Double Integral Characterization for Bergman Spaces

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Abstract. In this paper we characterize Bergman spaces with respect to double integral of the functions $|f(z)-f(w)|/|z-w|$, $|f(z)-f(w)|/\rho(z,w)$ and $|f(z)-f(w)|/\beta(z,w)$, where $\rho$ and $\beta$ are the pseudo-hyperbolic and hyperbolic metrics. We prove some necessary and sufficient conditions that implies a function to be in Bergman spaces.

Keywords: Bergman spaces, Pseudo-hyperbolic metric, Hyperbolic metric, Double integral.


1. Introduction

For $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ in $\mathbb{C}^n$, we define $\langle z, w \rangle = z_1 \bar{w}_1 + \cdots + z_n \bar{w}_n$, where $\bar{w}_k$ is the complex conjugate of $w_k$. We also write $|z| = \sqrt{\langle z, z \rangle} = \sqrt{|z_1|^2 + \cdots + |z_n|^2}$. Let $\mathbb{B}_n$ denotes the open unit ball of $\mathbb{C}^n$, that is

$$\mathbb{B}_n = \{ z \in \mathbb{C}^n : |z| < 1 \}.$$ 

For any $a \in \mathbb{B}_n - \{0\}$, we define

$$\varphi_a(z) = \frac{a - P_n(z) - s_nQ_n(z)}{1 - \langle z, a \rangle} \quad z \in \mathbb{B}_n,$$

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where $s_a = \sqrt{1 - |a|^2}$, $P_a$ is the orthogonal projection from $\mathbb{C}^n$ onto the subspace $[a]$ generated by $a$, and $Q_a$ is the orthogonal projection from $\mathbb{C}^n$ onto $\mathbb{C}^n - [a]$. When $a = 0$, we define $\phi_a(z) = -z$. These functions are called involutions. (see [9] for more information about these functions)

The hyperbolic metric (Bergman metric) is defined by

$$\beta(z, w) = \frac{1}{2} \log \frac{1 + |\phi_z(w)|}{1 - |\phi_z(w)|}, \quad z, w \in \mathbb{B}_n.$$ 

For any $z \in \mathbb{B}_n$ and $r > 0$, we denote Bergman metric ball at $z$ by $D(z, r)$. That is

$$D(z, r) = \{ w \in \mathbb{B}_n : \beta(z, w) < r \}.$$ 

Also, pseudo-hyperbolic metric is defined by $\rho(z, w) = |\phi_z(w)|$.

For $\alpha > -1$ let

$$dv_\alpha(z) = c_\alpha(1 - |z|^2)^\alpha dv(z)$$

where $dv(z)$ is the Lebesgue volume measure on $\mathbb{B}_n$ and $c_\alpha$ is a positive constant with $v_\alpha(\mathbb{B}_n) = 1$. For $0 < p < \infty$ and $\alpha > -1$, the weighted Bergman space $A^p_\alpha$ consists of all holomorphic functions in $L^p(\mathbb{B}_n, dv_\alpha)$, that is

$$A^p_\alpha = \left\{ f \in H(\mathbb{B}_n) : ||f||_{p, \alpha}^p = \int_{\mathbb{B}_n} |f(z)|^p dv_\alpha(z) < \infty \right\}.$$ 

Wulan and Zhu [8], characterized Bergman spaces with standard weights in terms of Lipschitz type conditions in the Euclidean, hyperbolic, and pseudo-hyperbolic metrics. In [4] Li et al. proved that a holomorphic function $f$ belongs to the $A^p_\alpha$, $p > n + 1 + \alpha$, if and only if the function $|f(z) - f(w)|/|1 - \langle z, w \rangle|$ is in $L^p(\mathbb{B}_n \times \mathbb{B}_n, dv_\alpha \times dv_\gamma)$, where $\gamma = (p + \alpha - n - 1)/2$.

