

## Double Integral Characterization for Bergman Spaces

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ABSTRACT. In this paper we characterize Bergman spaces with respect to double integral of the functions  $|f(z) - f(w)|/|z - w|$ ,  $|f(z) - f(w)|/\rho(z, w)$  and  $|f(z) - f(w)|/\beta(z, w)$ , where  $\rho$  and  $\beta$  are the pseudo-hyperbolic and hyperbolic metrics. We prove some necessary and sufficient conditions that implies a function to be in Bergman spaces.

**Keywords:** Bergman spaces, Pseudo-hyperbolic metric, Hyperbolic metric, Double integral.

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### 1. INTRODUCTION

For  $z = (z_1, \dots, z_n)$  and  $w = (w_1, \dots, w_n)$  in  $\mathbb{C}^n$ , we define  $\langle z, w \rangle = z_1 \bar{w}_1 + \dots + z_n \bar{w}_n$ , where  $\bar{w}_k$  is the complex conjugate of  $w_k$ . We also write  $|z| = \sqrt{\langle z, z \rangle} = \sqrt{|z_1|^2 + \dots + |z_n|^2}$ . Let  $\mathbb{B}_n$  denotes the open unit ball of  $\mathbb{C}^n$ , that is

$$\mathbb{B}_n = \{z \in \mathbb{C}^n : |z| < 1\}.$$

For any  $a \in \mathbb{B}_n - \{0\}$ , we define

$$\varphi_a(z) = \frac{a - P_a(z) - s_a Q_a(z)}{1 - \langle z, a \rangle} \quad z \in \mathbb{B}_n,$$

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where  $s_a = \sqrt{1 - |a|^2}$ ,  $P_a$  is the orthogonal projection from  $\mathbb{C}^n$  onto the subspace  $[a]$  generated by  $a$ , and  $Q_a$  is the orthogonal projection from  $\mathbb{C}^n$  onto  $\mathbb{C}^n - [a]$ . When  $a = 0$ , we define  $\varphi_a(z) = -z$ . These functions are called involutions. (see [9] for more information about these functions)

The hyperbolic metric (Bergman metric) is defined by

$$\beta(z, w) = \frac{1}{2} \log \frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|}, \quad z, w \in \mathbb{B}_n.$$

For any  $z \in \mathbb{B}_n$  and  $r > 0$ , we denote Bergman metric ball at  $z$  by  $D(z, r)$ . That is

$$D(z, r) = \{w \in \mathbb{B}_n : \beta(z, w) < r\}.$$

Also, pseudo-hyperbolic metric is defined by  $\rho(z, w) = |\varphi_z(w)|$ .

For  $\alpha > -1$  let

$$dv_\alpha(z) = c_\alpha(1 - |z|^2)^\alpha dv(z)$$

where  $dv(z)$  is the Lebesgue volume measure on  $\mathbb{B}_n$  and  $c_\alpha$  is a positive constant with  $v_\alpha(\mathbb{B}_n) = 1$ . For  $0 < p < \infty$  and  $\alpha > -1$ , the weighted Bergman space  $A_\alpha^p$  consists of all holomorphic functions in  $L^p(\mathbb{B}_n, dv_\alpha)$ , that is

$$A_\alpha^p = \left\{ f \in H(\mathbb{B}_n) : \|f\|_{\alpha,p}^p = \int_{\mathbb{B}_n} |f(z)|^p dv_\alpha(z) < \infty. \right\}$$

Wulan and Zhu [8], characterized Bergman spaces with standard weights in terms of Lipschitz type conditions in the Euclidean, hyperbolic, and pseudo-hyperbolic metrics. In [4] Li et al. proved that a holomorphic function  $f$  belongs to the  $A_\alpha^p$ ,  $p > n + 1 + \alpha$ , if and only if the function  $|f(z) - f(w)|/|1 - \langle z, w \rangle|$  is in  $L^p(\mathbb{B}_n \times \mathbb{B}_n, dv_\gamma \times dv_\gamma)$ , where  $\gamma = (p + \alpha - n - 1)/2$ .

