Double Integral Characterization for Bergman Spaces

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\textbf{Abstract.} In this paper we characterize Bergman spaces with respect to double integral of the functions $|f(z) - f(w)|/|z - w|$, $|f(z) - f(w)|/\rho(z, w)$ and $|f(z) - f(w)|/\beta(z, w)$, where $\rho$ and $\beta$ are the pseudo-hyperbolic and hyperbolic metrics. We prove some necessary and sufficient conditions that imply a function to be in Bergman spaces.

\textbf{Keywords:} Bergman spaces, Pseudo-hyperbolic metric, Hyperbolic metric, Double integral.


1. \textbf{Introduction}

For $z = (z_1, \cdots, z_n)$ and $w = (w_1, \cdots, w_n)$ in $\mathbb{C}^n$, we define $\langle z, w \rangle = z_1\overline{w_1} + \cdots + z_n\overline{w_n}$, where $\overline{w_k}$ is the complex conjugate of $w_k$. We also write $|z| = \sqrt{\langle z, z \rangle} = \sqrt{|z_1|^2 + \cdots + |z_n|^2}$. Let $\mathbb{B}_n$ denotes the open unit ball of $\mathbb{C}^n$, that is

$$\mathbb{B}_n = \{ z \in \mathbb{C}^n : |z| < 1 \}.$$ 

For any $a \in \mathbb{B}_n - \{0\}$, we define

$$\varphi_a(z) = \frac{a - P_a(z) - s_a Q_a(z)}{1 - \langle z, a \rangle} \quad z \in \mathbb{B}_n,$$

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Received 13 July 2013; Accepted 22 November 2015

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where \( s_a = \sqrt{1 - |a|^2} \), \( P_a \) is the orthogonal projection from \( \mathbb{C}^n \) onto the subspace \([a]\) generated by \( a \), and \( Q_a \) is the orthogonal projection from \( \mathbb{C}^n \) onto \( \mathbb{C}^n - [a] \). When \( a = 0 \), we define \( \varphi_a(z) = -z \). These functions are called involutions. (see [9] for more information about these functions).

The hyperbolic metric (Bergman metric) is defined by

\[
\beta(z,w) = \frac{1}{2} \log \frac{1 + |\varphi_a(w)|}{1 - |\varphi_a(w)|},
\]

for any \( z, w \in \mathbb{B}_n \).

For any \( z \in \mathbb{B}_n \) and \( r > 0 \), we denote Bergman metric ball at \( z \) by \( D(z, r) \). That is

\[
D(z, r) = \{ w \in \mathbb{B}_n : \beta(z, w) < r \}.
\]

Also, pseudo-hyperbolic metric is defined by \( \rho(z, w) = |\varphi_a(w)| \).

For \( \alpha > -1 \) let

\[
dv_{\alpha}(z) = c_{\alpha}(1 - |z|^2)^{\alpha} dv(z)
\]

where \( dv(z) \) is the Lebesgue volume measure on \( \mathbb{B}_n \) and \( c_{\alpha} \) is a positive constant with \( v_0(\mathbb{B}_n) = 1 \). For \( 0 < p < \infty \) and \( \alpha > -1 \), the weighted Bergman space \( A^p_{\alpha} \) consists of all holomorphic functions in \( L^p(\mathbb{B}_n, dv_{\alpha}) \), that is

\[
A^p_{\alpha} = \left\{ f \in H(\mathbb{B}_n) : ||f||_{p,\alpha}^p = \int_{\mathbb{B}_n} |f(z)|^p dv_{\alpha}(z) < \infty \right\}
\]

Wulan and Zhu [8], characterized Bergman spaces with standard weights in terms of Lipschitz type conditions in the Euclidean, hyperbolic, and pseudo-hyperbolic metrics. In [4] Li et al. proved that a holomorphic function \( f \) belongs to the \( A^p_{\alpha} \), \( p > n + 1 + \alpha \), if and only if the function \( |f(z) - f(w)|/|1 - \langle z, w \rangle| \) is in \( L^p(\mathbb{B}_n \times \mathbb{B}_n, dv_{\alpha} \times dv_{\alpha}) \), where \( \gamma = (p + \alpha - n - 1)/2 \).

Also, it was shown in [5] that for the case \( 0 < p < n + 1 + \alpha \), \( f \in A^p_{\alpha} \) if and only if the function \( |f(z) - f(w)|/|1 - \langle z, w \rangle| \) is in \( L^p(\mathbb{B}_n \times \mathbb{B}_n, dv_{\alpha} \times dv_{\alpha}) \) if and only if the function \( |f(z) - f(w)|/|z - w| \) is in \( L^p(\mathbb{B}_n \times \mathbb{B}_n, dv_{\alpha} \times dv_{\alpha}) \) for any \( \gamma > \alpha \). Our results are applicable for studying the action of symmetric lifting operator on \( A^p_{\alpha} \) in all cases especially for the case \( p = \alpha + 2 \).

