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Tangent Bundle of the Hypersurfaces in a Euclidean Space

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Abstract. Let M be an orientable hypersurface in the Euclidean space R^{2n} with induced metric g and TM be its tangent bundle. It is known that the tangent bundle TM has induced metric $\overline{g}$ as submanifold of the Euclidean space R^{4n} which is not a natural metric in the sense that the submersion $\pi : (TM, \overline{g}) \to (M, g)$ is not the Riemannian submersion. In this paper, we use the fact that R^{4n} is the tangent bundle of the Euclidean space R^{2n} to define a special complex structure $\overline{J}$ on the tangent bundle R^{4n} so that $(R^{4n}, \overline{J}, \langle,\rangle)$ is a Kaehler manifold, where $\langle,\rangle$ is the Euclidean metric which is also the Sasaki metric of the tangent bundle R^{4n}. We study the structure induced on the tangent bundle $(TM, \overline{g})$ of the hypersurface M, which is a submanifold of the Kaehler manifold $(R^{4n}, \overline{J}, \langle,\rangle)$. We show that the tangent bundle TM is a CR-submanifold of the Kaehler manifold $(R^{4n}, \overline{J}, \langle,\rangle)$. We find conditions under which certain special vector fields on the tangent bundle $(TM, \overline{g})$ are Killing vector fields. It is also shown that the tangent bundle TS^{2n-1} of the unit sphere S^{2n-1} admits a Riemannian metric $\overline{g}$ and that there exists a nontrivial Killing vector field on the tangent bundle $(TS^{2n-1}, \overline{g})$.

Keywords: Tangent bundle, Hypersurface, Kaehler manifold, Almost contact structure, Killing vector field, CR-Submanifold, Second fundamental form, Wiengarten map.

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1. Introduction

Recently efforts are made to study the geometry of the tangent bundle of a hypersurface M in the Euclidean space R^{n+1} (cf. [3]), where the authors have shown that the induced metric on its tangent bundle TM as submanifold of the Euclidean space R^{2n+2} is not a natural metric. In [4], we have extended the study initiated in [3] on the geometry of the tangent bundle TM of an immersed orientable hypersurface M in the Euclidean space R^{n+1}. It is well known that Killing vector fields play an important role in shaping the geometry of a Riemannian manifold, for instance the presence of nonzero Killing vector field on a compact Riemannian manifold forces its Ricci curvature to be non-negative and this in particular implies that on a compact Riemannian manifolds of negative Ricci curvature there does not exist a nonzero Killing vector field. The study of Killing vector fields becomes more interesting on the tangent bundle TM of a Riemannian manifold (M, g) as the tangent bundle TM is noncompact. It is known that if the tangent bundle TM of a Riemannian manifold (M, g) is equipped with Sasaki metric, then the verticle lift of a parallel vector field on M is a Killing vector field (cf. [15]). However if the Sasaki metric is replaced by the Cheeger-Gromoll metric, then the vertical lift of any nonzero vector field on M is never Killing (cf. [14]). Note that both Sasaki metric as well as Cheeger-Gromoll metrics are natural metrics. We consider an orientable real hypersurface M of the Euclidean space R^{2n} with the induced metric g. Then as the tangent bundle TM of M is a submanifold of codimension two in R^{4n}, it has induced metric $\overline{g}$ and this metric $\overline{g}$ on TM is not a natural metric as the submersion $\pi : (TM, \overline{g}) \to (M, g)$ is not the Riemannian submersion (cf. [3]). Let N be the unit normal vector field to the hypersurface M and J be the natural complex structure on the Euclidean space R^{2n}. Then we have a globally defined unit vector field ξ on the hypersurface given by $\xi = -JN$ called the characteristic vector field of the real hypersurface (cf. [1, 2, 5, 6, 7, 8, 9]), and this vector field ξ gives rise to two vector fields ξ^h (the horizontal lift) and ξ^v (the vertical lift) on the tangent bundle $(TM, \overline{g})$. In this paper, we use the fact that R^{4n} is the tangent bundle of the Euclidean space R^{2n} and that the projection $\overline{\pi} : R^{4n} \to R^{2n}$ is a Riemannian submersion, to define a special almost complex structure $\overline{J}$ on the tangent bundle R^{4n} which is different from the canonical complex structure of the Euclidean space R^{4n} and show that $(R^{4n}, \overline{J}, \langle,\rangle)$ is a Kaehler manifold, where $\langle,\rangle$ is the Euclidean metric on R^{4n}. It is shown that the codimension two submanifold $(TM, \overline{g})$ of the Kaehler manifold $(R^{4n}, \overline{J}, \langle,\rangle)$ is a CR-submanifold (cf. [10]) and it naturally inherits certain special vector fields other than ξ^h and ξ^v, and in this paper we are interested in finding conditions under which these special vector fields are Killing vector fields on $(TM, \overline{g})$. One of the interesting outcome of this study is, we have shown that the tangent bundle TS^{2n-1} of the unit sphere S^{2n-1} as submanifold of R^{4n} admits a nontrivial Killing vector field. It is worth pointing out that on the tangent bundle TS^{2n-1} with Sasakian metric no vertical or horizontal lift of a vector field is Killing as this will require the corresponding vector field on S^{2n-1} is parallel which is impossible as S^{2n-1} is space of constant curvature 1. Note that on even dimensional Riemannian manifolds which are irreducible, it is difficult to find Killing vector fields, where as on products like $S^{2k-1}\times S^{2l-1}$, $S^{2k-1}\times R^{2l-1}$, $R^{2k-1}\times R^{2l-1}$ one can easily find Killing vector fields. Since the tangent bundle TS^{2n-1} is trivial for $n=1,2,4$, finding Killing vector fields is easy in these dimensions, but for $n\geq 5$, it is not trivial.

