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Double Integral Characterization for Bergman Spaces

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Abstract. In this paper we characterize Bergman spaces with respect to double integral of the functions $|f(z)-f(w)|/|z-w|$, $|f(z)-f(w)|/\rho(z,w)$ and $|f(z)-f(w)|/\beta(z,w)$, where ρ and β are the pseudo-hyperbolic and hyperbolic metrics. We prove some necessary and sufficient conditions that implies a function to be in Bergman spaces.

Keywords: Bergman spaces, Pseudo-hyperbolic metric, Hyperbolic metric, Double integral.

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1. Introduction

For $z=(z_1,\cdots,z_n)$ and $w=(w_1,\cdots,w_n)$ in $\mathbb{C}^n$, we define $\langle z,w\rangle = z_1\overline{w_1}+\cdots+z_n\overline{w_n}$, where $\overline{w_k}$ is the complex conjugate of w_k. We also write $|z| = \sqrt{\langle z,z\rangle} = \sqrt{|z_1|^2+\cdots+|z_n|^2}$. Let $\mathbb{B}_n$ denotes the open unit ball of $\mathbb{C}^n$, that is

$$\mathbb{B}_n = \{z\in\mathbb{C}^n : |z|<1\}.$$

For any $a\in\mathbb{B}_n-\{0\}$, we define

$$\varphi_a(z) = \frac{a-P_a(z)-s_aQ_a(z)}{1-\langle z,a\rangle} \qquad z\in\mathbb{B}_n,$$

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where $s_a = \sqrt{1-|a|^2}$, P_a is the orthogonal projection from $\mathbb{C}^n$ onto the subspace $[a]$ generated by a, and Q_a is the orthogonal projection from $\mathbb{C}^n$ onto $\mathbb{C}^n - [a]$. When $a = 0$, we define $\varphi_a(z) = -z$. These functions are called involutions. (see [9] for more information about these functions)
The hyperbolic metric (Bergman metric) is defined by

$$\beta(z,w) = \frac{1}{2}\log\frac{1+|\varphi_z(w)|}{1-|\varphi_z(w)|}, \quad z,w \in \mathbb{B}_n.$$

For any $z \in \mathbb{B}_n$ and $r > 0$, we denote Bergman metric ball at z by $D(z,r)$. That is

$$D(z,r) = \{w \in \mathbb{B}_n : \beta(z,w) < r\}.$$

Also, pseudo-hyperbolic metric is defined by $\rho(z,w) = |\varphi_z(w)|$.
For $\alpha > -1$ let

$$dv_\alpha(z) = c_\alpha(1-|z|^2)^\alpha dv(z)$$

where $dv(z)$ is the Lebesgue volume measure on $\mathbb{B}_n$ and c_α is a positive constant with $v_\alpha(\mathbb{B}_n) = 1$. For $0 < p < \infty$ and $\alpha > -1$, the weighted Bergman space A^p_α consists of all holomorphic functions in $L^p(\mathbb{B}_n, dv_\alpha)$, that is

$$A^p_\alpha = \left\{ f \in H(\mathbb{B}_n) : ||f||^p_{\alpha,p} = \int_{\mathbb{B}_n} |f(z)|^p dv_\alpha(z) < \infty. \right\}$$

Wulan and Zhu [8], characterized Bergman spaces with standard weights in terms of Lipschitz type conditions in the Euclidean, hyperbolic, and pseudo-hyperbolic metrics. In [4] Li et al. proved that a holomorphic function f belongs to the A^p_α, $p > n+1+\alpha$, if and only if the function $|f(z)-f(w)|/|1-\langle z,w\rangle|$ is in $L^p(\mathbb{B}_n \times \mathbb{B}_n, dv_\gamma \times dv_\gamma)$, where $\gamma = (p+\alpha-n-1)/2$.
Also, it was shown in [5] that for the case $0 < p < n+1+\alpha$, $f \in A^p_\alpha$ if and only if the function $|f(z)-f(w)|/|1-\langle z,w\rangle|$ is in $L^p(\mathbb{B}_n \times \mathbb{B}_n, dv_\alpha \times dv_\alpha)$ if and only if the function $|f(z)-f(w)|/|z-w|$ is in $L^p(\mathbb{B}_n \times \mathbb{B}_n, dv_\alpha \times dv_\alpha)$.