Also we replace the Euclidean metric with pseudo-hyperbolic metric \( \rho \) and Bergman metric \( \beta \).

2. **Preliminaries**

**Lemma 2.1.** [9] There exists a positive constant \( C \) such that

\[
|f(z)|^p \leq \frac{C}{(1 - |z|^2)^{n+1+\alpha}} \int_{D(z,r)} |f(w)|^p \ dv_{\alpha}(w)
\]

for all \( f \in H(\mathbb{B}_n) \) and \( z \in \mathbb{B}_n \).
Lemma 2.2. [9] Suppose $s > -1$, $t$ is real, and
\[ I(z) = \int_{\mathbb{B}_n} \frac{(1 - |w|^2)^s}{|1 - \langle z, w \rangle|^{n+1+s+t}} \, dv(w), \quad z \in \mathbb{B}_n. \]
Then $I(z)$ is bounded in $\mathbb{B}_n$ whenever $t < 0$, and $I(z)$ is bounded by $(1 - |z|^2)^{-t}$ whenever $t > 0$.

Theorem 2.3. [8] Suppose that $p > 0$, $\alpha > -1$ and $f$ is analytic in $\mathbb{B}_n$. Then the following conditions are equivalent.

1. $f \in \mathcal{A}_n^p$.

2. There exists a continuous function $g$ in $L^p(\mathbb{B}_n, dv_\alpha)$ such that
\[ |f(z) - f(w)| \leq \rho(z, w)(g(z) + g(w)), \quad z, w \in \mathbb{B}_n. \]

3. There exists a continuous function $g$ in $L^p(\mathbb{B}_n, dv_\alpha)$ such that
\[ |f(z) - f(w)| \leq \beta(z, w)(g(z) + g(w)), \quad z, w \in \mathbb{B}_n. \]

4. There exists a continuous function $g$ in $L^p(\mathbb{B}_n, dv_{p+\alpha})$ such that
\[ |f(z) - f(w)| \leq |z - w|(g(z) + g(w)), \quad z, w \in \mathbb{B}_n. \]

Lemma 2.4. [4] Let $r > 0$. Then
\[ 1 - |z|^2 \sim 1 - |w|^2 \sim |1 - \langle z, w \rangle| \]
for all $z \in \mathbb{B}_n$ and $w \in D(z, r)$. Furthermore, there exists a positive constant $C$ such that
\[ (1 - |z|^2)^p |\nabla f(z)|^p \leq \frac{C}{(1 - |z|^2)^{n+1}} \int_{D(z, r)} |f(w) - f(z)|^p \, dv(w) \]
for all $z \in \mathbb{B}_n$ and $f \in H(\mathbb{B}_n)$.

3. PSEUDO-HYPERBOLIC METRIC

Lemma 3.1. Suppose $\alpha > -1$ and $f \in H(\mathbb{B}_n)$. Then there exists a positive constant $C$ such that
\[ \int_{\mathbb{B}_n} |f(z) - f(0)|^p \, dv_\alpha(z) \leq C \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z) - f(w)|^p}{\rho(z, w)^p} \, dv_\alpha(z)dv_\alpha(w). \]

Proof. Let
\[ J(f) = \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z) - f(w)|^p}{\rho(z, w)^p} \, dv_\alpha(z)dv_\alpha(w). \]
By making a change of variable, we have
\[
J(f) = \int_{\mathbb{B}_n} dv_\alpha(z) \int_{\mathbb{B}_n} \frac{|f(z) - f(\varphi_z(w))|^p}{\rho(z, \varphi_z(w))^p} \frac{(1 - |z|^2)^{n+1+\alpha}}{|1 - \langle z, w \rangle|^{2(\alpha+1+\alpha)}} \ dv_\alpha(w)
\]
\[
= \int_{\mathbb{B}_n} dv_\alpha(z) \int_{\mathbb{B}_n} \frac{|f(z) - f(\varphi_z(w))|^p}{|w|^p} \frac{(1 - |z|^2)^{n+1+\alpha}}{|1 - \langle z, w \rangle|^{2(\alpha+1+\alpha)}} \ dv_\alpha(w)
\]
\[
\geq \int_{\mathbb{B}_n} dv_\alpha(z) \int_{\mathbb{B}_n} |f(z) - f(\varphi_z(w))|^p \frac{(1 - |z|^2)^{n+1+\alpha}}{|1 - \langle z, w \rangle|^{2(\alpha+1+\alpha)}} \ dv_\alpha(w)
\]
\[
\geq \int_{\mathbb{B}_n} dv_\alpha(z) \int_{\mathbb{D}(z,r)} |f(z) - f(\varphi_z(w))|^p \frac{(1 - |z|^2)^{n+1+\alpha}}{|1 - \langle z, w \rangle|^{2(\alpha+1+\alpha)}} \ dv_\alpha(w).
\]
From the first part of Lemma 2.4, there exists a positive constant $C'$ such that
\[
J(f) \geq C' \int_{\mathbb{B}_n} dv_\alpha(z) \int_{\mathbb{D}(z,r)} |f(z) - f(\varphi_z(w))|^p \ dv_\alpha(z) - dv_\alpha(w).
\]
Then Lemma 2.1 implies that there exists another positive constant $C$ such that
\[
J(f) \geq C \int_{\mathbb{B}_n} |f(z) - f(\varphi_z(z))|^p \ dv_\alpha(z) = C \int_{\mathbb{B}_n} |f(z) - f(0)|^p \ dv_\alpha(z).
\]
The proof is complete.

