from \( \mathbb{C}^n \) onto the subspace \([a]\) generated by \( a \), and \( Q_a \) is the orthogonal projection from \( \mathbb{C}^n \) onto \( \mathbb{C}^n - [a] \). When \( a = 0 \), we define \( \varphi_a(z) = -z \). These functions are called involutions. (see [9] for more information about these functions)

The hyperbolic metric (Bergman metric) is defined by
\[
\beta(z, w) = \frac{1}{2} \log \frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|}, \quad z, w \in \mathbb{B}_n.
\]

For any \( z \in \mathbb{B}_n \) and \( r > 0 \), we denote Bergman metric ball at \( z \) by \( D(z, r) \). That is
\[
D(z, r) = \{ w \in \mathbb{B}_n : \beta(z, w) < r \}.
\]

Also, pseudo-hyperbolic metric is defined by \( \rho(z, w) = |\varphi_z(w)| \).

For \( \alpha > -1 \) let
\[
dv_\alpha(z) = c_\alpha(1 - |z|^2)\gamma dv(z)
\]
where \( dv(z) \) is the Lebesgue volume measure on \( \mathbb{B}_n \) and \( c_\alpha \) is a positive constant with \( c_\alpha(\mathbb{B}_n) = 1 \). For \( 0 < p < \infty \) and \( \alpha > -1 \), the weighted Bergman space \( A^p_\alpha \) consists of all holomorphic functions in \( L^p(\mathbb{B}_n, dv_\alpha) \), that is
\[
A^p_\alpha = \left\{ f \in H(\mathbb{B}_n) : ||f||_p^p = \int_{\mathbb{B}_n} |f(z)|^p dv_\alpha(z) < \infty \right\}
\]

Wulan and Zhu [8], characterized Bergman spaces with standard weights in terms of Lipschitz type conditions in the Euclidean, hyperbolic, and pseudo-hyperbolic metrics. In [4] Li et al. proved that a holomorphic function \( f \) belongs to the \( A^p_\alpha \), \( p > n + 1 + \alpha \), if and only if the function \( |f(z) - f(w)|/|1 - \langle z, w \rangle| \) is in \( L^p(\mathbb{B}_n \times \mathbb{B}_n, dv_\alpha \times dv_\alpha) \), where \( \gamma = (p + \alpha - n - 1)/2 \).

Also, it was shown in [5] that for the case \( 0 < p < \infty + 1 + \alpha \), \( f \in A^p_\alpha \) if and only if the function \( |f(z) - f(w)|/|1 - \langle z, w \rangle| \) is in \( L^p(\mathbb{B}_n \times \mathbb{B}_n, dv_\alpha \times dv_\alpha) \) if and only if the function \( |f(z) - f(w)|/|z - w| \) is in \( L^p(\mathbb{B}_n \times \mathbb{B}_n, dv_\alpha \times dv_\alpha) \).

Our aim in this paper is to prove, for \( f \in A^p_\alpha \), \( p > n + 1 + \alpha \), the function \( |f(z) - f(w)|/|z - w| \) is in \( L^p(\mathbb{B}_n \times \mathbb{B}_n, dv_\alpha \times dv_\alpha) \), where \( t = (p + \alpha - n - 1)/2 \) and if \( p = n + 1 + \alpha \), then \( |f(z) - f(w)|/|z - w| \) is in \( L^p(\mathbb{B}_n \times \mathbb{B}_n, dv_\alpha \times dv_\alpha) \), for any \( \gamma > \alpha \). Our results are applicable for studying the action of symmetric lifting operator on \( A^p_\alpha \) in all cases especially for the case \( p = \alpha + 2 \).

Also we replace the Euclidean metric with pseudo-hyperbolic metric \( \rho \) and Bergman metric \( \beta \).

2. Preliminaries

Lemma 2.1. [9] There exists a positive constant \( C \) such that
\[
|f(z)|^p \leq \frac{C}{(1 - |z|^2)^{n+1+\alpha}} \int_{D(z, r)} |f(w)|^p \ dv_\alpha(w)
\]
for all \( f \in H(\mathbb{B}_n) \) and \( z \in \mathbb{B}_n \).
Lemma 2.2. [9] Suppose \( s > -1 \), \( t \) is real, and
\[
I(z) = \int_{B_n} \frac{(1 - |w|^2)^s}{|1 - \langle z, w \rangle|^{n+1+s+t}} \, dv(w), \quad z \in B_n.
\]
Then \( I(z) \) is bounded in \( B_n \) whenever \( t < 0 \), and \( I(z) \) is bounded by \( (1 - |z|^2)^{-t} \) whenever \( t > 0 \).