2. Preliminaries

Let (M,g) be a Riemannian manifold and TM be its tangent bundle with projection map $\pi : TM \longrightarrow M$. Then for each $(p,u)\in TM$, the tangent space $T_{(p,u)}TM = \mathfrak{H}_{(p.u)}\oplus\mathfrak{V}_{(p,u)}$, where $\mathfrak{V}_{(p,u)}$ is the kernel of $d\pi_{(p,u)}:T_{(p,u)}(TM)\longrightarrow T_pM$ and $\mathfrak{H}_{(p.u)}$ is the kernel of the connection map $K_{(p,u)} : T_{(p,u)}(TM)\longrightarrow T_pM$ with respect to the Riemannian connection on (M,g). The subspaces $\mathfrak{H}_{(p.u)}, \mathfrak{V}_{(p,u)}$ are called the horizontal and vertical subspaces respectively. Consequently, the Lie algebra of smooth vector fields $\mathfrak{X}(TM)$ on the tangent bundle TM admits the decomposition $\mathfrak{X}(TM)=\mathfrak{H}\oplus\mathfrak{V}$ where $\mathfrak{H}$ is called the horizontal distribution and $\mathfrak{V}$ is called the vertical distribution on the tangent bundle TM. For each $X_p\in T_pM$, the horizontal lift of X_p to a point $z=(p,u)\in TM$ is the unique vector $X^h_z\in\mathfrak{H}_z$ such that $d\pi(X^h_z)=X_p\circ\pi$ and the vertical lift of X_p to a point $z=(p,u)\in TM$ is the unique vector $X^v_z\in\mathfrak{V}_z$ such that $X^v_z(df)=X_p(f)$ for all functions $f\in C^\infty(M)$, where df is the function defined by $(df)(p,u)=u(f)$. Also for a vector field $X\in\mathfrak{X}(M)$, the horizontal lift of X is a vector field $X^h\in\mathfrak{X}(TM)$ whose value at a point (p,u) is the horizontal lift of $X(p)$ to (p,u) , the vertical lift X^v of X is defined similarly. For $X\in\mathfrak{X}(M)$ the horizontal and vertical lifts X^h, X^v of X are uniquely determined vector fields on TM satisfying

$$d\pi(X^h_z)=X_{\pi(z)}, K(X^h_z)=0, d\pi(X^v_z)=0, K(X^v_z)=X_{\pi(z)}$$

Also, we have for a smooth function $f\in C^\infty(M)$ and vector fields $X,Y\in\mathfrak{X}(M)$, that $(fX)^h=(f\circ\pi)X^h$, $(fX)^v=(f\circ\pi)X^v$, $(X+Y)^h=X^h+Y^h$ and $(X+Y)^v=X^v+Y^v$. If $\dim M=m$ and (U,φ) is a chart on M with local coordinates $x^1,x^2,\dots,x^m$, then $(\pi^{-1}(U),\varphi)$ is a chart on TM with local coordinates $x^1,x^2,\dots,x^m,y^1,y^2,\dots,y^m$, where $x^i=x^i\circ\pi$ and $y^i=dx^i$, $i=1,2,\dots,m$.

A Riemannian metric $\overline{g}$ on the tangent bundle TM is said to be natural metric with respect to g on M if $\overline{g}_{(p,u)}(X^h,Y^h)=g_p(X,Y)$ and $\overline{g}_{(p,u)}(X^h,Y^v)=0$, for all vectors fields $X,Y\in\mathfrak{X}(M)$ and $(p,u)\in TM$, that is the projection map $\pi : TM\longrightarrow M$ is a Riemannian submersion.

Let M be an orientable hypersurface of the Euclidean space R^{2n} with immersion $f : M \longrightarrow R^{2n}$ and TM be its tangent bundle. Then as $F = df : TM \longrightarrow R^{4n} = TR^{2n}$ is also an immersion, TM is an immersed submanifold of the Euclidean space R^{4n}. We denote the induced metrics on M, TM by g, $\overline{g}$ respectively and the Euclidean metric on R^{2n} as well as on R^{4n} by $\langle , \rangle$. Also, we denote by $\overline{\nabla}, \overline{\nabla}, D$ and $\overline{D}$ the Riemannian connections on M, TM, R^{2n}, and R^{4n} respectively. Let N and S be the unit normal vector field and the shape operator of the hypersurface M. For the hypersurface M of the Euclidean space R^{2n} we have the following Gauss and Weingarten formulae

$$D_X Y = \overline{\nabla}_X Y + \langle S(X), Y \rangle N, \; D_X N = -S(X), \quad X, Y \in \mathfrak{X}(M) \tag{2.1}$$

where S is the shape operator (Weingarten map). Similarly for the submanifold TM of the Euclidean space R^{4n} we have the Gauss and Weingarten formulae

$$\overline{D}_E F = \overline{\nabla}_E F + h(E, F), \; \overline{D}_E \overline{N} = -\bar{S}_{\overline{N}}(E) + \overline{\nabla}^{\perp}_E \overline{N} \tag{2.2}$$

where $E, F \in \mathfrak{X}(TM)$, $\overline{\nabla}^{\perp}$ is the connection in the normal bundle of TM and $\bar{S}_{\overline{N}}$ denotes the Weingarten map in the direction of the normal $\overline{N}$ and is related to the second fundamental form h by

$$\left\langle h(X, Y), \overline{N} \right\rangle = \overline{g}(\bar{S}_{\overline{N}}(X), Y) \tag{2.3}$$

Also we observe that for $X \in \mathfrak{X}(M)$ the vertical lift X^v of X to TM, as $X^v \in \ker d\pi$, where $\pi : TM \to M$ is the natural submersion, we have $d\pi(X^v) = 0$ that is $df(\ d\pi(X^v)) = 0$ or equivalently we get $d(f \circ \pi)(X^v) = 0$, that is $d(\tilde{\pi} \circ F)(X^v) = 0$ $(\overline{\pi} : TR^{2n} \to R^{2n})$, which gives $dF(X^v) \in \ker d\tilde{\pi} = \bar{\mathfrak{V}}$.