Also, it was shown in [5] that for the case  $0 < p < n + 1 + \alpha$ ,  $f \in A_\alpha^p$  if and only if the function  $|f(z) - f(w)|/|1 - \langle z, w \rangle|$  is in  $L^p(\mathbb{B}_n \times \mathbb{B}_n, dv_\alpha \times dv_\alpha)$  if and only if the function  $|f(z) - f(w)|/|z - w|$  is in  $L^p(\mathbb{B}_n \times \mathbb{B}_n, dv_\alpha \times dv_\alpha)$ .

Our aim in this paper is to prove, for  $f \in A_\alpha^p$ ,  $p > n + 1 + \alpha$ , the function  $|f(z) - f(w)|/|z - w|$  is in  $L^p(\mathbb{B}_n \times \mathbb{B}_n, dv_t \times dv_t)$ , where  $t = (p + \alpha - n - 1)/2$  and if  $p = n + 1 + \alpha$ , then  $|f(z) - f(w)|/|z - w|$  is in  $L^p(\mathbb{B}_n \times \mathbb{B}_n, dv_\gamma \times dv_\gamma)$ , for any  $\gamma > \alpha$ . Our results are applicable for studying the action of symmetric lifting operator on  $A_\alpha^p$  in all cases especially for the case  $p = \alpha + 2$ .

Also we replace the Euclidean metric with pseudo-hyperbolic metric  $\rho$  and Bergman metric  $\beta$ .

## 2. PRELIMINARIES

**Lemma 2.1.** [9] *There exists a positive constant  $C$  such that*

$$|f(z)|^p \leq \frac{C}{(1 - |z|^2)^{n+1+\alpha}} \int_{D(z,r)} |f(w)|^p dv_\alpha(w)$$

for all  $f \in H(\mathbb{B}_n)$  and  $z \in \mathbb{B}_n$ .

**Lemma 2.2.** [9] *Suppose  $s > -1$ ,  $t$  is real, and*

$$I(z) = \int_{\mathbb{B}_n} \frac{(1 - |w|^2)^s}{|1 - \langle z, w \rangle|^{n+1+s+t}} dv(w), \quad z \in \mathbb{B}_n.$$

*Then  $I(z)$  is bounded in  $\mathbb{B}_n$  whenever  $t < 0$ , and  $I(z)$  is bounded by  $(1 - |z|^2)^{-t}$  whenever  $t > 0$ .*

**Theorem 2.3.** [8] *Suppose that  $p > 0$ ,  $\alpha > -1$  and  $f$  is analytic in  $\mathbb{B}_n$ . Then the following conditions are equivalent.*

- (1)  $f \in A_\alpha^p$ .
- (2) *There exists a continuous function  $g$  in  $L^p(\mathbb{B}_n, dv_\alpha)$  such that*

$$|f(z) - f(w)| \leq \rho(z, w)(g(z) + g(w)), \quad z, w \in \mathbb{B}_n.$$

- (3) *There exists a continuous function  $g$  in  $L^p(\mathbb{B}_n, dv_\alpha)$  such that*

$$|f(z) - f(w)| \leq \beta(z, w)(g(z) + g(w)), \quad z, w \in \mathbb{B}_n.$$

- (4) *There exists a continuous function  $g$  in  $L^p(\mathbb{B}_n, dv_{p+\alpha})$  such that*

$$|f(z) - f(w)| \leq |z - w|(g(z) + g(w)), \quad z, w \in \mathbb{B}_n.$$

**Lemma 2.4.** [4] *Let  $r > 0$ . Then*

$$1 - |z|^2 \sim 1 - |w|^2 \sim |1 - \langle z, w \rangle|$$

*for all  $z \in \mathbb{B}_n$  and  $w \in D(z, r)$ . Furthermore, there exists a positive constant  $C$  such that*

$$(1 - |z|^2)^p |\nabla f(z)|^p \leq \frac{C}{(1 - |z|^2)^{n+1}} \int_{D(z, r)} |f(w) - f(z)|^p dv(w)$$