Also, it was shown in [5] that for the case $0 < p < n + 1 + \alpha$, $f \in A^p_\alpha$ if and only if the function $|f(z) - f(w)|/|1 - \langle z, w \rangle|$ is in $L^p(\mathbb{B}_n \times \mathbb{B}_n, dv_\alpha \times dv_\gamma)$ if and only if the function $|f(z) - f(w)|/|z - w|$ is in $L^p(\mathbb{B}_n \times \mathbb{B}_n, dv_\alpha \times dv_\gamma)$.

Our aim in this paper is to prove, for $f \in A^p_\alpha$, $p > n + 1 + \alpha$, the function $|f(z) - f(w)|/|z - w|$ is in $L^p(\mathbb{B}_n \times \mathbb{B}_n, dv_\alpha \times dv_\gamma)$, where $t = (p + \alpha - n - 1)/2$ and if $p = n + 1 + \alpha$, then $|f(z) - f(w)|/|z - w|$ is in $L^p(\mathbb{B}_n \times \mathbb{B}_n, dv_\alpha \times dv_\gamma)$, for any $\gamma > \alpha$. Our results are applicable for studying the action of symmetric lifting operator on $A^p_\alpha$ in all cases especially for the case $p = \alpha + 2$.

Also we replace the Euclidean metric with pseudo-hyperbolic metric $\rho$ and Bergman metric $\beta$.

2. Preliminaries

Lemma 2.1. [9] There exists a positive constant $C$ such that

$$|f(z)|^p \leq \frac{C}{(1 - |z|^2)^{n+1+\alpha}} \int_{D(z, r)} |f(w)|^p \ dv_\alpha(w)$$

for all $f \in H(\mathbb{B}_n)$ and $z \in \mathbb{B}_n$. 

Lemma 2.2. [9] Suppose $s > -1$, $t$ is real, and

$$I(z) = \int_{\mathbb{B}_n} \frac{(1 - |w|^2)^s}{|1 - \langle z, w \rangle|^{n+1+s+t}} \, dv(w), \quad z \in \mathbb{B}_n.$$ 

Then $I(z)$ is bounded in $\mathbb{B}_n$ whenever $t < 0$, and $I(z)$ is bounded by $(1 - |z|^2)^{-t}$ whenever $t > 0$.

Theorem 2.3. [8] Suppose that $p > 0$, $\alpha > -1$ and $f$ is analytic in $\mathbb{B}_n$. Then the following conditions are equivalent.

1. $f \in A^p_\alpha$.
2. There exists a continuous function $g$ in $L^p(\mathbb{B}_n, dv_\alpha)$ such that
   $$|f(z) - f(w)| \leq \rho(z, w)(g(z) + g(w)), \quad z, w \in \mathbb{B}_n.$$
3. There exists a continuous function $g$ in $L^p(\mathbb{B}_n, dv_\alpha)$ such that
   $$|f(z) - f(w)| \leq \beta(z, w)(g(z) + g(w)), \quad z, w \in \mathbb{B}_n.$$
4. There exists a continuous function $g$ in $L^p(\mathbb{B}_n, dv_{p+\alpha})$ such that
   $$|f(z) - f(w)| \leq |z - w|(g(z) + g(w)), \quad z, w \in \mathbb{B}_n.$$

Lemma 2.4. [4] Let $r > 0$. Then

$$1 - |z|^2 \sim 1 - |w|^2 \sim |1 - \langle z, w \rangle|$$

for all $z \in \mathbb{B}_n$ and $w \in D(z, r)$. Furthermore, there exists a positive constant $C$ such that

$$\frac{(1 - |z|^2)^p}{|1 - |z|^2|^{n+1}} \int_{D(z, r)} |f(w) - f(z)|^p dv(w) \leq C \int_{\mathbb{B}_n} \frac{|f(z) - f(0)|^p}{\rho(z, w)^p} dv_\alpha(z)dv_\alpha(w).$$

for all $z \in \mathbb{B}_n$ and $f \in H(\mathbb{B}_n)$.