Theorem 2.3. [8] Suppose that \( p > 0 \), \( \alpha > -1 \) and \( f \) is analytic in \( B_n \). Then the following conditions are equivalent.

1. \( f \in A_p^\alpha \).
2. There exists a continuous function \( g \) in \( L^p(B_n, dv_\alpha) \) such that
\[
|f(z) - f(w)| \leq \rho(z, w)(g(z) + g(w)), \quad z, w \in B_n.
\]
3. There exists a continuous function \( g \) in \( L^p(B_n, dv_\alpha) \) such that
\[
|f(z) - f(w)| \leq \beta(z, w)(g(z) + g(w)), \quad z, w \in B_n.
\]
4. There exists a continuous function \( g \) in \( L^p(B_n, dv_{p+\alpha}) \) such that
\[
|f(z) - f(w)| \leq |z - w|(g(z) + g(w)), \quad z, w \in B_n.
\]

Lemma 2.4. [4] Let \( r > 0 \). Then
\[
1 - |z|^2 \sim 1 - |w|^2 \sim 1 - \langle z, w \rangle
\]
for all \( z \in B_n \) and \( w \in D(z, r) \). Furthermore, there exists a positive constant \( C \) such that
\[
(1 - |z|^2)^p|\nabla f(z)|^p \leq \frac{C}{(1 - |z|^2)^{n+1}} \int_{D(z, r)} |f(w) - f(z)|^p dv(w)
\]
for all \( z \in B_n \) and \( f \in H(B_n) \).

3. Pseudo-Hyperbolic Metric

Lemma 3.1. Suppose \( \alpha > -1 \) and \( f \in H(B_n) \). Then there exists a positive constant \( C \) such that
\[
\int_{B_n} |f(z) - f(0)|^p \, dv_\alpha(z) \leq C \int_{B_n} \int_{B_n} \frac{|f(z) - f(w)|^p}{\rho(z, w)^p} \, dv_\alpha(z)dv_\alpha(w).
\]

Proof. Let
\[
J(f) = \int_{B_n} \int_{B_n} \frac{|f(z) - f(w)|^p}{\rho(z, w)^p} \, dv_\alpha(z)dv_\alpha(w).
\]
By making a change of variable, we have
\[
J(f) = \int_{\mathbb{B}_n} dv_{\alpha}(z) \int_{\mathbb{B}_n} \frac{|f(z) - f(\varphi_z(w))|^p}{\rho(z, \varphi_z(w))^p} \frac{(1 - |z|^2)^{n+1+\alpha}}{|1 - \langle z, w \rangle|^{2(\alpha+1+\alpha)}} dv_{\alpha}(w)
\]
\[
= \int_{\mathbb{B}_n} dv_{\alpha}(z) \int_{\mathbb{B}_n} \frac{|f(z) - f(\varphi_z(w))|^p}{|w|^p} \frac{(1 - |z|^2)^{n+1+\alpha}}{|1 - \langle z, w \rangle|^{2(\alpha+1+\alpha)}} dv_{\alpha}(w)
\]
\[
\geq \int_{\mathbb{B}_n} dv_{\alpha}(z) \int_{D(z,r)} |f(z) - f(\varphi_z(w))|^p \frac{(1 - |z|^2)^{n+1+\alpha}}{|1 - \langle z, w \rangle|^{2(\alpha+1+\alpha)}} dv_{\alpha}(w).
\]

From the first part of Lemma 2.4, there exists a positive constant $C'$ such that
\[
J(f) \geq C' \int_{\mathbb{B}_n} dv_{\alpha}(z) \int_{D(z,r)} |f(z) - f(\varphi_z(w))|^p dv_{\alpha}(z).
\]
Then Lemma 2.1 implies that there exists another positive constant $C$ such that
\[
J(f) \geq C \int_{\mathbb{B}_n} |f(z) - f(\varphi_z(z))|^p dv_{\alpha}(z) = C \int_{\mathbb{B}_n} |f(z) - f(0)|^p dv_{\alpha}(z).
\]
The proof is complete. \(\square\)

**Lemma 3.2.** Suppose $\alpha > -1$ and $f \in A_p^\alpha$. Then
\[
\int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z) - f(w)|^p}{\rho(z, w)^p} dv_{\alpha}(z)dv_{\alpha}(w) < \infty.
\]