Our aim in this paper is to prove, for $f \in A^p_\alpha$, $p > n+1+\alpha$, the function $|f(z)-f(w)|/|z-w|$ is in $L^p(\mathbb{B}_n \times \mathbb{B}_n, dv_t \times dv_t)$, where $t = (p+\alpha-n-1)/2$ and if $p = n+1+\alpha$, then $|f(z)-f(w)|/|z-w|$ is in $L^p(\mathbb{B}_n \times \mathbb{B}_n, dv_\gamma \times dv_\gamma)$, for any $\gamma > \alpha$. Our results are applicable for studying the action of symmetric lifting operator on A^p_α in all cases especially for the case $p = \alpha + 2$.
Also we replace the Euclidean metric with pseudo-hyperbolic metric ρ and Bergman metric β.

2. Preliminaries

Lemma 2.1. [9] *There exists a positive constant C such that*

$$|f(z)|^p \leq \frac{C}{(1-|z|^2)^{n+1+\alpha}} \int_{D(z,r)} |f(w)|^p \, dv_\alpha(w)$$

for all $f \in H(\mathbb{B}_n)$ and $z \in \mathbb{B}_n$.

Lemma 2.2. [9] *Suppose $s > -1$, t is real, and*

$$I(z) = \int_{\mathbb{B}_n} \frac{(1-|w|^2)^s}{|1-\langle z,w\rangle|^{n+1+s+t}}\, dv(w), \quad z \in \mathbb{B}_n.$$

Then $I(z)$ is bounded in $\mathbb{B}_n$ whenever $t < 0$, and $I(z)$ is bounded by $(1-|z|^2)^{-t}$ whenever $t > 0$.

Theorem 2.3. [8] *Suppose that $p > 0$, $\alpha > -1$ and f is analytic in $\mathbb{B}_n$. Then the following conditions are equivalent.*

(1) $f \in A^p_\alpha$.

(2) *There exists a continuous function g in $L^p(\mathbb{B}_n, dv_\alpha)$ such that*

$$|f(z) - f(w)| \le \rho(z,w)(g(z) + g(w)), \quad z, w \in \mathbb{B}_n.$$

(3) *There exists a continuous function g in $L^p(\mathbb{B}_n, dv_\alpha)$ such that*

$$|f(z) - f(w)| \le \beta(z,w)(g(z) + g(w)), \quad z, w \in \mathbb{B}_n.$$

(4) *There exists a continuous function g in $L^p(\mathbb{B}_n, dv_{p+\alpha})$ such that*

$$|f(z) - f(w)| \le |z-w|(g(z) + g(w)), \quad z, w \in \mathbb{B}_n.$$

Lemma 2.4. [4] *Let $r > 0$. Then*

$$1 - |z|^2 \sim 1 - |w|^2 \sim |1 - \langle z, w\rangle|$$

for all $z \in \mathbb{B}_n$ and $w \in D(z,r)$. Furthermore, there exists a positive constant C such that

$$(1-|z|^2)^p |\nabla f(z)|^p \le \frac{C}{(1-|z|^2)^{n+1}} \int_{D(z,r)} |f(w) - f(z)|^p dv(w)$$

for all $z \in \mathbb{B}_n$ and $f \in H(\mathbb{B}_n)$.

3. Pseudo-Hyperbolic Metric

Lemma 3.1. *Suppose $\alpha > -1$ and $f \in H(\mathbb{B}_n)$. Then there exists a positive constant C such that*

$$\int_{\mathbb{B}_n} |f(z) - f(0)|^p \, dv_\alpha(z) \le C \int_{\mathbb{B}_n}\int_{\mathbb{B}_n} \frac{|f(z)-f(w)|^p}{\rho(z,w)^p}\, dv_\alpha(z) dv_\alpha(w).$$

Proof. Let

$$J(f) = \int_{\mathbb{B}_n}\int_{\mathbb{B}_n} \frac{|f(z)-f(w)|^p}{\rho(z,w)^p}\, dv_\alpha(z) dv_\alpha(w).$$