Now we state the following results which are needed in our work.

Lemma 2.1. [3] *Let N be the unit normal vector field to the hypersurface M of R^{2n} and $P = (p, X_p) \in TM$. Then the horizontal and vertical lifts Y^h_P, Y^c_P of $Y_p \in T_p M$ satisfy*

$$dF_P(Y^h_P) = (df_p(Y_p))^h + V_P, \qquad dF_P(Y^v_P) = (df_p(Y_p))^v$$

where $V_P \in \mathfrak{V}_P$ is given by $V_P = \langle S_p(X_p), Y_p \rangle N^v_P$, N^v_P being the vertical lift of the unit normal N to with respect to the tangent bundle $\overline{\pi} : R^{4n} \to R^{2n}$.

Lemma 2.2. [3] *If (M, g) is an orientable hypersurface of R^{2n}, and $(TM, \overline{g})$ is its tangent bundle as submanifold of R^{4n}, then the metric $\overline{g}$ on TM for $P = (p, u) \in TM$, satisfies:*

(*i*) $\overline{g}_P(X^h_P, Y^h_P) = g_p(X_p, Y_p) + g_p(S_p(X_p), u) g_p(S_p(Y_p), u)$.
(*ii*) $\overline{g}_P(X^h_P, Y^v_P) = 0$.
(*ii*) $\overline{g}(X^v, Y^v) = g_p(X_p, Y_p)$.

Remark 2.3. It is well known that a metric $\overline{g}$ defined on TM using the Riemannian metric g of M (such as Sasaki metric, Cheeger-Gromoll metric) are natural metrics in the sense that the submersion $\pi : (TM, \overline{g}) \longrightarrow (M, g)$ becomes a Riemannian submersion with respect to these metrics. However, as seen from above Lemmas, the induced metric on the tangent bundle TM of a hypersurface M of the Euclidean space R^{2n}, as a submanifold of R^{4n} is not a natural metric because of the present of the term $g_p(S_p(X_p), u)g_p(S_p(Y_p), u)$ in the inner product of horizontal vectors on TM. Moreover, note that the for an orientable hypersurface M of the Euclidean space R^{2n}, the vertical lift N^v of the unit normal is tangential to the submanifold TM of R^{4n} as seen in 2.1

In what follows, we drop the suffixes like in $g_p(S_p(X_p), u)$ and and it will be understood from the context of the entities appearing in the equations.

Theorem 2.4. [3] *Let (M, g) be an orientable hypersurface of R^{2n}, and $(TM, \overline{g})$ be its tangent bundle as submanifold of R^{4n}. If ∇ and $\overline{\nabla}$ denote the Riemannian connections on (M, g) and $(TM, \overline{g})$ respectively, then*

(i) $\overline{\nabla}_{X^h} Y^h = (\overline{\nabla}_X Y)^h - \frac{1}{2}(R(X, Y)u)^v$,
(ii) $\overline{\nabla}_{X^v} Y^h = g(S(X), Y) \circ \pi N^v$
(iii) $\overline{\nabla}_{X^v} Y^v = 0$, ***(iv)*** $\overline{\nabla}_{X^h} Y^v = (\overline{\nabla}_X Y)^v + g(S(X), Y) \circ \pi \ N^v$.

Lemma 2.5. [4] *Let TM be the tangent bundle of an orientable hypersurface M of R^{2n}. Then for $X, Y \in \mathfrak{X}(M)$,*

(i) $h(X^v, Y^v) = 0$,
(ii) $h(X^v, Y^h) = 0$,
(iii) $h(X^h, Y^h) = g(S(X), Y) \circ \pi \ N^h$.

Lemma 2.6. [4] *For the tangent bundle TM of an orientable hypersurface M of R^{2n} and $X \in \mathfrak{X}(M)$, we have*

(i) $\overline{D}_{X^v} N^v = 0$,
(ii) $\overline{D}_{X^v} N^h = 0$,
(iii) $\overline{D}_{X^h} N^v = -(S(X))^v$, ***(iv)*** $\overline{D}_{X^h} N^h = -(S(X))^h$.

Let J be the natural complex structure on the Euclidean space R^{2n}, which makes $(R^{2n}, J, \langle, \rangle)$ a Kaehler manifold. Then on an orientable real hypersurface M of R^{2n} with unit normal N, we define a unit vector field $\xi \in \mathfrak{X}(M)$ by $\xi = -JN$, with its dual 1-form $\eta(X) = g(X, \xi)$, where g is the induced metric on M. For $X \in \mathfrak{X}(M)$, we express $JX = \varphi(X) + \eta(X)N$, where $\varphi(X)$ is the tangential component of JX, and it follows that φ is a $(1, 1)$ tensor field on M, and that (φ, ξ, η, g) defines an almost contact metric structure on M (cf. [5], [8], [9]), that is

$$\varphi^2 X = -X + \eta(X)\xi, \ \eta(\xi) = 1, \ \eta \circ \varphi = 0, \ \varphi(\xi) = 0$$

and

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad X, Y \in \mathfrak{X}(M)$$

Moreover, we have the following.