**Lemma 3.2.** Suppose $\alpha > -1$ and $f \in A^p_\beta$. Then
\[
\int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z) - f(w)|^p}{\rho(z, w)^p} \ dv_\alpha(z)dv_\alpha(w) < \infty.
\]

**Proof.** Given $f \in A^p_\beta$, from Theorem 2.3, there exists a continuous function $g \in L^p(\mathbb{B}_n, dv_\alpha)$ such that for all $z, w \in \mathbb{B}_n$,
\[
|f(z) - f(w)| \leq \rho(z, w)(g(z) + g(w)).
\]
There exists a positive constant $C$ such that
\[
|f(z) - f(w)|^p \rho(z, w)^p \leq C(g(z)^p + g(w)^p).
\]
So,
\[
\int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(w) - f(z)|^p}{\rho(z, w)^p} \ dv_\alpha(z)dv_\alpha(w)
\]
\[
\leq C \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} (g(z)^p + g(w)^p) \ dv_\alpha(z)dv_\alpha(w)
\]
\[
= 2C \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} g(z)^p \ dv_\alpha(z)dv_\alpha(w) < \infty.
\]

We can combine these two lemmas and obtain the following theorem.
Theorem 3.3. Suppose that $\alpha > -1$. Then $f \in A^p_\alpha$ if and only if
\[
\int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z) - f(w)|^p}{\rho(z,w)^p} \, dv_\alpha(z)dv_\alpha(w) < \infty.
\]

4. BERGMAN METRIC

Now, we replace metric $\rho$ by Bergman metric $\beta$.

Lemma 4.1. Suppose that $\alpha > -1$ and $f \in H(\mathbb{B}_n)$. If
\[
\int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z) - f(w)|^p}{\beta(z,w)^p} \, dv_\alpha(z)d\tau(w) < \infty,
\]
then $f \in A^p_\alpha$, where
\[
d\tau(w) = \frac{dw(w)}{(1 - |w|^2)^{n+1}}
\]
is the Mobius invariant volume measure on $\mathbb{B}_n$.

Proof. By Lemma 2.4, there exists a positive constant $C$ such that
\[
(1 - |z|^2)^p|\nabla f(z)|^p \leq \frac{C}{(1 - |z|^2)^{n+1+\alpha}} \int_{D(z,r)} |f(z) - f(w)|^p dv(w)
\leq \frac{C}{(1 - |z|^2)^{n+1+\alpha}} \int_{D(z,r)} |f(z) - f(w)|^p dv_\alpha(w).
\]
Since $D(z,r)$ is open unit ball in metric $\beta$, we have
\[
(1 - |z|^2)^p|\nabla f(z)|^p \leq \frac{Cr^p}{(1 - |z|^2)^{n+1+\alpha}} \int_{D(z,r)} \frac{|f(z) - f(w)|^p}{\beta(z,w)^p} \, dv_\alpha(w).
\]
After integrating
\[
\int_{\mathbb{B}_n} (1 - |z|^2)^p|\nabla f(z)|^p dv_\alpha(z) \leq Cr^p \int_{\mathbb{B}_n} \int_{D(z,r)} \frac{|f(z) - f(w)|^p}{\beta(z,w)^p} \, dv_\alpha(w)d\tau(z)
\leq Cr^p \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z) - f(w)|^p}{\beta(z,w)^p} \, dv_\alpha(z)d\tau(w).
\]
Therefore $(1 - |z|^2)|\nabla f(z)| \in A^p_\alpha$. It follows from Theorem 2.16 of [9] that
$f \in A^p_\alpha$.

By the same reason as in Lemma 3.2, we can prove the following lemma.

