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On Direct Sum of Branches in Hyper BCK-algebras

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Abstract. In this paper, the notion of direct sum of branches in hyper BCK-algebras is introduced and some related properties are investigated. Applying this notion to lower hyper BCK-semi lattice, a necessary condition for a hyper BCK-ideal to be prime is given. Some properties of hyper BCK-chain are studied. It is proved that if H is a hyper BCK-chain and $[a)$ is finite for any $a \in H$, then $\mid Aut(H) \mid= 1$.

Keywords: Hyper BCK-algebra, (weak, strong) Hyper BCK-ideal, Direct sum of branches, Hyper BCK-chain.

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1. Introduction

The study of BCK-algebras was initiated by Y. Imai and K. Iséki [6] in 1966 as a generalization of the concept of set-theoretic difference and propositional calcului. Since then a great deal of literature has been produced on the theory of BCK-algebras. The hyper structures theory(called also multi algebras) was introduced in 1934 by F. Marty at the 8th congress of Scandinavian Mathematicians [10]. In [9], Y. B. Jun et al. applied the hyper structures to BCK-algebras, and introduced the notion of a hyper BCK-algebra which is a generalization of BCK-algebra, and investigated some related properties. Now, we follow [9] and [7] and introduce the notion of branch of a hyper BCK-algebra. We consider a hyper BCK-algebra that is a direct sum of branches, and investigate related properties. We give a necessary condition for a hyper

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BCK-ideal of lower hyper BCK-semilattice to be prime. Also, we define the hyper BCK-chain and obtain a condition for $[a)$ to be a hyper BCK-ideal. We prove that if H is a hyper BCK-chain and $[a)$ is finite for any $a \in H$, then $\mid Aut(H) \mid= 1$. Finally, we state the relation between the branches of two isomorphic hyper BCK-algebras.

2. Preliminaries

We first present some elementary aspects of hyper BCK-algebras that are necessary for this paper, and for more details we refer to [9] and [8]. Let H be a nonempty set endowed with a hyper operation " $\circ$ ", that is, $\circ$ is a function from $H \times H$ to $\mathcal{P}^*(\mathcal{H}) = \mathcal{P}(\mathcal{H}) \setminus \phi$. For two subsets A and B of H and $x, y \in H$, denote by $A \circ B$, $x \circ B$ and $A \circ y$ the sets $\bigcup\limits_{a \in A, b \in B} a \circ b$, $\bigcup\limits_{b \in B} x \circ b$ and $\bigcup\limits_{a \in A} a \circ y$, respectively.

Definition 2.1. [9] A nonempty set H endowed with a hyper operation " $\circ$ " and a constant 0 is said to be a hyper BCK-algebra if it satisfies the following axioms:

(H1) $(x \circ z) \circ (y \circ z) \ll x \circ y$,

(H2) $(x \circ y) \circ z = (x \circ z) \circ y$,

(H3) $x \circ H \ll \{x\}$,

(H4) $x \ll y$ and $y \ll x$ imply $x = y$,

where $x \ll y$ is defined by $0 \in x \circ y$, and for every $A, B \subseteq H$, $A \ll B$ is defined by $\forall a \in A$, $\exists b \in B$ such that $a \ll b$. In such case, we call "$\ll$" the *hyper order* in H.

Theorem 2.2. [9] *In any hyper BCK-algebra H, we have*

(a1) $0 \circ x = \{0\}$,
(a2) $x \circ 0 = \{x\}$,
(a3) $x \ll x$,
(a4) $x \circ y \ll \{x\}$,
(a5) $A \ll A$,
(a6) $A \ll 0$ *implies* $A = \{0\}$,
(a7) $A \subseteq B$ *implies* $A \ll B$,
(a8) $y \ll z$ *implies* $x \circ z \ll x \circ y$,
(a9) $x \circ y = \{0\}$ *implies* $(x \circ z) \circ (y \circ z) = \{0\}$,
for any $x, y, z \in H$ *and* $A, B \subseteq H$.

Definition 2.3. [9] Let H be a hyper BCK-algebra. Then a nonempty subset S of H is called a *hyper subalgebra* of H if S is a hyper BCK-algebra with respect to the hyper operation "$\circ$" on H. If S is a nonempty subset of a hyper

BCK-algebra H, then S is a hyper subalgebra of H if and only if $x \circ y \subseteq S$ for all $x, y \in S$.

Theorem 2.4. [9] *Let H be a hyper BCK-algebra. Then the set $S(H) := \{x \in H|\ x \circ x = \{0\}\}$ is a hyper subalgebra of H, which is called BCK-part of H. Moreover, H is a BCK-algebra if and only if $S(H) = H$.*

Definition 2.5. [9, 8] Let H be a hyper BCK-algebra. Then, a nonempty subset I of H with $0 \in I$ is called a weak hyper BCK-ideal of H if it satisfies: $(\forall x, y \in H)(x \circ y \subseteq I$ and $y \in I \Longrightarrow x \in I)$; hyper BCK-ideal of H if it satisfies: $(\forall x, y \in H)(x \circ y \ll I$ and $y \in I \Longrightarrow x \in I)$; reflexive hyper BCK-ideal of H if it is a hyper BCK-ideal of H and satisfies: $(\forall x \in H)\ x \circ x = \{0\}$; strong hyper BCK-ideal of H if it satisfies: $(\forall x, y \in H)(x \circ y) \cap I \neq \phi$ and $y \in I \Longrightarrow x \in I)$.