Lemma 2.7. [8] *Let M be an orientable real hypersurface of R^{2n}. Then the structure (φ, ξ, η, g) on M satisfies*

(*i*) $(\overline{\nabla}_X \varphi)(Y) = \eta(Y)SX - g(SX, Y)\xi,$

(*ii*) $\overline{\nabla}_X \xi = \varphi SX,\ X, Y \in \mathfrak{X}(M).$

3. A Structure on $(TM, \overline{g})$

We know that the Euclidean space R^{4n} has many complex structures, however in this section we treat R^{4n} as the tangent bundle of R^{2n} and consider a specific complex structure on the Euclidean space R^{4n}. Let $\overline{\pi} : R^{4n} = TR^{2n} \to R^{2n}$ be the submersion of the tangent bundle of R^{2n}. Then it is easy to show that the Euclidean metric $\langle , \rangle$ on the tangent bundle R^{4n} is Sasaki metric and using the canonical almost complex structure J of R^{2n}, we define $\overline{J} : \mathfrak{X}(R^{4n}) \to \mathfrak{X}(R^{4n})$ by

$$\overline{J}(E^h) = (JE)^h, \quad \overline{J}(E^v) = (JE)^v, \qquad E \in \mathfrak{X}(R^{2n})$$

and it is easily follows that $\overline{J}$ is an almost complex structure, satisfying $\langle \overline{J}E, \overline{J}F \rangle = \langle E, F \rangle$ with respect to the Euclidean metric $\langle , \rangle$ on R^{4n} and that $(\overline{D}_E \overline{J})(F) = 0,\ E, F \in \mathfrak{X}(R^{4n})$ that is $(R^{4n}, \overline{J}, \langle , \rangle)$ is a Kaehler manifold. Regarding the complex structure $\overline{J}$ defined above, we have the following

Lemma 3.1. *Let $\overline{\pi} : R^{4n} \to R^{2n}$ be the submersion of the tangent bundle $R^{4n} = TR^{2n}$. Then complex structure $\overline{J}$ on R^{4n} satisfies*

$$J \circ d\overline{\pi} = d\overline{\pi} \circ \overline{J}$$

Proof. Take $X \in \mathfrak{X}(R^{2n})$, then for the horizontal lift X^h, we have:

$$J \circ d\overline{\pi}(X^h) = J(d\overline{\pi}(X^h)) = JX \circ \overline{\pi}$$

and

$$d\overline{\pi} \circ \overline{J}(X^h) = d\overline{\pi}(JX)^h = JX \circ \overline{\pi}$$

which proves

$$J \circ d\overline{\pi}(X^h) = d\overline{\pi} \circ \overline{J}(X^h)$$

Similarly for the vertical lift X^vwe have

$$J \circ d\overline{\pi}(X^v) = J(d\overline{\pi}(X^v)) = 0$$

and

$$d\overline{\pi} \circ \overline{J}(X^v) = d\overline{\pi}(JX)^v = 0$$

This proves the Lemma. □

Remark 3.2. If M is an orientable real hypersurface of the Euclidean space R^{2n} with immersion f, then $F = df$ is the immersion of the tangent bundle TM into the Euclidean space R^{4n} and as immersions are local embeddings, in general, we identify the local quantities on submanifold with those of the ambient space for instance we identify $df(X)$ with X for $X \in \mathfrak{X}(M)$. However, while dealing with the immersion F of TM in R^{4n} one need to be cautious specially while dealing with the horizontal lifts (cf. 2.1). Therefore in what follows, we shall bring dF in to play whenever it is needed specially in the case of horizontal lifts.

Observe that if M is an orientable real hypersurface of the Euclidean space R^{2n} with unit normal vector field N, then we know that horizontal lift N^h is a unit normal vector field to the submanifold TM of R^{4n} and that the vertical lift $N^v \in \mathfrak{X}(TM)$ (cf.[1]). We have

$$\overline{J}N^h = (JN)^h = -(df(\xi))^h = -dF(\xi^h) + g(S(\xi), u)N^v \in \mathfrak{X}(TM) \tag{3.1}$$

and

$$\overline{J}N^v = (JN)^v = -\xi^v \in \mathfrak{X}(TM) \tag{3.2}$$

Let M be an orientable real hypersurface of the Kaehler manifold $(R^{2n}, J, \langle,\rangle)$. Then as TM is submanifold of the Kaehler manifold $(R^{4n}, \overline{J}, \langle,\rangle)$, we denote by $\Gamma(T^{\perp}TM)$ the space of smooth normal vector fields to TM. The restriction of the complex structure $\overline{J}$ on R^{4n} to $\mathfrak{X}(TM)$ and $\Gamma(T^{\perp}TM)$ can be expressed as

$$\overline{J}(E) = \overline{\varphi}(E) + \overline{\psi}(E),\ \overline{J}(\overline{N}) = \overline{G}(\overline{N}) + \overline{\chi}(\overline{N}), \quad E \in \mathfrak{X}(TM),\ \overline{N} \in \Gamma(T^{\perp}TM)$$

where $\overline{\varphi}(E)$, $\overline{G}(\overline{N})$ are the tangential and $\overline{\psi}(E)$, $\overline{\chi}(\overline{N})$ are the normal components of $\overline{J}E$, and $\overline{J}(\overline{N})$ respectively. Note that the horizontal lift N^h of the unit normal N to the hypersurface M is normal to TM that is $N^h \in \Gamma(T^{\perp}TM)$, where as the vertical lift $N^v \in \mathfrak{X}(TM)$.

Lemma 3.3. *Let TM be the tangent bundle of an orientable real hypersurface of R^{2n}. Then for $X \in \mathfrak{X}(M)$,*

$$\overline{\varphi}(X^h) = (\varphi(X))^h - g(S(X), u)\xi^v, \quad \overline{\varphi}(X^v) = (\varphi(X))^v + \eta(X) \circ \pi N^v$$
$$\overline{\psi}(X^h) = \eta(X) \circ \pi N^h, \quad \overline{\psi}(X^v) = 0$$

Proof. Note that for the horizontal lift X^h we have

$$\begin{aligned}
\overline{J}X^h &= \overline{J}dF(X^h) = \overline{J}\left((df(X))^h + g(SX, u) \circ \pi N^v\right) \\
&= (Jdf(X))^h + g(SX, u) \circ \pi\, (JN)^v \\
&= (\varphi X + \eta(X)N)^h - g(SX, u) \circ \pi \xi^v \\
&= (\varphi(X))^h - g(SX, u) \circ \pi \xi^v + \eta(X) \circ \pi N^h
\end{aligned}$$

which together with the definition $\overline{J}X^h = \overline{\varphi}(X^h) + \overline{\psi}(X^h)$, on equating tangential and normal components give

$$\overline{\varphi}(X^h) = (\varphi(X))^h - g(S(X), u)\xi^v \text{ and } \overline{\psi}(X^h) = \eta(X) \circ \pi N^h$$