Lemma 4.2. Suppose $\alpha > -1$ and $f \in H(\mathbb{B}_n)$. If $f \in A^p_\alpha$, then
\[
\int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z) - f(w)|^p}{\beta(z,w)^p} \, dv_\alpha(z)dv_\alpha(w) < \infty.
\]
5. Euclidean metric

Theorem 5.1. Suppose $\alpha > -1$, $p = n + 1 + \alpha$ and $f \in A^p_\alpha$, then

$$I(f) = \int_{B_n} \int_{B_n} \frac{|f(z) - f(w)|^p}{|z - w|^p} \, dv_\gamma(z) dv_\gamma(w) < \infty,$$

for any $\gamma > \alpha$.

Proof. Given $f \in A^p_\alpha$, from Theorem 2.3, there exists a continuous function $g \in L^p(B_n, dv_\alpha)$ such that for all $z, w \in B_n$,

$$|f(z) - f(w)| \leq \rho(z, w)(g(z) + g(w)) \leq \frac{|z - w|}{|1 - \langle z, w \rangle|}(g(z) + g(w)).$$

There exists a positive constant $C$ such that

$$I(f) \leq 2C \int_{B_n} g(z)^p \, dv_\gamma(z) \int_{B_n} \frac{dv_\gamma(w)}{|1 - \langle z, w \rangle|^p}.$$

Since $\alpha - \gamma < 0$, by Lemma 2.2, the last integral is bounded. Then there exists another positive constant $M$ such that

$$I(f) \leq M \int_{B_n} g(z)^p dv_\gamma(z) \int_{B_n} \frac{dv_\gamma(w)}{|1 - \langle z, w \rangle|^{n+1+\gamma+\delta - \alpha}} < \infty.$$

Lemma 5.2. Suppose $\alpha > -1$, $f \in H(B_n)$ and $\delta$ and $\gamma$ are real parameters such that

$$\delta + \gamma = p + \alpha - (n+1), \quad -1 < \gamma < p - (n+1).$$

If $f \in A^p_\alpha$, then

$$I(f) = \int_{B_n} \int_{B_n} \frac{|f(z) - f(w)|^p}{|z - w|^p} \, dv_\delta(z) dv_\gamma(w) < \infty.$$

Proof. By the proof of the previous lemma, there exists a positive constant $C$ such that

$$I(f) \leq 2C \int_{B_n} g(z)^p \, dv_\delta(z) \int_{B_n} \frac{dv_\gamma(w)}{|1 - \langle z, w \rangle|^p}.$$
Since \( \delta - \alpha > 0 \), by Lemma 2.2, there exists another positive constant \( M \) such that
\[
I(f) \leq M \int_{B_n} \frac{g(z)^p}{(1 - |z|^2)^{\delta - \alpha}} \ dv_\delta(z) = M \int_{B_n} g(z)^p \ dv_\alpha(z) < \infty.
\]

\[\square\]

**Corollary 5.3.** Suppose that \( \alpha > -1, p > n + 1 + \alpha \) and \( f \in A_\alpha^p \), then
\[
\int_{B_n} \int_{B_n} \frac{|f(z) - f(w)|^p}{|z - w|^p} \ dv_\delta(z)dv_\delta(w) < \infty,
\]
where \( t = \frac{p+\alpha-(n+1)}{2} \).

If \( n = 1 \), then we obtain the following corollary.

**Corollary 5.4.** Suppose that \( \alpha > -1, p > \alpha + 2 \) and \( f \in A_\alpha^p(D) \), then
\[
\int_{D} \int_{D} \frac{|f(z) - f(w)|^p}{|z - w|^p} \ dA_t(z)dA_t(w) < \infty,
\]
where \( t = \frac{p+\alpha-2}{2} \).

The symmetric lifting operator \( L : H(D) \to H(D \times D) \) is defined by
\[
L(f)(z, w) = \frac{f(z) - f(w)}{z - w}.
\]

The action of symmetric lifting operator on \( A_\alpha^p(D) \) in the cases \( p > \alpha + 2 \) and \( p < \alpha + 2 \) was studied in [8]. In the case \( p = \alpha + 2 \), we have the following result.

**Corollary 5.5.** Suppose that \( \alpha > -1, p = \alpha + 2 \). Then the symmetric lifting operator maps \( A_\alpha^p(D) \) into \( A_\gamma^p(D^2) \), for any \( \gamma > \alpha \).

**Proof.** The result follows by letting \( n = 1 \) in Theorem 5.1. \( \square \)

If \( \alpha > -1, p > \alpha + 2 \) and \( f \in A_\alpha^p(D) \), then by Corollary 5.4, \( L(f) \in A_\gamma^p(D^2) \), which means that the symmetric lifting operator maps \( f \in A_\alpha^p(D) \) into \( A_\gamma^p(D^2) \), for \( t = \frac{p+\alpha-2}{2} \). This is the Theorem 4.4 in [8].

**Acknowledgments**

The authors would like to thank the referees for their useful comments.

**References**