Theorem 2.6. [9, 8] *(i) Every strong hyper BCK-ideal of a hyper BCK-algebra is a hyper BCK-ideal.*

(ii) Every hyper BCK-ideal of a hyper BCK-algebra is a weak hyper BCK-ideal.

Theorem 2.7. [8] *Let A be a subset of a hyper BCK-algebra H. If I is a hyper BCK-ideal of H such that $A \ll I$, then $A \subseteq I$.*

Definition 2.8. [1] Let H be a hyper BCK-algebra, ρ be an equivalence relation on H and $A, B \subseteq H$. Then

(i) we write $A\rho B$, if there exist $a \in A$ and $b \in B$ such that $a\rho b$,

(ii) we write $A\bar{\rho}B$, if for all $a \in A$ there exists $b \in B$ such that $a\rho b$ and for all $b \in B$ there exists $a \in A$ such that $a\rho b$,

(iii) ρ is called a congruence relation on H, if $x\rho y$ and $x'\rho y'$, then $x \circ x' \bar{\rho} y \circ y'$, for all $x, y, x', y' \in H$,

(iv) ρ is called regular on H, if $x \circ y\rho\{0\}$ and $y \circ x\rho\{0\}$, then $x\rho y$, for all $x, y \in H$.

Theorem 2.9. [1] *Let ρ be a regular congruence relation on H and $\frac{H}{\rho} = \{[x]_\rho \mid x \in H\}$. Then $\frac{H}{\rho}$ with hyperoperation "$\circ$" and hyperorder "$\ll$" which is defined as follows:*

$$[x]_\rho \circ [y]_\rho = \{[z]_\rho \mid \ for\ some\ z \in x \circ y\}, \quad [x]_\rho \ll [y]_\rho \Longleftrightarrow [0]_\rho \in [x]_\rho \circ [y]_\rho$$

is a hyper BCK-algebra which is called quotient hyper BCK-algebra.

3. Direct Sum of Branches

Definition 3.1. [7] An element a of a hyper BCK-algebra H is called a hyperatom if it satisfies

$$(\forall x \in H)(x \ll a \Longrightarrow x = 0\ or\ x = a.)$$

Denote by $A(H)$ the set of all hyperatoms of H. Obviously, $0 \in A(H)$.

Lemma 3.2. *Let H be a hyper BCK-algebra. Then the set $A(H)$ is a hyper subalgebra of H.*

Proof. Let $a, b \in A(H)$. Then by Theorem 2.2(a_4), we have $a \circ b \ll \{a\}$. This implies that $t \ll a$, for any $t \in a \circ b$. Since a is a hyperatom, we get $t = 0$ or $t = a$. Hence $a \circ b \subseteq \{0, a\}$ and so $a \circ b \subseteq A(H)$. Therefore, $A(H)$ is a hyper subalgebra of H. □

Definition 3.3. Let H be a hyper BCK-algebra. For any hyperatom $a \in A^*(H) = A(H) - \{0\}$, the set

$$B(a) := \{x \in H \mid a \ll x\} \cup \{0\}$$

is called the branch of H generated by a.

Definition 3.4. A hyper BCK-algebra $(H, \ll)$ is called ordered if the hyper-order " $\ll$ "is transitive.

For any element a of a hyper BCK-algebra H, we denote

$$[a) := \{x \in H \mid x \ll a\}.$$

Lemma 3.5. *Let H be an ordered hyper BCK-algebra. If the set $[a)$ is finite for any $a \in H$, then*

$$H = \bigcup_{a \in A^*(H)} B(a).$$

Proof. It suffices to show that for any $0 \neq x \in H$, there exists $a \in A^*(H)$ such that $x \in B(a)$. Let $0 \neq x \in H$. If x is a hyperatom , then, since $x \in B(x)$, the result holds. Otherwise, there is $x_1 \in H$ such that $x_1 \ll x$ and $x_1 \neq x$. If $x_1 \in A^*(H)$, then $x \in B(x_1)$, and so the proof is complete. Otherwise, there exists $x_2 \in H$ such that $x_2 \ll x_1$ and $x_2 \neq x_1$. By transitivity of $\ll$, we get $x_2 \ll x_1 \ll x$. Since $[x)$ is finite, this process will be stopped in a hyperatom element, e.i., there exists $x_n \in A^*(H)$ such that $x_n \ll x_{n-1} \ll .... \ll x_1 \ll x$. Thus $x \in B(x_n)$. This completes the proof. □

Definition 3.6. An ordered hyper BCK-algebra H is called a direct sum of branches and denoted by $H = \bigoplus_{a \in A^*(H)} B(a)$ if it satisfies the following:

(i) $H = \bigcup_{a \in A^*(H)} B(a)$,

(ii) $B(a) \cap B(b) = \{0\}$, for any $a, b \in A^*(H)$ with $a \neq b$.