Similarly for the vertical lift X^v, we have

$$\overline{J}X^v = \overline{\varphi}(X^v) + \overline{\psi}(X^v) = (JX)^v = (\varphi X + \eta(X)N)^v$$

which gives

$$(\overline{\varphi}(X^v)) + \overline{\psi}(X^v) = (\varphi X)^v + \eta(X) \circ \pi N^v$$

Comparing the tangential and normal components we conclude

$$\overline{\varphi}(X^v) = (\varphi(X))^v + \eta(X) \circ \pi N^v, \quad \text{and} \quad \overline{\psi}(X^v) = 0.$$

□

We choose a unit normal vector field $N^* \in \Gamma(T^{\perp}TM)$ such that $\left\{N^*, N^h\right\}$ is a local orthonormal frame of normals for the submanifold TM. It is known that N^* is vertical vector field on the tangent bundle R^{4n} (cf. [1]). Since, $\left\langle \overline{J}N^*, N^* \right\rangle = 0$, $\left\langle \overline{J}N^*, N^h \right\rangle = \left\langle N^*, \xi^h \right\rangle = 0$, it follows that $\overline{J}N^* \in \mathfrak{X}(TM)$ and we define unit vector field $\zeta \in \mathfrak{X}(TM)$ by

$$\zeta = -\overline{J}N^* \tag{3.3}$$

Now, for any normal vector field $\overline{N} \in \Gamma(T^{\perp}TM)$, we have

$$\overline{N} = \left\langle \overline{N}, N^* \right\rangle N^* + \left\langle \overline{N}, N^h \right\rangle N^h$$

which together with equations (3.1), (3.2) and (3.3) gives $\overline{\chi}(\overline{N}) = 0$ and that $\overline{J}(\overline{N}) \in \mathfrak{X}(TM)$, isgiven by

$$\overline{J}(\overline{N}) = \left\langle \overline{J}(\overline{N}), \zeta \right\rangle \zeta + \left\langle \overline{J}(\overline{N}), T \right\rangle T \tag{3.4}$$

where $T \in \mathfrak{X}(TM)$, is given by

$$T = \xi^h - g(S(\xi), u)N^v = -\overline{J}N^h \tag{3.5}$$

Also, using equation (3.2), we have

$$-\xi^v = \overline{J}N^v = \overline{\varphi}(N^v) + \overline{\psi}(N^v)$$

which gives

$$\overline{\varphi}(N^v) = -\xi^v \text{ and } \overline{\psi}(N^v) = 0 \tag{3.6}$$

Moreover, we have

$$\overline{\varphi}(\zeta) = 0 \text{ and } \overline{\psi}(\zeta) = N^*,\ \overline{\psi}(\xi^h) = N^h \tag{3.7}$$

If we denote by α, β the smooth 1-forms on TM dual to the vector field ζ and T respectively, then for $E \in \mathfrak{X}(TM)$, it follows that

$$\overline{J}\left(\overline{\psi}(E)\right) = -\alpha(E)\zeta - \beta(E)T$$

and consequently, operating $\overline{J}$ on $\overline{J}(E) = \overline{\varphi}(E) + \overline{\psi}(E)$, $E \in \mathfrak{X}(TM)$, we get

$$\overline{\varphi}^2 = -I + \alpha \otimes \zeta + \beta \otimes T \text{ and } \overline{\psi} \circ \overline{\varphi} = 0 \tag{3.8}$$

Using Lemma 2.1 and equations (3.3), (3.5), (3.6), (3.8), we see that the vector fields ζ, T and 1-fomrs α, βsatisfy

$$\overline{\varphi}(\zeta) = 0,\ \overline{\varphi}(T) = 0,\ \overline{g}(\zeta, T) = 0,\ \alpha \circ \overline{\varphi} = 0,\ \beta \circ \overline{\varphi} = 0 \tag{3.9}$$

Also, as $\overline{g}$ is the induced metric on the submanifold TM and $\overline{J}$ is skew symmetric with respect to the Hermitian metric $\langle,\rangle$, we have

$$\overline{g}\left(\overline{\varphi}(E),F\right)=-\overline{g}(E,\overline{\varphi}(F)),\quad E,F\in\mathfrak{X}(TM) \tag{3.10}$$

Then using equations (3.8), (3.9) and (3.10), we have

$$\overline{g}\left(\overline{\varphi}(E),\overline{\varphi}(F)\right)=\overline{g}(E,F)-\alpha(E)\alpha(F)-\beta(E)\beta(F),\quad E,F\in\mathfrak{X}(TM) \tag{3.11}$$

Thus we have proved the following

Lemma 3.4. *Let TM be the tangent bundle of an orientable real hypersurface of R^{2n}. Then there is a structure $(\overline{\varphi},\zeta,T,\alpha,\beta,\overline{g})$ similar to contact metric structure on TM, where $\overline{\varphi}$ is a tensor field of type $(1,1)$, ζ,T are smooth vector fields and α, β are smooth 1-forms dual to ζ,T with respect to the Riemannian metric $\overline{g}$ satisfying*

$$\overline{\varphi}^2=-I+\alpha\otimes\zeta+\beta\otimes T,\ \overline{\varphi}(\zeta)=0,\ \overline{\varphi}(T)=0,\ \alpha\circ\overline{\varphi}=0,\ \beta\circ\overline{\varphi}=0,\ \overline{g}(\zeta,T)=0$$
$$\overline{g}\left(\overline{\varphi}(E),\overline{\varphi}(F)\right)=\overline{g}(E,F)-\alpha(E)\alpha(F)-\beta(E)\beta(F),\quad E,F\in\mathfrak{X}(TM).$$

In the next Lemma, we compute the co-variant derivatives of the tensor $\overline{\varphi}$.