**Proof.** Given $f \in A_p^\alpha$, from Theorem 2.3, there exists a continuous function $g \in L^p(\mathbb{B}_n, dv_{\alpha})$ such that for all $z, w \in \mathbb{B}_n$,
\[
|f(z) - f(w)| \leq \rho(z, w)(g(z) + g(w)).
\]
There exists a positive constant $C$ such that
\[
\frac{|f(z) - f(w)|^p}{\rho(z, w)^p} \leq C(g(z)^p + g(w)^p).
\]
So,
\[
\int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(w) - f(z)|^p}{\rho(z, w)^p} dv_{\alpha}(z)dv_{\alpha}(w)
\]
\[
\leq C \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} (g(z)^p + g(w)^p) dv_{\alpha}(z)dv_{\alpha}(w)
\]
\[
= 2C \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} g(z)^p dv_{\alpha}(z)dv_{\alpha}(w) < \infty.
\]
\(\square\)

We can combine these two lemmas and obtain the following theorem.
Theorem 3.3. Suppose that \( \alpha > -1 \). Then \( f \in A^p_\alpha \) if and only if
\[
\int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z) - f(w)|^p}{\rho(z, w)^p} \, dv_\alpha(z) dv_\alpha(w) < \infty.
\]

4. BERGMAN METRIC

Now, we replace metric \( \rho \) by Bergman metric \( \beta \).

Lemma 4.1. Suppose that \( \alpha > -1 \) and \( f \in H(\mathbb{B}_n) \). If
\[
\int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z) - f(w)|^p}{\beta(z, w)^p} \, dv_\alpha(z) d\tau(w) < \infty,
\]
then \( f \in A^p_\alpha \), where
\[
d\tau(w) = \frac{dv(w)}{(1 - |w|^2)^{n+1}}
\]
is the Mobius invariant volume measure on \( \mathbb{B}_n \).

Proof. By Lemma 2.4, there exists a positive constant \( C \) such that
\[
(1 - |z|^2)^p|\nabla f(z)|^p \leq \frac{C}{(1 - |z|^2)^{n+1+\alpha}} \int_{D(z, r)} |f(z) - f(w)|^p dv(w)
\]
\[
\leq \frac{C}{(1 - |z|^2)^{n+1+\alpha}} \int_{D(z, r)} |f(z) - f(w)|^p dv_\alpha(w).
\]
Since \( D(z, r) \) is open unit ball in metric \( \beta \), we have
\[
(1 - |z|^2)^p|\nabla f(z)|^p \leq \frac{Cr^p}{(1 - |z|^2)^{n+1+\alpha}} \int_{D(z, r)} \frac{|f(z) - f(w)|^p}{\beta(z, w)^p} \, dv_\alpha(w).
\]
After integrating
\[
\int_{\mathbb{B}_n} (1 - |z|^2)^p|\nabla f(z)|^p dv_\alpha(z) \leq Cr^p \int_{\mathbb{B}_n} \int_{D(z, r)} \frac{|f(z) - f(w)|^p}{\beta(z, w)^p} \, dv_\alpha(w) d\tau(z)
\]
\[
\leq Cr^p \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z) - f(w)|^p}{\beta(z, w)^p} \, dv_\alpha(z) d\tau(w).
\]
Therefore \( (1 - |z|^2)\nabla f(z) \in A^p_\alpha \). It follows from Theorem 2.16 of [9] that \( f \in A^p_\alpha \). \( \square \)

By the same reason as in Lemma 3.2, we can prove the following lemma.

Lemma 4.2. Suppose \( \alpha > -1 \) and \( f \in H(\mathbb{B}_n) \). If \( f \in A^p_\alpha \), then
\[
\int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z) - f(w)|^p}{\beta(z, w)^p} \, dv_\alpha(z) dv_\alpha(w) < \infty.
\]
5. Euclidean metric

**Theorem 5.1.** Suppose \( \alpha > -1 \), \( p = n + 1 + \alpha \) and \( f \in A_p^\alpha \), then

\[
I(f) = \int_{B_n} \int_{B_n} |f(z) - f(w)|^p \frac{dv_\gamma(z)dv_\gamma(w)}{|z - w|^p} < \infty,
\]

for any \( \gamma > \alpha \).