Example 3.7. (i) Let $H = \{0, a, b, c\}$. Consider the following table:

$\circ$	0	a	b	c
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0, a\}$	$\{0\}$	$\{a\}$
b	$\{b\}$	$\{b\}$	$\{0, a, b\}$	$\{b\}$
c	$\{c\}$	$\{c\}$	$\{c\}$	$\{0, c\}$

Then $(H;\circ,0)$ is an ordered hyper BCK-algebra. It is routine to check that $A^*(H)=\{a,c\}$, $B(a)=\{0,a,b\}$ and $B(c)=\{0,c\}$. Therefore $H=B(a)\oplus B(c)$ and so H is a direct sum of branches.

(ii) Let $K=\{0,1,2,3\}$, with hyper operation " $\circ$ " given by the following table:

$\circ$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$	$\{0\}$
2	$\{2\}$	$\{2\}$	$\{0\}$	$\{0\}$
3	$\{3\}$	$\{2\}$	$\{1\}$	$\{0\}$

Then $(K;\circ,0)$ is an ordered hyper BCK-algebra but it is not a direct sum of branches since $B(1)\cap B(2)\neq\{0\}$.

Lemma 3.8. *Let $H=\bigoplus_{a\in A^*(H)} B(a)$. Then the following statements hold for any $x,y\in H$, $T\subseteq H$ and $a\in A^*(H)$.*

(i) *$\{x\}\ll B(a)$ implies $x\in B(a)$,*

(ii) *$T\ll B(a)$ implies $T\subseteq B(a)$,*

(iii) *$(x\circ y)\cap B(a)\neq\phi,\{0\}$ implies $x\circ y\subseteq B(a)$.*

Proof. (i) Let $x\in H$ be such that $\{x\}\ll B(a)$ for some $a\in A^*(H)$. If $x=0$, then clearly $x\in B(a)$. Assume that $x\neq 0$. Thus there exists $z\in B(a)$ such that $x\ll z$. Since $H=\bigoplus_{a\in A^*(H)} B(a)$, there exists $b\in A^*(H)$ such that $x\in B(b)$. Hence $b\ll x$ and so $b\ll z$. Then $z\in B(a)\cap B(b)$. It follows from $x\ll z$ and $x\neq 0$ that $z\neq 0$. Hence $B(a)\cap B(b)\neq\{0\}$ and so by Definition 3.6, $a=b$. Therefore $x\in B(a)$.

(ii) It is a consequence from (i).

(iii) Let $x,y\in H$ and let $a\in A^*(H)$ be such that $(x\circ y)\cap B(a)\neq\phi,\{0\}$. Then there exists $0\neq t\in x\circ y$ such that $t\in B(a)$. Hence $a\ll t$ and so from $x\circ y\ll x$, we get $a\ll x$. This implies that $x\in B(a)$ and so $x\circ y\ll B(a)$. Then by (ii), we get $x\circ y\subseteq B(a)$, which completes the proof. □

Proposition 3.9. *Let $H=\bigoplus_{a\in A^*(H)} B(a)$. Then for any $a\in A^*(H)$, we have*

(i) $B(a)$ is a hyper subalgebra of H,

(ii) $B(a)$ is a weak hyper BCK-ideal of H,

(iii) $B(a)$ is a strong hyper BCK-ideal of H.

Proof. (i) Let $a\in A^*(H)$ and $x,y\in B(a)$. Since $x\circ y\ll\{x\}\in B(a)$, it follows that $x\circ y\ll B(a)$. Hence by Lemma 3.8(ii), we get $x\circ y\subseteq B(a)$. Therefore $B(a)$ is a hyper subalgebra of H.

(ii) Let $x,y\in H$ be such that $x\circ y\subseteq B(a)$ and $y\in B(a)$, for some $a\in A^*(H)$. We will show that $x\in B(a)$. If $x\circ y=\{0\}$, then $x\ll y$ and so $x\ll B(a)$. Hence by Lemma 3.8 (i), $x\in B(a)$. If $x\circ y\neq\{0\}$, then there exists $(0\neq)t\in x\circ y$

and so $a \ll t$. Since $t \ll x$, we get $x \in B(a)$. Therefore $B(a)$ is a weak hyper BCK-ideal of H.