By making a change of variable, we have

$$\begin{aligned}J(f) &= \int_{\mathbb{B}_n} dv_\alpha(z) \int_{\mathbb{B}_n} \frac{|f(z)-f(\varphi_z(w))|^p}{\rho(z,\varphi_z(w))^p} \frac{(1-|z|^2)^{n+1+\alpha}}{|1-\langle z,w\rangle|^{2(n+1+\alpha)}}\, dv_\alpha(w)\\ &= \int_{\mathbb{B}_n} dv_\alpha(z) \int_{\mathbb{B}_n} \frac{|f(z)-f(\varphi_z(w))|^p}{|w|^p} \frac{(1-|z|^2)^{n+1+\alpha}}{|1-\langle z,w\rangle|^{2(n+1+\alpha)}}\, dv_\alpha(w)\\ &\geq \int_{\mathbb{B}_n} dv_\alpha(z) \int_{\mathbb{B}_n} |f(z)-f(\varphi_z(w))|^p \frac{(1-|z|^2)^{n+1+\alpha}}{|1-\langle z,w\rangle|^{2(n+1+\alpha)}}\, dv_\alpha(w)\\ &\geq \int_{\mathbb{B}_n} dv_\alpha(z) \int_{D(z,r)} |f(z)-f(\varphi_z(w))|^p \frac{(1-|z|^2)^{n+1+\alpha}}{|1-\langle z,w\rangle|^{2(n+1+\alpha)}}\, dv_\alpha(w).\end{aligned}$$

From the first part of Lemma 2.4, there exists a positive constant C' such that

$$J(f) \geq C' \int_{\mathbb{B}_n} dv_\alpha(z) \int_{D(z,r)} \frac{|f(z)-f(\varphi_z(w))|^p}{(1-|z|^2)^{n+1+\alpha}}\, dv_\alpha(w).$$

Then Lemma 2.1 implies that there exists another positive constant C such that

$$J(f) \geq C \int_{\mathbb{B}_n} |f(z)-f(\varphi_z(z))|^p\, dv_\alpha(z) = C \int_{\mathbb{B}_n} |f(z)-f(0)|^p\, dv_\alpha(z).$$

The proof is complete. □

Lemma 3.2. *Suppose $\alpha > -1$ and $f \in A^p_\alpha$. Then*

$$\int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z)-f(w)|^p}{\rho(z,w)^p}\, dv_\alpha(z) dv_\alpha(w) < \infty.$$

Proof. Given $f \in A^p_\alpha$, from Theorem 2.3, there exists a continuous function $g \in L^p(\mathbb{B}_n, dv_\alpha)$ such that for all $z, w \in \mathbb{B}_n$,

$$|f(z)-f(w)| \leq \rho(z,w)(g(z)+g(w)).$$

There exists a positive constant C such that

$$\frac{|f(z)-f(w)|^p}{\rho(z,w)^p} \leq C(g(z)^p + g(w)^p).$$

So,

$$\begin{aligned}&\int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(w)-f(z)|^p}{\rho(z,w)^p}\, dv_\alpha(z) dv_\alpha(w)\\ &\qquad\leq C \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} (g(z)^p + g(w)^p)\, dv_\alpha(z) dv_\alpha(w)\\ &\qquad= 2C \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} g(z)^p\, dv_\alpha(z) dv_\alpha(w) < \infty.\end{aligned}$$

□

We can combine these two lemmas and obtain the following theorem.

Theorem 3.3. *Suppose that $\alpha > -1$. Then $f \in A^p_\alpha$ if and only if*

$$\int_{\mathbb{B}_n}\int_{\mathbb{B}_n} \frac{|f(z)-f(w)|^p}{\rho(z,w)^p}\, dv_\alpha(z)dv_\alpha(w) < \infty.$$

4. Bergman metric

Now, we replace metric ρ by Bergman metric β.