Lemma 3.5. *Let $(\overline{\varphi},\zeta,T,\alpha,\beta,\overline{g})$ be the structure on the tangent bundle TM of an orientable real hypersurface M of the Euclidean space R^{2n}. Then*

(i) $(\overline{\nabla}_{X^h}\overline{\varphi})(Y^h)=\left\{(\nabla_X\varphi)(Y)\right\}^h-\left\{X\left(g(SY,u)+g(SY,u)JSX\right)\right\}^v$

(ii) $(\overline{\nabla}_{X^h}\overline{\varphi})(Y^v)=0$,

(iii) $(\overline{\nabla}_{X^v}\overline{\varphi})(Y^v)=0$, $(\overline{\nabla}_{X^v}\overline{\varphi})(Y^h)=g\left(SX,\varphi Y\right)\circ\pi N^v+g\left(SX,Y\right)\circ\pi\xi^v$.

Proof. Using the definition of $\overline{J}$, Lemma 2.1 and Lemma 3.3 together with equation (3.1), we get for $X,Y\in\mathfrak{X}(M)$

$$\begin{aligned}\overline{J}Y^h &= \overline{J}dF(Y^h)=\overline{J}\left((df(Y))^h+g(SY,u)\circ\pi N^v\right)\\ &= (\varphi Y+\eta(Y)N)^h-g(SY,u)\circ\pi\xi^v\\ &= \overline{\varphi}\left(Y^h\right)+\eta(Y)\circ\pi N^h\end{aligned}$$

which gives

$$\begin{aligned}\overline{D}_{X^h}\overline{J}Y^h &= \overline{D}_{\left((df(X))^h+g(SX,u)\circ\pi N^v\right)}\left(\overline{\varphi}\left(Y^h\right)+\eta(Y)\circ\pi N^h\right)\\ &= \overline{D}_{(df(X))^h}\overline{\varphi}\left(Y^h\right)+X(\eta(Y))\circ\pi N^h+\eta(Y)\circ\pi\overline{D}_{(df(X))^h}N^h\\ &\quad +g(SX,u)\circ\pi\overline{D}_{N^v}\left((\varphi(Y))^h-g(S(Y),u)\circ\pi\xi^v\right)+0\\ &\quad +g(SX,u)\circ\pi\eta(Y)\circ\pi\overline{D}_{N^v}N^h\end{aligned}$$

Note that the tangent bundle $TR^{2n}=R^{4n}$ has Sasaki metric and thus using Lemma 7.2 of [10] (keeping in view that R^{2n} is flat), in the above equation, we get

$$\overline{D}_{X^h}\overline{J}Y^h=\overline{\nabla}_{X^h}\overline{\varphi}\left(Y^h\right)+h(X^h,\overline{\varphi}\left(Y^h\right)+X(\eta(Y))\circ\pi N^h-\eta(Y)\circ\pi\left(SX\right)^h \tag{3.12}$$

Similarly we have

$$\begin{aligned}
\overline{JD}_{X^h}Y^h &= \overline{J}\left(\overline{D}_{\left((df(X))^h+g(SX,u)\circ\pi N^v\right)}\left((df(Y))^h+g(SY,u)\circ\pi N^v\right)\right)\\
&= \overline{J}\left\{\overline{\nabla}_{X^h}Y^h+h(X^h,Y^h)+X(g(SY,u)\circ\pi N^v+g(SY,u)\circ\pi\left(D_XN\right)^v\right.\\
&\quad\left.+\,g(SX,u)\circ\pi\overline{D}_{N^v}\left(dfY\right)^h+0+0\right\}\\
&= \overline{\varphi}\left(\overline{\nabla}_{X^h}Y^h\right)+\overline{\psi}\left(\overline{\nabla}_{X^h}Y^h\right)+\overline{J}h(X^h,Y^h)-X(g(SY,u)\circ\pi\xi^v\\
&\quad -g(SY,u)\circ\pi\overline{J}\left(SX\right)^v\\
&= \overline{\varphi}\left(\overline{\nabla}_{X^h}Y^h\right)+\overline{\psi}\left(\overline{\nabla}_{X^h}Y^h\right)-g(SX,Y)\circ\pi\xi^h-X(g(SY,u)\circ\pi\xi^v\\
&\quad -g(SY,u)\circ\pi\left(\varphi SX\right)^v-g(SY,u)\circ\pi\eta(SX)N^v \quad (3.13)
\end{aligned}$$

where we used Lemmas 2.3, 2.4 and Lemma 7.2 in [10]. Now as $\left(R^{4n},\overline{J},\langle,\rangle\right)$ is a Kaehler manifold, the equations (3.12) and (3.13) on comparing tangential we get

$$\left(\overline{\nabla}_{X^h}\overline{\varphi}\right)\left(Y^h\right)=\left\{\left(\nabla_X\varphi\right)(Y)\right\}^h-\left\{X\left(g(SY,u)+g(SY,u)JSX\right)\right\}^v$$

which proves (i).