(iii) Let $x, y \in H$ be such that $y \in B(a)$ and $(x \circ y) \cap B(a) \neq \phi$ for some $a \in A^*(H)$. If $(x \circ y) \cap B(a) = \{0\}$, then $0 \in x \circ y$ and so $x \ll y$, which implies that $x \in B(a)$. If $(x \circ y) \cap B(a) \neq \{0\}$, then by Lemma 3.8 (iii), $x \circ y \subseteq B(a)$. By (ii), $B(a)$ is a weak hyper BCK-ideal. Hence from $x \circ y \subseteq B(a)$ and $y \in B(a)$, we get $x \in B(a)$. Therefore $B(a)$ is a strong hyper BCK-ideal. □

Theorem 3.10. *Let* $H = \bigoplus_{a \in A^*(H)} B(a)$. *Then for any nonempty subset* Q *of* $A^*(H)$, $\bigoplus_{a \in Q} B(a)$ *is a strong hyper* BCK*-ideal of* H.

Proof. The proof is similar to the proof of Proposition 3.9(iii) by some modification. □

Since every strong hyper BCK-ideal is a (weak) hyper BCK-ideal, we have the following corollary.

Corollary 3.11. *Let* $H = \bigoplus_{a \in A^*(H)} B(a)$. *Then for any nonempty subset* Q *of* $A^*(H)$, $\bigoplus_{a \in Q} B(a)$ *is a (week) hyper* BCK*-ideal of* H.

The following proposition shows that the union of two direct sum of branches is a direct sum of branches too.

Proposition 3.12. *Let* $H = \bigoplus_{a \in A^*(H)} B(a)$ *and* $K = \bigoplus_{b \in A^*(K)} B(b)$. *If* $H \cap K = (0)$, *then* $H \oplus K$ *is a direct sum of branches, where* $H \oplus K = H \cup K$ *and its hyperoperation "* $\circ$ *" is defined as follows:*

$$x \circ y := \begin{cases} x \circ_H y & \text{if } x, y \in H, \\ x \circ_K y & \text{if } x, y \in K, \\ \{x\} & \text{otherwise} \end{cases} \tag{3.1}$$

for all $x, y \in H \cup K$.

Proof. From [9], it is known that $H \oplus K$ is a hyper BCK-algebra. By (3.1), $x \ll_{H \oplus K} y$ if and only if $x, y \in H$ or $x, y \in K$. This implies that $H \oplus K$ is ordered. Let $a \in A^*(H)$ and let $(0 \neq) x \in H \oplus K$ be such that $x \ll a$. Then $0 \in x \circ a$ and so from $x \neq 0$ we conclude $x \circ a \neq \{x\}$. Hence, it follows from (3.1) and $a \in H$ that $x \circ a = x \circ_H a$. Then $x \ll_H a$. Since a is a hyperatom of H, we get $x = a$. Hence a is a hyperatom of $H \oplus K$ and so $A^*(H) \subseteq A^*(H \oplus K)$. Similarly, we can show that $A^*(K) \subseteq A^*(H \oplus K)$. Thus $A^*(H) \cup A^*(K) \subseteq A^*(H \oplus K)$. Obviously, since $H, K \subseteq H \oplus K$, we get $A^*(H \oplus K) \subseteq A^*(H) \cup A^*(K)$. Hence $A^*(H \oplus K) = A^*(H) \cup A^*(K)$. It is clear that $H \cup K = (\bigoplus_{a \in A^*(H)} B(a)) \cup (\bigoplus_{b \in A^*(K)} B(b))$. Since $B(a) \cap B(b) = (0)$ for any $a \in A^*(H)$ and $b \in A^*(K)$, we obtain $H \oplus K = \bigoplus_{c \in A^*(H) \cup A^*(K)} B(c)$ and so $H \oplus K = \bigoplus_{c \in A^*(H \oplus K)} B(c)$. Therefore $H \oplus K$ is a direct sum of branches. □

We recall that an ordered hyper BCK-algebra is said to be a lower hyper BCK-semilattice if $x \wedge y$, the greatest lower bound of x and y, exists for any $x, y \in H$. Also, a proper hyper BCK-ideal P of a lower hyper BCK-semilattice is said to be prime if $x \wedge y \in P$ implies $x \in P$ or $y \in P$ for any $x, y \in H$ (see [5]).

Proposition 3.13. *Let $H = \bigoplus_{a \in A^*(H)} B(a)$ be a lower hyper BCK-semilattice and $\mid P \mid \geq 2$. Then for any $b \in A^*(H)$, $Q_b := \bigcup_{b \neq a \in A^*(H)} B(a)$ is a prime hyper BCK-ideal of H.*

Proof. Let $b \in A^*(H)$. By Corollary 3.11, Q_b is a hyper BCK-ideal of H. Let $x, y \in H$ be such that $x \notin Q_b$ and $y \notin Q_b$. Hence, we get $x, y \in B(b)$, $x \neq 0$ and $y \neq 0$. Thus $b \ll x, y$ and so $b \ll x \wedge y$. It follows that $x \wedge y \in B(b)$. Moreover, $x \wedge y \neq 0$ because $b \neq 0$. Hence $x \wedge y \notin Q_b$. Therefore Q_b is prime. □

The following theorem gives a necessary condition for a hyper BCK-ideal to be prime.