3. PSEUDO-HYPERBOLIC METRIC

Lemma 3.1. Suppose $\alpha > -1$ and $f \in H(\mathbb{B}_n)$. Then there exists a positive constant $C$ such that

$$\int_{\mathbb{B}_n} |f(z) - f(0)|^p \, dv_\alpha(z) \leq C \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z) - f(w)|^p}{\rho(z, w)^p} dv_\alpha(z)dv_\alpha(w).$$

Proof. Let

$$J(f) = \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z) - f(w)|^p}{\rho(z, w)^p} dv_\alpha(z)dv_\alpha(w).$$
By making a change of variable, we have
\[ J(f) = \int_{\mathbb{B}_n} dv_\alpha(z) \int_{\mathbb{B}_n} \frac{|f(z) - f(\varphi_z(w))|^p}{\rho(z,\varphi_z(w))^p} \frac{(1 - |z|^2)^{n+1+\alpha}}{|1 - \langle z, w \rangle|^{2(\alpha+1+\alpha)}} dv_\alpha(w) \]
\[ = \int_{\mathbb{B}_n} dv_\alpha(z) \int_{\mathbb{B}_n} \frac{|f(z) - f(\varphi_z(w))|^p}{|w|^p} \frac{(1 - |z|^2)^{n+1+\alpha}}{|1 - \langle z, w \rangle|^{2(\alpha+1+\alpha)}} dv_\alpha(w) \]
\[ \geq \int_{\mathbb{B}_n} dv_\alpha(z) \int_{\mathbb{B}_n} |f(z) - f(\varphi_z(w))|^p \frac{(1 - |z|^2)^{n+1+\alpha}}{|1 - \langle z, w \rangle|^{2(\alpha+1+\alpha)}} dv_\alpha(w) \]
\[ \geq \int_{\mathbb{B}_n} dv_\alpha(z) \int_{D(z,r)} |f(z) - f(\varphi_z(w))|^p \frac{(1 - |z|^2)^{n+1+\alpha}}{|1 - \langle z, w \rangle|^{2(\alpha+1+\alpha)}} dv_\alpha(w). \]

From the first part of Lemma 2.4, there exists a positive constant \( C' \) such that
\[ J(f) \geq C' \int_{\mathbb{B}_n} dv_\alpha(z) \int_{D(z,r)} |f(z) - f(\varphi_z(w))|^p \frac{(1 - |z|^2)^{n+1+\alpha}}{|1 - \langle z, w \rangle|^{2(\alpha+1+\alpha)}} dv_\alpha(w). \]

Then Lemma 2.1 implies that there exists another positive constant \( C \) such that
\[ J(f) \geq C \int_{\mathbb{B}_n} |f(z) - f(\varphi_z(z))^p dv_\alpha(z) = C \int_{\mathbb{B}_n} |f(z) - f(0)|^p dv_\alpha(z). \]

The proof is complete. \( \square \)

**Lemma 3.2.** Suppose \( \alpha > -1 \) and \( f \in A^p_{\mathbb{B}_n} \). Then
\[ \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z) - f(w)|^p}{\rho(z,w)^p} dv_\alpha(z) dv_\alpha(w) < \infty. \]

**Proof.** Given \( f \in A^p_{\mathbb{B}_n} \), from Theorem 2.3, there exists a continuous function \( g \in L^p(\mathbb{B}_n, dv_\alpha) \) such that for all \( z, w \in \mathbb{B}_n \),
\[ |f(z) - f(w)| \leq \rho(z,w)(g(z) + g(w)). \]

There exists a positive constant \( C \) such that
\[ \frac{|f(z) - f(w)|^p}{\rho(z,w)^p} \leq C(g(z)^p + g(w)^p). \]

So,
\[ \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(w) - f(z)|^p}{\rho(z,w)^p} dv_\alpha(z) dv_\alpha(w) \]
\[ \leq C \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} (g(z)^p + g(w)^p) dv_\alpha(z) dv_\alpha(w) \]
\[ = 2C \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} g(z)^p dv_\alpha(z) dv_\alpha(w) < \infty. \]

\( \square \)

We can combine these two lemmas and obtain the following theorem.
Theorem 3.3. Suppose that $\alpha > -1$. Then $f \in A^p_\alpha$ if and only if

\[ \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z) - f(w)|^p}{\rho(z,w)^p} \, d\nu_{\alpha}(z) d\nu_{\alpha}(w) < \infty. \]

### 4. Bergman metric

Now, we replace metric $\rho$ by Bergman metric $\beta$.