Lemma 4.1. *Suppose that $\alpha > -1$ and $f \in H(\mathbb{B}_n)$. If*

$$\int_{\mathbb{B}_n}\int_{\mathbb{B}_n} \frac{|f(z)-f(w)|^p}{\beta(z,w)^p}\, dv_\alpha(z)d\tau(w) < \infty,$$

then $f \in A^p_\alpha$, where

$$d\tau(w) = \frac{dv(w)}{(1-|w|^2)^{n+1}}$$

is the Mobius invariant volume measure on $\mathbb{B}_n$.

Proof. By Lemma 2.4, there exists a positive constant C such that

$$\begin{aligned}(1-|z|^2)^p|\nabla f(z)|^p \le& \frac{C}{(1-|z|^2)^{n+1}}\int_{D(z,r)} |f(z)-f(w)|^p dv(w)\\ \le& \frac{C}{(1-|z|^2)^{n+1+\alpha}}\int_{D(z,r)} |f(z)-f(w)|^p dv_\alpha(w).\end{aligned}$$

Since $D(z,r)$ is open unit ball in metric β, we have

$$(1-|z|^2)^p|\nabla f(z)|^p \le \frac{Cr^p}{(1-|z|^2)^{n+1+\alpha}}\int_{D(z,r)} \frac{|f(z)-f(w)|^p}{\beta(z,w)^p}\, dv_\alpha(w).$$

After integrating

$$\begin{aligned}\int_{\mathbb{B}_n}(1-|z|^2)^p|\nabla f(z)|^p dv_\alpha(z) \le& Cr^p\int_{\mathbb{B}_n}\int_{D(z,r)} \frac{|f(z)-f(w)|^p}{\beta(z,w)^p}\, dv_\alpha(w)d\tau(z)\\ \le& Cr^p\int_{\mathbb{B}_n}\int_{\mathbb{B}_n} \frac{|f(z)-f(w)|^p}{\beta(z,w)^p}\, dv_\alpha(w)d\tau(z).\end{aligned}$$

Therefore $(1-|z|^2)\nabla f(z) \in A^p_\alpha$. It follows from Theorem 2.16 of [9] that $f \in A^p_\alpha$. □

By the same reason as in Lemma 3.2, we can prove the following lemma.

Lemma 4.2. *Suppose $\alpha > -1$ and $f \in H(\mathbb{B}_n)$. If $f \in A^p_\alpha$, then*

$$\int_{\mathbb{B}_n}\int_{\mathbb{B}_n} \frac{|f(z)-f(w)|^p}{\beta(z,w)^p}\, dv_\alpha(z)dv_\alpha(w) < \infty.$$

5. Euclidean metric

Theorem 5.1. *Suppose $\alpha > -1$, $p = n+1+\alpha$ and $f \in A^p_\alpha$, then*

$$I(f) = \int_{\mathbb{B}_n}\int_{\mathbb{B}_n} \frac{|f(z)-f(w)|^p}{|z-w|^p}\, dv_\gamma(z)dv_\gamma(w) < \infty,$$

for any $\gamma > \alpha$.

Proof. Given $f \in A^p_\alpha$, from Theorem 2.3, there exists a continuous function $g \in L^p(\mathbb{B}_n, dv_\alpha)$ such that for all $z, w \in \mathbb{B}_n$,

$$|f(z)-f(w)| \le \rho(z,w)(g(z)+g(w)) \le \frac{|z-w|}{|1-\langle z,w\rangle|}(g(z)+g(w)).$$

There exists a positive constant C such that

$$\begin{aligned} I(f) \le & 2C\int_{\mathbb{B}_n} g(z)^p\, dv_\gamma(z)\int_{\mathbb{B}_n} \frac{dv_\gamma(w)}{|1-\langle z,w\rangle|^p} \\ = & 2C\int_{\mathbb{B}_n} g(z)^p\, dv_\gamma(z)\int_{\mathbb{B}_n} \frac{dv_\gamma(w)}{|1-\langle z,w\rangle|^{n+1+\alpha}}. \end{aligned}$$