Theorem 3.14. *Let $H = \bigoplus_{a \in A^*(H)} B(a)$ be a lower hyper BCK-semilattice. If I is a prime hyper BCK-ideal of H, then $H - I \subseteq B(a)$ for some $a \in A^*(H)$.*

Proof. Let I be a prime hyper BCK-ideal of H. If $\mid A^*(H) \mid = 1$, then $B(a) = H$ for $a \in A^*(H)$ and so clearly $H - I \subseteq B(a)$. Assume that $\mid A^*(H) \mid \geq 2$. Suppose on the contrary, $H - I \nsubseteq B(a)$ for any $a \in A^*(H)$. Then, since $H - I \subseteq \bigcup_{a \in A^*(H)} B(a)$ and $\mid A^*(H) \mid \geq 2$, there exist $b, c \in A^*(H)$ with $b \neq c$ such that $(H - I) \cap B(b) \neq \phi$ and $(H - I) \cap B(c) \neq \phi$. Hence there are $x \in (H - I) \cap B(b)$ and $y \in (H - I) \cap B(c)$. This imply $x \in B(b)$, $y \in B(c)$ and $x, y \notin I$. By Corollary 3.11, $B(b)$ is a hyper BCK-ideal of H. Then it follows from $x \wedge y \ll x \in B(b)$ and Lemma 3.8(i) that $x \wedge y \in B(b)$. Similarly, we have $x \wedge y \in B(c)$. Hence $x \wedge y \in B(b) \cap B(c)$ and so $x \wedge y = 0$. On the other hand, since $x, y \notin I$ and I is prime, we have $x \wedge y \notin I$. Hence $0 \notin I$, which a contradiction. Therefore $H - I \subseteq B(a)$ for some $a \in A^*(H)$. □

Proposition 3.15. *Let $H = \bigoplus_{a \in A^*(H)} B(a)$. If the branch $B(a)$ is bounded such that $S(H) \cap B(a) = \{0, c_a\}$ where c_a is an upper bound of $B(a)$, for some $a \in A^*(H)$, then $I := H - \{c_a\}$ is a hyper BCK-ideal of H.*

Proof. Obviously, $0 \in I$. It suffices to show that the inequality $c_a \circ x \ll I$ does not hold for any $x \in I$. Suppose on the contrary that $c_a \circ x \ll I$ for some $x \in I$.

Then for any $t \in c_a \circ x$ there exists $i_t \in I$ such that $t \ll i_t$. Since $c_a \in S(H)$, we get $c_a \circ c_a = \{0\}$. Applying the axiom $(H1)$, we have

$$t \circ t \subseteq (c_a \circ x) \circ (c_a \circ x) \ll c_a \circ c_a = \{0\}.$$

This implies that $t \circ t = \{0\}$. Hence $t \in S(H)$. By Lemma 3.8(i), since $t \ll c_a \in B(a)$, we have $t \in B(a)$. Therefore $t \in S(H) \cap B(a)$ and so $t = 0$ or $t = c_a$. If $t = 0$, then $c_a \ll x$, which implies that $x \in B(a)$. Thus, since c_a is an upper bound of $B(a)$, we have $c_a = x$, which a contradiction. If $t = c_a$, then $c_a \ll i_t$. This implies that $i_t \in B(a)$ and so $c)a = i_t$. Hence $c_a \in I$, which a contradiction. Therefore the assumption is false and so I is a hyper BCK-ideal. □

Theorem 3.16. *Let* $H = \bigoplus_{a \in A^*(H)} B(a)$*, and let all branches of* H *be bounded. Assume that* $S(H) = U \cup \{0\}$*, where* U *is the set of upper bounds of branches. Then* M *is a maximal hyper* BCK*-ideal of* H *if and only if* $M = H - \{c_a\}$*, for some* $c_a \in U$*.*

Proof. Let M be a maximal hyper BCK-ideal of H. Then M is prime and so by Theorem 3.14, there exists $a \in A^*(H)$ such that $H - M \subseteq B(a)$. Hence there exists $T \subseteq B(a)$ such that $M = \bigoplus_{a \neq b \in A^*(H)} B(b) \cup T$. We note that if $\mid A^* \mid = 1$, then we have $M = T$. Assume that $c_a \in B(a)$ is an upper bound of $B(a)$. If $c_a \in T$, then $B(a) \subseteq T$ and so $M = H$, which a contradiction. Hence $c_a \notin T$. This implies that $M \subseteq H - \{c_a\}$. By Proposition 3.15, $H - \{c_a\}$ is a hyper BCK-ideal. Then by maximality of M, we get $M = H - \{c_a\}$. Conversely, by Theorem 3.15, the result holds. □

Definition 3.17. A hyper BCK-algebra H is said to be *hyperatomic* if each its element is hyperatom, i.e., $A(H) = H$.