Lemma 4.1. Suppose that $\alpha > -1$ and $f \in H(\mathbb{B}_n)$. If

\[ \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z) - f(w)|^p}{\beta(z,w)^p} \, d\nu_{\alpha}(z) d\tau(w) < \infty, \]

then $f \in A^p_\alpha$, where

\[ d\tau(w) = \frac{dv(w)}{(1 - |w|^2)^{n+1}} \]

is the Mobius invariant volume measure on $\mathbb{B}_n$.

Proof. By Lemma 2.4, there exists a positive constant $C$ such that

\[ (1 - |z|^2)^p |\nabla f(z)|^p \leq \frac{C}{(1 - |z|^2)^{n+1}} \int_{D(z,r)} |f(z) - f(w)|^p dv(w) \]

\[ \leq \frac{C}{(1 - |z|^2)^{n+1+\alpha}} \int_{D(z,r)} |f(z) - f(w)|^p d\nu_{\alpha}(w). \]

Since $D(z,r)$ is open unit ball in metric $\beta$, we have

\[ (1 - |z|^2)^p |\nabla f(z)|^p \leq \frac{Cr^p}{(1 - |z|^2)^{n+1+\alpha}} \int_{D(z,r)} \frac{|f(z) - f(w)|^p}{\beta(z,w)^p} \, d\nu_{\alpha}(w). \]

After integrating

\[ \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} (1 - |z|^2)^p |\nabla f(z)|^p d\nu_{\alpha}(z) \leq Cr^p \int_{\mathbb{B}_n} \int_{D(z,r)} \frac{|f(z) - f(w)|^p}{\beta(z,w)^p} \, d\nu_{\alpha}(w) d\tau(z) \]

\[ \leq Cr^p \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z) - f(w)|^p}{\beta(z,w)^p} \, d\nu_{\alpha}(z) d\tau(z). \]

Therefore $(1 - |z|^2) \nabla f(z) \in A^p_\alpha$. It follows from Theorem 2.16 of [9] that $f \in A^p_\alpha$. \qed

By the same reason as in Lemma 3.2, we can prove the following lemma.

Lemma 4.2. Suppose $\alpha > -1$ and $f \in H(\mathbb{B}_n)$. If $f \in A^p_\alpha$, then

\[ \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z) - f(w)|^p}{\beta(z,w)^p} \, d\nu_{\alpha}(z) d\nu_{\alpha}(w) < \infty. \]
5. Euclidean metric

Theorem 5.1. Suppose $\alpha > -1$, $p = n + 1 + \alpha$ and $f \in A^p_{\alpha}$, then

$$I(f) = \int_{B_n} \int_{B_n} \frac{|f(z) - f(w)|^p}{|z - w|^p} dv_{\gamma}(z)dv_{\gamma}(w) < \infty,$$

for any $\gamma > \alpha$.