*for all  $z \in \mathbb{B}_n$  and  $f \in H(\mathbb{B}_n)$ .*

### 3. PSEUDO-HYPERBOLIC METRIC

**Lemma 3.1.** *Suppose  $\alpha > -1$  and  $f \in H(\mathbb{B}_n)$ . Then there exists a positive constant  $C$  such that*

$$\int_{\mathbb{B}_n} |f(z) - f(0)|^p dv_\alpha(z) \leq C \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z) - f(w)|^p}{\rho(z, w)^p} dv_\alpha(z) dv_\alpha(w).$$

*Proof.* Let

$$J(f) = \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z) - f(w)|^p}{\rho(z, w)^p} dv_\alpha(z) dv_\alpha(w).$$

By making a change of variable, we have

$$\begin{aligned} J(f) &= \int_{\mathbb{B}_n} dv_\alpha(z) \int_{\mathbb{B}_n} \frac{|f(z) - f(\varphi_z(w))|^p}{\rho(z, \varphi_z(w))^p} \frac{(1 - |z|^2)^{n+1+\alpha}}{|1 - \langle z, w \rangle|^{2(n+1+\alpha)}} dv_\alpha(w) \\ &= \int_{\mathbb{B}_n} dv_\alpha(z) \int_{\mathbb{B}_n} \frac{|f(z) - f(\varphi_z(w))|^p}{|w|^p} \frac{(1 - |z|^2)^{n+1+\alpha}}{|1 - \langle z, w \rangle|^{2(n+1+\alpha)}} dv_\alpha(w) \\ &\geq \int_{\mathbb{B}_n} dv_\alpha(z) \int_{\mathbb{B}_n} |f(z) - f(\varphi_z(w))|^p \frac{(1 - |z|^2)^{n+1+\alpha}}{|1 - \langle z, w \rangle|^{2(n+1+\alpha)}} dv_\alpha(w) \\ &\geq \int_{\mathbb{B}_n} dv_\alpha(z) \int_{D(z,r)} |f(z) - f(\varphi_z(w))|^p \frac{(1 - |z|^2)^{n+1+\alpha}}{|1 - \langle z, w \rangle|^{2(n+1+\alpha)}} dv_\alpha(w). \end{aligned}$$

From the first part of Lemma 2.4, there exists a positive constant  $C'$  such that

$$J(f) \geq C' \int_{\mathbb{B}_n} dv_\alpha(z) \int_{D(z,r)} \frac{|f(z) - f(\varphi_z(w))|^p}{(1 - |z|^2)^{n+1+\alpha}} dv_\alpha(w).$$

Then Lemma 2.1 implies that there exists another positive constant  $C$  such that

$$J(f) \geq C \int_{\mathbb{B}_n} |f(z) - f(\varphi_z(z))|^p dv_\alpha(z) = C \int_{\mathbb{B}_n} |f(z) - f(0)|^p dv_\alpha(z).$$

The proof is complete.  $\square$

**Lemma 3.2.** *Suppose  $\alpha > -1$  and  $f \in A_\alpha^p$ . Then*

$$\int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z) - f(w)|^p}{\rho(z, w)^p} dv_\alpha(z) dv_\alpha(w) < \infty.$$

*Proof.* Given  $f \in A_\alpha^p$ , from Theorem 2.3, there exists a continuous function  $g \in L^p(\mathbb{B}_n, dv_\alpha)$  such that for all  $z, w \in \mathbb{B}_n$ ,

$$|f(z) - f(w)| \leq \rho(z, w)(g(z) + g(w)).$$

There exists a positive constant  $C$  such that

$$\frac{|f(z) - f(w)|^p}{\rho(z, w)^p} \leq C(g(z)^p + g(w)^p).$$

So,

$$\begin{aligned} \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(w) - f(z)|^p}{\rho(z, w)^p} dv_\alpha(z) dv_\alpha(w) &\leq C \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} (g(z)^p + g(w)^p) dv_\alpha(z) dv_\alpha(w) \\ &= 2C \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} g(z)^p dv_\alpha(z) dv_\alpha(w) < \infty. \end{aligned}$$

$\square$

We can combine these two lemmas and obtain the following theorem.

**Theorem 3.3.** *Suppose that  $\alpha > -1$ . Then  $f \in A_\alpha^p$  if and only if*

$$\int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z) - f(w)|^p}{\rho(z, w)^p} dv_\alpha(z) dv_\alpha(w) < \infty.$$

4. BERGMAN METRIC

Now, we replace metric  $\rho$  by Bergman metric  $\beta$ .