Proposition 3.18. *Let* $H = \bigoplus_{a \in A^*(H)} B(a)$*. Then there exists a regular congruence* ρ *on* H *such that the quotient hyper* BCK*-algebra* $\frac{H}{\rho}$ *is hyperatomic.*

Proof. Let $H = \bigoplus_{a \in A^*(H)} B(a)$. Define the relation ρ on H as follows:

$$x \rho y \ \Leftrightarrow x = y = 0 \text{ or } a \ll x, y, \text{ for some } a \in A^*(H).$$

Putting $B^*(a) = B(a) - \{0\}$, it is easy to see that the sets $\{0\}$ and $B^*(a)$ for any $a \in A^*(H)$ form a partition of H. This implies that ρ is an equivalence relation on H. It is clear that $[0]_\rho = \{0\}$ and $[a]_\rho = B^*(a)$. Let $x, y \in H$ be such that $x \rho y$. Then $x = y = 0$ or $x, y \in B^*(a)$, for some $a \in A^*(H)$. Hence for any $z \in H$, $x \circ z = y \circ z$ or $x \circ z, y \circ z \subseteq B^*(a)$. This implies $x \circ z \overline{\rho} y \circ z$. Similarly, we can show that if $x \rho y$, then $z \circ x \overline{\rho} z \circ y$, for any $z \in H$. Thus ρ is congruence. To proof the regularity of ρ assume that $x \circ y \rho 0$ and $y \circ x \rho 0$ for some $x, y \in H$. Then there are $t \in x \circ y$ and $s \in y \circ x$ such that $t \rho 0$ and $s \rho 0$.

Then, from $[0]_\rho = \{0\}$, we get $t = s = 0$, and so $x \ll y$ and $y \ll x$. Hence $x = y$ and consequently, $x\rho y$. Thus ρ is regular and so by Theorem 2.9, $\frac{H}{\rho}$ is a hyper BCK-algebra. Let $[a]_\rho \in \frac{H}{\rho}$. If $[x]_\rho \ll [a]_\rho$, for some $[x]_\rho \in \frac{H}{\rho}$, then $[0]_\rho \in [x]_\rho \circ [a]_\rho$ and so $[0]_\rho = [t]_\rho$ for some $t \in x \circ a$. Hence $t = 0$ and so $x \ll a$. Since a is a hyperatom, we get $x = 0$ or $x = a$, which implies $[x]_\rho = [0]_\rho$ or $[x]_\rho = [a]_\theta$. Therefore $\frac{H}{\rho}$ is hyperatomic. □

Now, we recall the definition of hypercondition and consider $H = \bigoplus_{a\in A^*(H)} B(a)$ satisfying the hypercondition.

Definition 3.19. [9] A hyper BCK-algebra H is said to satisfy the hypercondition if, for every $a, b \in H$, the set $\bigtriangledown(a, b) := \{x \in H \mid 0 \in (x \circ a) \circ b\}$ has the greatest hyperelement. This greatest hyperelement is denoted by $a \ominus b$. Obviously, $0, a, b \in \bigtriangledown(a, b)$.

Lemma 3.20. *If $H = \bigoplus_{a\in A^*(H)} B(a)$ satisfies the hypercondition, then $H = B(a)$, for some $a \in A^*(H)$.*

Proof. Let t be a non-zero element of H. Then there exists $a \in A^*(H)$ such that $t \in B(a)$. By the hypothesis, $t \ominus x$ exists, for all $x \in H$. Since $t \ll t \ominus x$, we get $t \ominus x \in B(a)$. By Lemma 3.8(i), it follows from $x \ll t \ominus x$ that $x \in B(a)$, for any $x \in H$. This implies that $H = B(a)$. □

4. Hyper BCK-Chain

Definition 4.1. An ordered hyper BCK-algebra H is said to be a hyper BCK-chain if $x \ll y$ or $y \ll x$, for any $x, y \in H$.

Example 4.2. (i) Let $N = \{0, 1, 2, ...\}$ and define a hyperoperation "$\circ$" on N as follows:

$$x \circ y := \begin{cases} \{0, x\} & \text{if } x \leq y, \\ \{x\} & \text{otherwise} \end{cases}$$

for all $x, y \in H$. Then $(N; \circ, 0)$ is a hyper BCK-chain. In fact $0 \ll 1 \ll 2 \ll ...$; Then H is not a hyper BCK-chain since neither $2 \ll 3$ nor $3 \ll 2$.

(ii) Consider a hyper BCK-algebra $H = \{0, 1, 2, 3\}$ with the following Cayley table:

$\circ$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{0\}$	$\{0,1\}$
2	$\{2\}$	$\{2\}$	$\{0\}$	$\{2\}$
3	$\{3\}$	$\{3\}$	$\{3\}$	$\{0\}$

Then H is not a hyper BCK-chain since neither $2 \ll 3$ nor $3 \ll 2$.