**Proof.** Given \( f \in A_p^\alpha \), from Theorem 2.3, there exists a continuous function \( g \in L^p(\mathbb{B}_n, dv_\alpha) \) such that for all \( z, w \in \mathbb{B}_n \),

\[
|f(z) - f(w)| \leq \rho(z, w)(g(z) + g(w)) \leq \frac{|z - w|}{|1 - \langle z, w \rangle|}(g(z) + g(w)).
\]

There exists a positive constant \( C \) such that

\[
I(f) \leq 2C \int_{B_n} g(z)^p dv_\gamma(z) \int_{B_n} \frac{dv_\gamma(w)}{|1 - \langle z, w \rangle|^p}.
\]

Since \( \alpha - \gamma < 0 \), by Lemma 2.2, the last integral is bounded. Then there exists another positive constant \( M \) such that

\[
I(f) \leq M \int_{B_n} g(z)^p dv_\gamma(z) = M c_\gamma \int_{B_n} g(z)^p (1 - |z|^2)\gamma^{-\alpha}(1 - |z|^2)\alpha dv(z) < \infty.
\]

\( \square \)

**Lemma 5.2.** Suppose \( \alpha > -1 \), \( f \in H(\mathbb{B}_n) \) and \( \delta \) and \( \gamma \) are real parameters such that

\[
\delta + \gamma = p + \alpha - (n + 1), \quad -1 < \gamma < p - (n + 1).
\]

If \( f \in A_p^\alpha \), then

\[
I(f) = \int_{B_n} \int_{B_n} |f(z) - f(w)|^p \frac{dv_\delta(z)dv_\gamma(w)}{|z - w|^p} < \infty.
\]

**Proof.** By the proof of the previous lemma, there exists a positive constant \( C \) such that

\[
I(f) \leq 2C \int_{B_n} g(z)^p dv_\delta(z) \int_{B_n} \frac{dv_\gamma(w)}{|1 - \langle z, w \rangle|^p} = 2C \int_{B_n} g(z)^p dv_\delta(z) \int_{B_n} \frac{dv_\gamma(w)}{|1 - \langle z, w \rangle|^{n+1+\gamma+(\delta-\alpha)}}.
\]
Since $\delta - \alpha > 0$, by Lemma 2.2, there exists another positive constant $M$ such that

$$I(f) \leq M \int_{B_n} \frac{g(z)^p}{(1 - |z|^2)^{\delta - \alpha}} \, dv(z) = M \int_{B_n} g(z)^p \, dv(z) < \infty.$$ 

\[ \square \]

**Corollary 5.3.** Suppose that $\alpha > -1$, $p > n + 1 + \alpha$ and $f \in A^p_0$, then

$$\int_{B_n} \int_{B_n} \frac{|f(z) - f(w)|^p}{|z - w|^p} \, dv(z) dv(w) < \infty,$$

where $t = \frac{p + \alpha - (n+1)}{2}$.

If $n = 1$, then we obtain the following corollary.

**Corollary 5.4.** Suppose that $\alpha > -1$, $p > \alpha + 2$ and $f \in A^p_0(\mathbb{D})$, then

$$\int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|f(z) - f(w)|^p}{|z - w|^p} \, dA(z) dA(w) < \infty,$$

where $t = \frac{p + \alpha - 2}{2}$.

The symmetric lifting operator $L : H(\mathbb{D}) \rightarrow H(\mathbb{D} \times \mathbb{D})$ is defined by

$$L(f)(z, w) = \frac{f(z) - f(w)}{z - w}.$$

The action of symmetric lifting operator on $A^p_0(\mathbb{D})$ in the cases $p > \alpha + 2$ and $p < \alpha + 2$ was studied in [8]. In the case $p = \alpha + 2$, we have the following result.

**Corollary 5.5.** Suppose that $\alpha > -1$, $p = \alpha + 2$. Then the symmetric lifting operator maps $A^p_0(\mathbb{D})$ into $A^p(\mathbb{D}^2)$, for any $\gamma > \alpha$.

**Proof.** The result follows by letting $n = 1$ in Theorem 5.1. \[ \square \]

If $\alpha > -1$, $p > \alpha + 2$ and $f \in A^p_0(\mathbb{D})$, then by Corollary 5.4, $L(f) \in A^p(\mathbb{D}^2)$, which means that the symmetric lifting operator maps $f \in A^p_0(\mathbb{D})$ into $A^p(\mathbb{D}^2)$, for $t = \frac{p + \alpha - 2}{2}$. This is the Theorem 4.4 in [8].

**Acknowledgments**

The authors would like to thank the referees for their useful comments.

**References**