**Lemma 4.1.** *Suppose that  $\alpha > -1$  and  $f \in H(\mathbb{B}_n)$ . If*

$$\int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z) - f(w)|^p}{\beta(z, w)^p} dv_\alpha(z) d\tau(w) < \infty,$$

then  $f \in A_\alpha^p$ , where

$$d\tau(w) = \frac{dv(w)}{(1 - |w|^2)^{n+1}}$$

is the Mobius invariant volume measure on  $\mathbb{B}_n$ .

*Proof.* By Lemma 2.4, there exists a positive constant  $C$  such that

$$\begin{aligned} (1 - |z|^2)^p |\nabla f(z)|^p &\leq \frac{C}{(1 - |z|^2)^{n+1}} \int_{D(z,r)} |f(z) - f(w)|^p dv(w) \\ &\leq \frac{C}{(1 - |z|^2)^{n+1+\alpha}} \int_{D(z,r)} |f(z) - f(w)|^p dv_\alpha(w). \end{aligned}$$

Since  $D(z, r)$  is open unit ball in metric  $\beta$ , we have

$$(1 - |z|^2)^p |\nabla f(z)|^p \leq \frac{Cr^p}{(1 - |z|^2)^{n+1+\alpha}} \int_{D(z,r)} \frac{|f(z) - f(w)|^p}{\beta(z, w)^p} dv_\alpha(w).$$

After integrating

$$\begin{aligned} \int_{\mathbb{B}_n} (1 - |z|^2)^p |\nabla f(z)|^p dv_\alpha(z) &\leq Cr^p \int_{\mathbb{B}_n} \int_{D(z,r)} \frac{|f(z) - f(w)|^p}{\beta(z, w)^p} dv_\alpha(w) d\tau(z) \\ &\leq Cr^p \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z) - f(w)|^p}{\beta(z, w)^p} dv_\alpha(w) d\tau(z). \end{aligned}$$

Therefore  $(1 - |z|^2) \nabla f(z) \in A_\alpha^p$ . It follows from Theorem 2.16 of [9] that  $f \in A_\alpha^p$ . □

By the same reason as in Lemma 3.2, we can prove the following lemma.

**Lemma 4.2.** *Suppose  $\alpha > -1$  and  $f \in H(\mathbb{B}_n)$ . If  $f \in A_\alpha^p$ , then*

$$\int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z) - f(w)|^p}{\beta(z, w)^p} dv_\alpha(z) dv_\alpha(w) < \infty.$$

5. EUCLIDEAN METRIC

**Theorem 5.1.** *Suppose  $\alpha > -1$ ,  $p = n + 1 + \alpha$  and  $f \in A_\alpha^p$ , then*

$$I(f) = \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z) - f(w)|^p}{|z - w|^p} dv_\gamma(z)dv_\gamma(w) < \infty,$$

for any  $\gamma > \alpha$ .

*Proof.* Given  $f \in A_\alpha^p$ , from Theorem 2.3, there exists a continuous function  $g \in L^p(\mathbb{B}_n, dv_\alpha)$  such that for all  $z, w \in \mathbb{B}_n$ ,

$$|f(z) - f(w)| \leq \rho(z, w)(g(z) + g(w)) \leq \frac{|z - w|}{|1 - \langle z, w \rangle|} (g(z) + g(w)).$$

There exists a positive constant  $C$  such that

$$\begin{aligned} I(f) &\leq 2C \int_{\mathbb{B}_n} g(z)^p dv_\gamma(z) \int_{\mathbb{B}_n} \frac{dv_\gamma(w)}{|1 - \langle z, w \rangle|^p} \\ &= 2C \int_{\mathbb{B}_n} g(z)^p dv_\gamma(z) \int_{\mathbb{B}_n} \frac{dv_\gamma(w)}{|1 - \langle z, w \rangle|^{n+1+\alpha}}. \end{aligned}$$