(iii) Let $(H;\circ,0)$ be a hyper BCK-chain, and let $\alpha \notin H$. Then the Iseki's hyper BCK-algebra $K := (H \cup \{\alpha\}; \circ^{'}, 0)$ which $\circ^{'}$ is defined by

$$\alpha\circ^{'}\alpha = \{0\},\ x\circ^{'}\alpha = \{0\},\ \alpha\circ^{'}x = \{\alpha\},\ \text{and } x\circ^{'}y = x\circ y, \text{for any } x, y \in H,$$

is a hyper BCK-chain.

Definition 4.3. [4] Let H be a hyper BCK-algebra. We say that H satisfies the condition right-multiple (for short, condition r-m) if the following implication holds:

$$(\forall x, y, z \in H)\ (x \ll y \Longrightarrow x \circ z \ll y \circ z).$$

The following theorem gives a condition for the set $[a)$ to be a hyper BCK-ideal.

Theorem 4.4. *Let H be a hyper BCK-chain satisfying the condition r-m. If $[a) \cup S(H) = H$ and $[a) \cap S(H) = \{0\}$, then $[a)$ is a hyper BCK-ideal of H.*

Proof. Obviously, $0 \in [a)$. It suffices to show that the inequality $x \circ b \ll [a)$ does not hold for any $0 \neq x \in S(H)$ and $b \in [a)$. Suppose on the contrary that the inequality holds for some $0 \neq x \in S(H)$ and $b \in [a)$. Then for any $t \in x \circ b$ there exists $d_t \in [a)$ such that $t \ll d_t$. Since $x \in S(H)$, we get $x \circ x = \{0\}$. Then, using axiom $H1$, we obtain $t \circ t = \{0\}$. Hence $t \in S(H)$. On the other hand, it follows from $t \ll d_t \in [a)$ that $t \in [a)$. Thus $t \in [a) \cap S(H) = \{0\}$, and so $t = 0$. Hence $0 \in x \circ b$. This implies that $x \ll b$. Since $b \ll a$, we get $x \in [a)$ and so $(0 \neq)x \in [a) \cap S(H)$, which a contradiction. Therefore $[a)$ is a hyper BCK-ideal of H. □

The following example shows that the condition $[a) \cap S(H) = \{0\}$ in Theorem 4.4 is necessary.

Example 4.5. Consider a hyper BCK-chain $H = \{0, a, b, c\}$ with the following Cayley table:

$\circ$	0	a	b	c
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{0\}$	$\{0\}$
b	$\{b\}$	$\{a\}$	$\{0, a\}$	$\{0, a\}$
c	$\{c\}$	$\{a\}$	$\{a\}$	$\{0, a\}$

Then $[a) = \{0, a\} = S(H)$. Obviously, $b \circ a = \{a\} \ll [a)$ and $b \notin [a)$. It follows that $[a)$ is not a hyper BCK-ideal of H.

Definition 4.6. [4] Let H be a hyper BCK-algebra. We say that H satisfies the condition $\ll$-scalar if the following implication holds:

$$(\forall x, y \in H)\ (x \ll y \Longrightarrow x \circ y = \{0\}).$$

Lemma 4.7. *If a hyper BCK-algebra satisfies the condition $\ll$-scalar, then it satisfies the condition r-m.*

Proof. Let $x, y \in H$ be such that $x \ll y$. Then by hypothesis, we have $x \circ y = \{0\}$ and so by Theorem 2.2(a_9), we have $(x \circ z) \circ (y \circ z) = \{0\}$. This implies that $x \circ z \ll y \circ z$. Therefore H satisfies the condition r-m. □

Proposition 4.8. *Every hyper BCK-algebra satisfying condition $\ll$-scalar is a ordered hyper BCK-algebra.*

Proof. Let H be a hyper BCK-algebra satisfying the condition $\ll$-scalar, and let $x, y, z \in H$ be such that $x \ll y$ and $y \ll z$. Then by hypothesis, we get $x \circ y = \{0\}$ and $y \circ z = \{0\}$. By Theorem 2.3(a_9), it follows from $x \circ y = \{0\}$ that $(x \circ z) \circ (y \circ z) = \{0\}$. Hence $x \circ z = \{0\}$, that is, $x \ll z$. Therefore H is a ordered hyper BCK-algebra. □

Applying Lemma 4.7 and Theorem 4.4, we have the following corollary.

Corollary 4.9. *Let H be a hyper BCK-chain satisfying the condition $\ll$-scalar. If $[a) \cup S(H) = H$ and $[a) \cap S(H) = \{0\}$, then $[a)$ is a hyper BCK-ideal of H.*

Theorem 4.10. *Let H be a hyper BCK-chain. If the set $[a)$ is finite for any $a \in H$, then $\mid Aut(H) \mid = 1$.*