Now, using $h(X^v,Y^v)=0$ and $\bar{S}_{\overline{\psi}(Y^v)}X^v=0$ together with $\overline{D}_{X^v}\overline{J}Y^v=\overline{JD}_{X^v}Y^v$, and comparing tangential components, we immediately arrive at

$$(\overline{\nabla}_{X^v}\overline{\varphi})(Y^v)=0$$

Next, we have $\overline{\nabla}_{X^v}\overline{\varphi}(Y^h)=\overline{\nabla}_{X^v}\left((\varphi Y)^h-g(SX,u)\circ\pi\xi^v\right)=\overline{\nabla}_{X^v}\left(\varphi Y\right)^h=g\left(SX,\varphi Y\right)\circ\pi N^v$ and $\overline{\varphi}\left(\overline{\nabla}_{X^v}Y^h\right)=g\left(SX,\varphi Y\right)\circ\pi\overline{\varphi}\left(N^v\right)=-g\left(SX,Y\right)\circ\pi\xi^v$. Thus, we get

$$\left(\overline{\nabla}_{X^v}\overline{\varphi}\right)\left(Y^h\right)=g\left(SX,\varphi Y\right)\circ\pi N^v+g\left(SX,Y\right)\circ\pi\xi^v$$

Finally, using $h(X^h,Y^v)=0$ and $\bar{S}_{\overline{\psi}(Y^v)}X^h=0$ together with $\overline{D}_{X^h}\overline{J}Y^v=\overline{JD}_{X^h}Y^v$, and comparing tangential components, we immediately arrive at

$$(\overline{\nabla}_{X^h}\overline{\varphi})(Y^v)=0$$

□

Lemma 3.6. *Let* $(\overline{\varphi},\zeta,T,\alpha,\beta,\overline{g})$ *be the structure on the tangent bundle* TM *of an orientable real hypersurface* M *of the Euclidean space* R^{2n}*. Then for* $E\in\mathfrak{X}(TM)$,

$$\overline{\nabla}_E\zeta=\overline{\varphi}\left(\overline{S}_{N^*}(E)\right)-\overline{J}\left(\overline{\nabla}_E^{\perp}N^*\right),\quad h(E,\zeta)=\overline{\psi}\left(\overline{S}_{N^*}(E)\right)$$

$$\overline{\nabla}_ET=\overline{\varphi}\left(\overline{S}_{N^h}(E)\right)-\overline{J}\left(\overline{\nabla}_E^{\perp}N^h\right),\quad h(E,T)=\overline{\psi}\left(\overline{S}_{N^h}(E)\right)$$

Proof. Using equation (2.2), we have

$$\begin{aligned}\overline{\nabla}_E\zeta &= \overline{D}_E\zeta - h(E,\zeta)\\ &= -\overline{J}\overline{D}_E N^* - h(E,\zeta)\\ &= \overline{J}\left(\overline{S}_{N^*}(E)\right) - \overline{J}\left(\overline{\nabla}^{\perp}_E N^*\right) - h(E,\zeta)\\ &= \overline{\varphi}\left(\overline{S}_{N^*}(E)\right) + \overline{\psi}\left(\overline{S}_{N^*}(E)\right) - \overline{J}\left(\overline{\nabla}^{\perp}_E N^*\right) - h(E,\zeta)\end{aligned}$$

Since $\overline{J}(\overline{N}) \in \mathfrak{X}(TM)$ for each normal $\overline{N} \in \Gamma(T^{\perp}TM)$, equation tangential and normal components in above equation, we get the first part. The second part follows similarly using $T = -\overline{J}N^h$. □

Now, we prove the following:

Theorem 3.7. *The tangent bundle TM of an orientable real hypersurface M of the Euclidean space R^{2n} is a CR-submanifold of the Kaehler manifold $(R^{4n}, \overline{J}, \langle,\rangle)$.*

Proof. Use the structure $(\overline{\varphi}, \zeta, T, \alpha, \beta, \overline{g})$ on the submanifold TM of R^{4n} to define the distribution D by

$$D = \{E \in \mathfrak{X}(TM) : \alpha(E) = \beta(E) = 0\}$$

and $D^{\perp}$ be the distribution spanned by the orthogonal vector fields ζ and T. Note that ζ is unit vector field on TM and the length of the vector field T satisfies

$$\|T\|^2 = 1 + 2g(S(\xi), u)^2 \geq 1$$

which shows that $D^{\perp}$ is 2-dimensional distribution on TM and that $\overline{J}D^{\perp} = \Gamma(T^{\perp}TM)$. It is easy to see that D and $D^{\perp}$ are orthogonal complementary distributions and that $\dim D = 4(n-1)$. Note that for $E \in \mathfrak{X}(TM)$, we have

$$\overline{\psi}(E) = \left\langle \overline{\psi}(E), N^* \right\rangle N^* + \left\langle \overline{\psi}(E), N^h \right\rangle N^h = \alpha(E)N^* + \beta(E)N^h$$

and consequently if $E \in D$, then above equation gives $\overline{J}E = \overline{\varphi}E$ which is orthogonal to both ζ and T and that $\overline{J}E \in D$, which implies $\overline{J}D = D$. This proves that TM is a CR-submanifold of the Kaehler manifold $(R^{4n}, \overline{J}, \langle,\rangle)$ (cf. [8]). □

4. Killing Vector Fields on TM

Let TM be the tangent bundle of an orientable real hypersurface M of the Euclidean space R^{2n}. Recall that a vector field $\varsigma \in \mathfrak{X}(TM)$ on the Riemannian manifold $(TM, \overline{g})$ is said to be Killing if

$$\left(\pounds_{\varsigma}\overline{g}\right)(E,F) = 0, \quad E, F \in \mathfrak{X}(TM)$$

where $\pounds_{\varsigma}$ is the Lie derivative with respect to the vector field ς. We have seen in previous section that the tangent bundle $(TM, \overline{g})$ admits a structure $(\overline{\varphi}, \zeta, T, \alpha, \beta, \overline{g})$, that is similar to the almost contact structure. In this section we are interested in finding conditions under which the special vector fields ζ and T are Killing vector fields and as a particular case we get that the tangent bundle $\left(TS^{2n-1}, \overline{g}\right)$ of the unit sphere S^{2n-1} in the Euclidean space R^{2n} admits a nontrivial Killing vector field.