Since  $\alpha - \gamma < 0$ , by Lemma 2.2, the last integral is bounded. Then there exists another positive constant  $M$  such that

$$\begin{aligned} I(f) &\leq M \int_{\mathbb{B}_n} g(z)^p dv_\gamma(z) \\ &= M c_\gamma \int_{\mathbb{B}_n} g(z)^p (1 - |z|^2)^{\gamma-\alpha} (1 - |z|^2)^\alpha dv(z) \\ &< M \frac{c_\gamma}{c_\alpha} \int_{\mathbb{B}_n} g(z)^p dv_\alpha(z) < \infty. \end{aligned}$$

□

**Lemma 5.2.** *Suppose  $\alpha > -1$ ,  $f \in H(\mathbb{B}_n)$  and  $\delta$  and  $\gamma$  are real parameters such that*

$$\delta + \gamma = p + \alpha - (n + 1), \quad -1 < \gamma < p - (n + 1).$$

If  $f \in A_\alpha^p$ , then

$$I(f) = \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z) - f(w)|^p}{|z - w|^p} dv_\delta(z)dv_\gamma(w) < \infty.$$

*Proof.* By the proof of the previous lemma, there exists a positive constant  $C$  such that

$$\begin{aligned} I(f) &\leq 2C \int_{\mathbb{B}_n} g(z)^p dv_\delta(z) \int_{\mathbb{B}_n} \frac{dv_\gamma(w)}{|1 - \langle z, w \rangle|^p} \\ &= 2C \int_{\mathbb{B}_n} g(z)^p dv_\delta(z) \int_{\mathbb{B}_n} \frac{dv_\gamma(w)}{|1 - \langle z, w \rangle|^{n+1+\gamma+(\delta-\alpha)}}. \end{aligned}$$

Since  $\delta - \alpha > 0$ , by Lemma 2.2, there exists another positive constant  $M$  such that

$$I(f) \leq M \int_{\mathbb{B}_n} \frac{g(z)^p}{(1 - |z|^2)^{\delta - \alpha}} dv_\delta(z) = M \int_{\mathbb{B}_n} g(z)^p dv_\alpha(z) < \infty.$$

□

**Corollary 5.3.** *Suppose that  $\alpha > -1$ ,  $p > n + 1 + \alpha$  and  $f \in A_\alpha^p$ , then*

$$\int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z) - f(w)|^p}{|z - w|^p} dv_t(z) dv_t(w) < \infty,$$

where  $t = \frac{p + \alpha - (n + 1)}{2}$ .

If  $n = 1$ , then we obtain the following corollary.

**Corollary 5.4.** *Suppose that  $\alpha > -1$ ,  $p > \alpha + 2$  and  $f \in A_\alpha^p(\mathbb{D})$ , then*

$$\int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|f(z) - f(w)|^p}{|z - w|^p} dA_t(z) dA_t(w) < \infty,$$

where  $t = \frac{p + \alpha - 2}{2}$ .

The symmetric lifting operator  $L : H(\mathbb{D}) \rightarrow H(\mathbb{D} \times \mathbb{D})$  is defined by

$$L(f)(z, w) = \frac{f(z) - f(w)}{z - w}.$$

The action of symmetric lifting operator on  $A_\alpha^p(\mathbb{D})$  in the cases  $p > \alpha + 2$  and  $p < \alpha + 2$  was studied in [8]. In the case  $p = \alpha + 2$ , we have the following result.

**Corollary 5.5.** *Suppose that  $\alpha > -1$ ,  $p = \alpha + 2$ . Then the symmetric lifting operator maps  $A_\alpha^p(\mathbb{D})$  into  $A_\gamma^p(\mathbb{D}^2)$ , for any  $\gamma > \alpha$ .*

*Proof.* The result follows by letting  $n = 1$  in Theorem 5.1. □

If  $\alpha > -1$ ,  $p > \alpha + 2$  and  $f \in A_\alpha^p(\mathbb{D})$ , then by Corollary 5.4,  $L(f) \in A_t^p(\mathbb{D}^2)$ , which means that the symmetric lifting operator maps  $f \in A_\alpha^p(\mathbb{D})$  into  $A_t^p(\mathbb{D}^2)$ , for  $t = \frac{p + \alpha - 2}{2}$ . This is the Theorem 4.4 in [8].

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