Proof. Assume that $f : H \to H$ is an isomorphism. It suffices to show that $f(x) = x$ for any $x \in H$. Suppose on the contrary that there exists $a \in H$ such that $f(a) \neq a$. Since $f(0) = 0$ and $[a)$ is finite, then we may suppose that $\mid [a) \mid = n$, where n is the least number with property $f(a) \neq a$ and $f(x) = x$ for any $(a \neq) x \in [a)$. Hence, we can assume that $[a) = \{x_i \in H \mid 0 = x_1 \ll x_2 \ll ... \ll x_{n-1} \ll x_n = a\}$. Therefore $f(x_i) = x_i$ for any $i = 1, 2, ..., n-1$. Since f is injective, we have $f(a) \neq x_i$ for any $i = 1, 2, ..., n$ and so $f(a) \notin [a)$. Assume that $f(a) = c$. Then from $c \notin [a)$ and the fact that H is a chain, we get $a \ll c$ and $a \neq c$. Since f is surjective, there exists $d \in H$ such that $a = f(d)$. Clearly, $d \neq a$. If $d \ll a$, then $d \in [a)$ and so $d = x_i$ for some $i = 1, 2, ..., n-1$. Hence $f(d) = d$ and so from $a = f(d)$ we get $a = d$. This implies $f(a) = f(d) = d = a$, that is, $f(a) = a$, which a contradiction. Thus $a \ll d$. If follows from f is isotone that $f(a) \ll f(d)$. Hence $c \ll a$, which a contradiction. Then $f(x) = x$ for any $x \in H$, that is, $f = id_H$. Therefore $\mid Aut(H) \mid = 1$. □

The following example shows that the finiteness assumption for $[a)$ in Theorem 4.10 is necessary.

Example 4.11. Let $H = \{0, 1, 2, ...\} \cup \{\frac{1}{n} : n = 2, 3, ...\}$. Define a hyperoperation "$\circ$" on H as follows:

$$x \circ y := \begin{cases} \{0, x\} & \text{if } x \leq y, \\ \{x\} & \text{otherwise} \end{cases}$$

for all $x, y \in H$. It is routine to check that H is a hyper BCK-chain. Clearly, $[a)$ is infinite for any $(0 \neq)a \in H$. Define a function $f : H \to H$ by $f(n) = n-1$ for $n = 2, 3, ...$; $f(\frac{1}{n}) = \frac{1}{n+1}$ for $n = 1, 2, ...$; and $f(0) = 0$. It can be verified that f is an isomorphism that is not the identity map. Therefore $\mid Aut(H) \mid > 1$.

The following proposition shows that the image of a branch of an isomorphism is a branch too.

Theorem 4.12. *Let* $H = \bigoplus_{a \in A^*(H)} B(a)$ *and* $K = \bigoplus_{b \in A^*(K)} B(b)$*. If all branches of* H *and* K *are chain, then the following statements hold:*

(i) If $f : H \to K$ *is a homomorphism, then* $f(B(a)) \cap B(b) \neq (0)$ *implies* $f(B(a)) \subseteq B(b)$*, for any* $a \in A^*(H)$ *and* $b \in A^*(K)$*;*

(ii) If $f : H \to K$ *is an isomorphism, then for any* $a \in A^*(H)$*, there exists* $b \in A^*(K)$ *such that* $f(a) = b$ *and* $f(B(a)) = B(b)$ *and consequently,* $\mid B(a) \mid = \mid B(b) \mid$.

Proof. (i) Assume that $f(B(a)) \cap B(b) \neq (0)$ for some $a \in A^*(H)$ and $b \in A^*(K)$. Then there exist $x \in B(a)$ and $y \in B(b)$ such that $y = f(x) \neq 0$. For any $t \in B(a)$, we have $t \ll x$ or $x \ll t$. Since f is isotone, we get $f(t) \ll f(x)$ or $f(x) \ll f(t)$. Hence $f(t) \ll y$ or $y \ll f(t)$. If $f(t) \ll y$, then by Lemma 3.8(i), we have $f(t) \in B(b)$. If $y \ll f(t)$, then it follows from $b \ll y$ that $b \ll f(t)$ and so $f(t) \in B(b)$. Therefore $f(B(a)) \subseteq B(b)$.

(ii) Let $a \in A^*(H)$. Clearly, $f(a) \in B(b)$ for some $b \in Q$. Since $a \neq 0$, we get $f(a) \neq 0$. Hence $b \ll f(a)$. Since f is epimorphism, $b = f(t)$ for some $t \in H$. Thus $f(t) \ll f(a)$ and so $t \ll a$ because f^{-1} is isotone. Since a is a hyperatom, we get $t = a$. Hence $f(a) = b$. To proof the second part (ii), we note that $0 \neq f(a) \in f(B(a)) \cap B(b)$. Using (i), we get $f(B(a)) \subseteq B(b)$. Let $0 \neq y \in B(b)$. Then $b \ll y$. But $b = f(a)$. Hence $f(a) \ll y$ and so $a \ll f^{-1}(y)$. This implies $f^{-1}(y) \in B(a)$. Hence $y = f(f^{-1}(y)) \in f(B(a))$, and consequently $B(b) \subseteq f(B(a))$. Therefore $B(b) = f(B(a))$. It is clear that $\mid B(a) \mid = \mid f(B(a)) \mid$. Therefore $\mid B(a) \mid = \mid B(b) \mid$. □

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