Proof. Given $f \in A^p_{\alpha}$, from Theorem 2.3, there exists a continuous function $g \in L^p(B_n, dv_{\alpha})$ such that for all $z, w \in B_n$,

$$|f(z) - f(w)| \leq \rho(z, w)(g(z) + g(w)) \leq \frac{|z - w|}{|1 - \langle z, w \rangle|} (g(z) + g(w)).$$

There exists a positive constant $C$ such that

$$I(f) \leq 2C \int_{B_n} g(z)^p dv_{\gamma}(z) \int_{B_n} \frac{dv_{\gamma}(w)}{|1 - \langle z, w \rangle|^p}.$$

Since $\alpha - \gamma < 0$, by Lemma 2.2, the last integral is bounded. Then there exists another positive constant $M$ such that

$$I(f) \leq M \int_{B_n} g(z)^p dv_{\gamma}(z) = Mc_\gamma \int_{B_n} g(z)^p (1 - |z|^2)^{\gamma-\alpha} (1 - |z|^2)^\alpha dv(z) < M c_\gamma c_\alpha < \infty.$$

□

Lemma 5.2. Suppose $\alpha > -1$, $f \in H(B_n)$ and $\delta$ and $\gamma$ are real parameters such that

$$\delta + \gamma = p + \alpha - (n + 1), \quad -1 < \gamma < p - (n + 1).$$

If $f \in A^p_{\alpha}$, then

$$I(f) = \int_{B_n} \int_{B_n} \frac{|f(z) - f(w)|^p}{|z - w|^p} dv_{\delta}(z)dv_{\gamma}(w) < \infty.$$

Proof. By the proof of the previous lemma, there exists a positive constant $C$ such that

$$I(f) \leq 2C \int_{B_n} g(z)^p dv_{\delta}(z) \int_{B_n} \frac{dv_{\gamma}(w)}{|1 - \langle z, w \rangle|^p}.$$

$$= 2C \int_{B_n} g(z)^p dv_{\delta}(z) \int_{B_n} \frac{dv_{\gamma}(w)}{|1 - \langle z, w \rangle|^{n+1+\gamma+(\delta-\alpha)}}.$$
Since $\delta - \alpha > 0$, by Lemma 2.2, there exists another positive constant $M$ such that
\[
I(f) \leq M \int_{B_n} \frac{g(z)^p}{(1 - |z|^2)^\delta - \alpha} \, dv\delta(z) = M \int_{B_n} g(z)^p \, dv\alpha(z) < \infty.
\]
\[
\square
\]

**Corollary 5.3.** Suppose that $\alpha > -1$, $p > n + 1 + \alpha$ and $f \in A^p_{\alpha}$, then
\[
\int_{B_n} \int_{B_n} \frac{|f(z) - f(w)|^p}{|z - w|^p} \, dv\alpha(z) dv\alpha(w) < \infty,
\]
where $t = \frac{p + \alpha - (n + 1)}{2}$.

If $n = 1$, then we obtain the following corollary.

**Corollary 5.4.** Suppose that $\alpha > -1$, $p > \alpha + 2$ and $f \in A^p_{\alpha}(D)$, then
\[
\int_{D} \int_{D} \frac{|f(z) - f(w)|^p}{|z - w|^p} \, dA_t(z) dA_t(w) < \infty,
\]
where $t = \frac{p + \alpha - 2}{2}$.

The symmetric lifting operator $L : H(D) \to H(D \times D)$ is defined by
\[
L(f)(z, w) = \frac{f(z) - f(w)}{z - w}.
\]
The action of symmetric lifting operator on $A^p_{\alpha}(D)$ in the cases $p > \alpha + 2$ and $p < \alpha + 2$ was studied in [8]. In the case $p = \alpha + 2$, we have the following result.

**Corollary 5.5.** Suppose that $\alpha > -1$, $p = \alpha + 2$. Then the symmetric lifting operator maps $A^p_{\alpha}(D)$ into $A^p_{\gamma}(D^2)$, for any $\gamma > \alpha$.

**Proof.** The result follows by letting $n = 1$ in Theorem 5.1. \[
\square
\]

If $\alpha > -1$, $p > \alpha + 2$ and $f \in A^p_{\alpha}(D)$, then by Corollary 5.4, $L(f) \in A^p_{t}(D^2)$, which means that the symmetric lifting operator maps $f \in A^p_{\alpha}(D)$ into $A^p_{t}(D^2)$, for $t = \frac{p + \alpha - 2}{2}$. This is the Theorem 4.4 in [8].

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**References**