Since $\alpha-\gamma < 0$, by Lemma 2.2, the last integral is bounded. Then there exists another positive constant M such that

$$\begin{aligned} I(f) \le & M\int_{\mathbb{B}_n} g(z)^p dv_\gamma(z) \\ = & Mc_\gamma\int_{\mathbb{B}_n} g(z)^p(1-|z|^2)^{\gamma-\alpha}(1-|z|^2)^\alpha dv(z) \\ < & M\frac{c_\gamma}{c_\alpha}\int_{\mathbb{B}_n} g(z)^p dv_\alpha(z) < \infty. \end{aligned}$$

□

Lemma 5.2. *Suppose $\alpha > -1$, $f \in H(\mathbb{B}_n)$ and δ and γ are real parameters such that*

$$\delta+\gamma = p+\alpha-(n+1), \quad -1 < \gamma < p-(n+1).$$

If $f \in A^p_\alpha$, then

$$I(f) = \int_{\mathbb{B}_n}\int_{\mathbb{B}_n} \frac{|f(z)-f(w)|^p}{|z-w|^p} dv_\delta(z)dv_\gamma(w) < \infty.$$

Proof. By the proof of the previous lemma, there exists a positive constant C such that

$$\begin{aligned} I(f) \le & 2C\int_{\mathbb{B}_n} g(z)^p\, dv_\delta(z)\int_{\mathbb{B}_n} \frac{dv_\gamma(w)}{|1-\langle z,w\rangle|^p} \\ = & 2C\int_{\mathbb{B}_n} g(z)^p\, dv_\delta(z)\int_{\mathbb{B}_n} \frac{dv_\gamma(w)}{|1-\langle z,w\rangle|^{n+1+\gamma+(\delta-\alpha)}}. \end{aligned}$$

Since $\delta - \alpha > 0$, by Lemma 2.2, there exists another positive constant M such that

$$I(f) \leq M \int_{\mathbb{B}_n} \frac{g(z)^p}{(1-|z|^2)^{\delta-\alpha}}\, dv_\delta(z) = M \int_{\mathbb{B}_n} g(z)^p \, dv_\alpha(z) < \infty.$$

□

Corollary 5.3. *Suppose that* $\alpha > -1$, $p > n+1+\alpha$ *and* $f \in A^p_\alpha$, *then*

$$\int_{\mathbb{B}_n}\int_{\mathbb{B}_n} \frac{|f(z)-f(w)|^p}{|z-w|^p}\, dv_t(z)dv_t(w) < \infty,$$

where $t = \frac{p+\alpha-(n+1)}{2}$.

If $n = 1$, then we obtain the following corollary.

Corollary 5.4. *Suppose that* $\alpha > -1$, $p > \alpha+2$ *and* $f \in A^p_\alpha(\mathbb{D})$, *then*

$$\int_{\mathbb{D}}\int_{\mathbb{D}} \frac{|f(z)-f(w)|^p}{|z-w|^p}\, dA_t(z)dA_t(w) < \infty,$$

where $t = \frac{p+\alpha-2}{2}$.

The symmetric lifting operator $L : H(\mathbb{D}) \to H(\mathbb{D}\times\mathbb{D})$ is defined by

$$L(f)(z,w) = \frac{f(z)-f(w)}{z-w}.$$

The action of symmetric lifting operator on $A^p_\alpha(\mathbb{D})$ in the cases $p > \alpha+2$ and $p < \alpha+2$ was studied in [8]. In the case $p = \alpha+2$, we have the following result.

Corollary 5.5. *Suppose that* $\alpha > -1$, $p = \alpha+2$. *Then the symmetric lifting operator maps* $A^p_\alpha(\mathbb{D})$ *into* $A^p_\gamma(\mathbb{D}^2)$, *for any* $\gamma > \alpha$.

Proof. The result follows by letting $n = 1$ in Theorem 5.1. □

If $\alpha > -1$, $p > \alpha+2$ and $f \in A^p_\alpha(\mathbb{D})$, then by Corollary 5.4, $L(f) \in A^p_t(\mathbb{D}^2)$, which means that the symmetric lifting operator maps $f \in A^p_\alpha(\mathbb{D})$ into $A^p_t(\mathbb{D}^2)$, for $t = \frac{p+\alpha-2}{2}$. This is the Theorem 4.4 in [8].

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