Theorem 4.1. *Let $(\overline{\varphi}, \zeta, T, \alpha, \beta, \overline{g})$ be the structure on the tangent bundle TM of an orientable real hypersurface M of the Euclidean space R^{2n}. Then the vector field ζ is Killing.*

Proof. First note that on taking inner product with N^* in each part of Lemma 2.5, we conclude that $\overline{S}_{N^*}(X^h) = 0$, $\overline{S}_{N^*}(X^v) = 0$, $X \in \mathfrak{X}(M)$ and consequently,

$$\overline{S}_{N^*}(E) = 0, \quad E \in \mathfrak{X}(TM) \tag{4.1}$$

Also using second part of equation (2.2) in (ii) and (iv) of Lemma 2.4, we conclude that $\overline{\nabla}^{\perp}_{E} N^h = 0$, $E \in \mathfrak{X}(TM)$, that is N^h is parallel on the normal bundle of TM. Moreover, we have

$$\overline{\nabla}^{\perp}_{E} N^* = \left\langle \overline{\nabla}^{\perp}_{E} N^*, N^h \right\rangle N^h = -\left\langle N^*, \overline{\nabla}^{\perp}_{E} N^h \right\rangle N^h = 0$$

that is N^* is parallel in the normal bundle of TM. Thus using equation (4.1) in Lemma 3.5, it follows that ζ is a parallel vector field and consequently, it is a Killing vector field. □

Theorem 4.2. *Let $(\overline{\varphi}, \zeta, T, \alpha, \beta, \overline{g})$ be the structure on the tangent bundle TM of an orientable real hypersurface M of the Euclidean space R^{2n}. Then the vector field T is Killing if and only if the following condition holds*

$$\overline{g}\left(\left(\overline{\varphi} \circ \overline{S}_{N^h} - \overline{S}_{N^h} \circ \overline{\varphi}\right)\left(X^h\right), Y^h\right) = 0, \quad X, Y \in \mathfrak{X}(M)$$

Proof. Since N^h is parallel in the normal bundle of TM, by Lemma 3.5, we have

$$\overline{\nabla}_E T = \overline{\varphi}\left(\overline{S}_{N^h}(E)\right), \quad E \in \mathfrak{X}(TM) \tag{4.2}$$

Also using Lemma 2.4, we conclude that

$$\overline{S}_{N^h}(X^v) = 0, \overline{S}_{N^h}(X^h) = (S(X))^h, \quad X \in \mathfrak{X}(M) \tag{4.3}$$

Then using skew-symmetry of the tensor $\overline{\varphi}$, and equations (4.2) and (4.3) together with Lemma 3.3, we immediately arrive at

$$(\pounds_T \overline{g})(X^v, Y^v) = 0 \tag{4.4}$$

$$\begin{aligned} (\pounds_T \overline{g})\left(X^h, Y^v\right) &= \overline{g}\left(\overline{\varphi} \circ \overline{S}_{N^h}\left(X^h\right), Y^v\right) = -\overline{g}\left(\overline{S}_{N^h}\left(X^h\right), \overline{\varphi}\left(Y^v\right)\right) \\ &= -\overline{g}\left(\overline{S}_{N^h}\left(X^h\right), \left(\varphi\left(Y\right)\right)^v + \eta(X) \circ \pi N^v\right) = 0 \quad (4.5) \end{aligned}$$

$$(\pounds_T \overline{g})\left(X^h, Y^h\right) = \overline{g}\left(\left(\overline{\varphi} \circ \overline{S}_{N^h} - \overline{S}_{N^h} \circ \overline{\varphi}\right)\left(X^h\right), Y^h\right) \tag{4.6}$$

and the equations (4.4)-(4.6) prove the Theorem. □

Consider the unit sphere S^{2n-1} in the Euclidean space R^{2n}, whose shape operator is given by $S=-I$. Using Lemma 2.4, we get on the tangent bundle TS^{2n-1} that

$$\overline{S}_{N^h}(X^h)=\left(S(X)\right)^h=-X^h,\quad \overline{S}_{N^h}(X^v)=0$$

Then the Lemma 3.3 together with above equation, gives

$$\begin{aligned}\left(\overline{\varphi}\circ\overline{S}_{N^h}-\overline{S}_{N^h}\circ\overline{\varphi}\right)\left(X^h\right) &= -\overline{\varphi}\left(X^h\right)-\overline{S}_{N^h}\left((\varphi(X))^h-g(S(X),u)\circ\pi\xi^v\right)\\ &= -g(X,u)\circ\pi\xi^v,\quad X\in\mathfrak{X}(S^{2n-1})\end{aligned}$$

and consequently,

$$\overline{g}\left(\left(\overline{\varphi}\circ\overline{S}_{N^h}-\overline{S}_{N^h}\circ\overline{\varphi}\right)\left(X^h\right),Y^h\right)=0,\quad X,Y\in\mathfrak{X}(S^{2n-1})$$

Thus as a particular case of the Theorem 4.2, we have

Corollary 4.3. *Let $(\overline{\varphi},\zeta,T,\alpha,\beta,\overline{g})$ be the structure on the tangent bundle TS^{2n-1} of the unit sphere S^{2n-1} in the Euclidean space R^{2n}, $n>1$. Then the vector field T is a nontrivial Killing vector field.*

Proof. It remains to be shown that T is nontrivial. Since, N^h is parallel in the normal bundle of TS^{2n-1}, by Lemmas 2.4 and 3.5, we have

$$\overline{\nabla}_{X^h}T=-\overline{\varphi}\left(X^h\right),\quad X\in\mathfrak{X}(S^{2n-1}) \tag{4.7}$$

where we used the fact that the shape operator S of the unit sphere S^{2n-1} is given by $S=-I$. The Lemma 3.4 gives the rank of $\overline{\varphi}$ is $4(n-1)$ and consequently, equation (4.7) gives that the Killing vector field T is not parallel, that is T is a nontrivial Killing vector field. □

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