Tangent Bundle of the Hypersurfaces in a Euclidean Space

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Abstract. Let $M$ be an orientable hypersurface in the Euclidean space $\mathbb{R}^{2n}$ with induced metric $g$ and $TM$ be its tangent bundle. It is known that the tangent bundle $TM$ has induced metric $\overline{g}$ as submanifold of the Euclidean space $\mathbb{R}^{4n}$ which is not a natural metric in the sense that the submersion $\pi : (TM, \overline{g}) \rightarrow (M, g)$ is not the Riemannian submersion. In this paper, we use the fact that $\mathbb{R}^{4n}$ is the tangent bundle of the Euclidean space $\mathbb{R}^{2n}$ to define a special complex structure $\mathcal{J}$ on the tangent bundle $\mathbb{R}^{4n}$ so that $(\mathbb{R}^{4n}, \mathcal{J}, \langle \cdot, \cdot \rangle)$ is a Kaehler manifold, where $\langle \cdot, \cdot \rangle$ is the Euclidean metric which is also the Sasaki metric of the tangent bundle $\mathbb{R}^{4n}$. We study the structure induced on the tangent bundle $(TM, \overline{g})$ of the hypersurface $M$, which is a submanifold of the Kaehler manifold $(\mathbb{R}^{4n}, \mathcal{J}, \langle \cdot, \cdot \rangle)$. We show that the tangent bundle $TM$ is a CR-submanifold of the Kaehler manifold $(\mathbb{R}^{4n}, \mathcal{J}, \langle \cdot, \cdot \rangle)$. We find conditions under which certain special vector fields on the tangent bundle $(TM, \overline{g})$ are Killing vector fields. It is also shown that the tangent bundle $TS^{2n-1}$ of the unit sphere $S^{2n-1}$ admits a Riemannian metric $\overline{g}$ and that there exists a nontrivial Killing vector field on the tangent bundle $(TS^{2n-1}, \overline{g})$.

Keywords: Tangent bundle, Hypersurface, Kaehler manifold, Almost contact structure, Killing vector field, CR-Submanifold, Second fundamental form, Wiegarten map.


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Received 29 May 2013; Accepted 15 November 2015
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1. Introduction

Recently efforts are made to study the geometry of the tangent bundle of a hypersurface $M$ in the Euclidean space $\mathbb{R}^{n+1}$ (cf. [3]), where the authors have shown that the induced metric on its tangent bundle $TM$ as submanifold of the Euclidean space $\mathbb{R}^{2n+2}$ is not a natural metric. In [4], we have extended the study initiated in [3] on the geometry of the tangent bundle $TM$ of an immersed orientable hypersurface $M$ in the Euclidean space $\mathbb{R}^{n+1}$. It is well known that Killing vector fields play an important role in shaping the geometry of a Riemannian manifold, for instance the presence of nonzero Killing vector field on a compact Riemannian manifold forces its Ricci curvature to be non-negative and this in particular implies that on a compact Riemannian manifolds of negative Ricci curvature there does not exist a nonzero Killing vector field.

The study of Killing vector fields becomes more interesting on the tangent bundle $TM$ of a Riemannian manifold $(M, g)$ as the tangent bundle $TM$ is noncompact. It is known that if the tangent bundle $TM$ of a Riemannian manifold $(M, g)$ is equipped with Sasaki metric, then the verticle lift of a parallel vector field on $M$ is a Killing vector field (cf. [15]). However if the Sasaki metric is replaced by the Cheeger-Gromoll metric, then the vertical lift of any nonzero vector field on $M$ is never Killing (cf. [14]). Note that both Sasaki metric as well as Cheeger-Gromoll metrics are natural metrics. We consider an orientable real hypersurface $M$ of the Euclidean space $\mathbb{R}^{2n}$ with the induced metric $g$. Then as the tangent bundle $TM$ of $M$ is a submanifold of codimension two in $\mathbb{R}^{4n}$, it has induced metric $\overline{g}$ and this metric $\overline{g}$ on $TM$ is not a natural metric as the submersion $\pi: (TM, \overline{g}) \to (M, g)$ is not the Riemannian submersion (cf. [3]). Let $N$ be the unit normal vector field to the hypersurface $M$ and $J$ be the natural complex structure on the Euclidean space $\mathbb{R}^{4n}$. Then we have a globally defined unit vector field $\xi$ on the hypersurface given by $\xi = -JN$ called the characteristic vector field of the real hypersurface (cf. [1, 2, 5, 6, 7, 8, 9]), and this vector field $\xi$ gives rise to two vector fields $\xi^h$ (the horizontal lift) and $\xi^v$ (the vertical lift) on the tangent bundle $(TM, \overline{g})$. In this paper, we use the fact that $T^{4n}$ is the tangent bundle of the Euclidean space $\mathbb{R}^{2n}$ and that the projection $\pi: T^{4n} \to R^{2n}$ is a Riemannian submersion, to define a special almost complex structure $J$ on the tangent bundle $T^{4n}$ which is different from the canonical complex structure of the Euclidean space $\mathbb{R}^{4n}$ and show that $(T^{4n}, J, \langle \cdot, \cdot \rangle)$ is a Kaehler manifold, where $\langle \cdot, \cdot \rangle$ is the Euclidean metric on $\mathbb{R}^{4n}$. It is shown that the codimension two submanifold $(TM, \overline{g})$ of the Kaehler manifold $(T^{4n}, J, \langle \cdot, \cdot \rangle)$ is a CR-submanifold (cf. [10]) and it naturally inherits certain special vector fields other than $\xi^h$ and $\xi^v$, and in this paper we are interested in finding conditions under which these special vector fields are Killing vector fields on $(TM, \overline{g})$. One of the interesting outcome of this study is, we have shown that the tangent bundle $TS^{2n-1}$ of the unit sphere $S^{2n-1}$ as
submanifold of \( R^{4n} \) admits a nontrivial Killing vector field. It is worth pointing out that on the tangent bundle \( TS^{2n-1} \) with Sasakian metric no vertical or horizontal lift of a vector field is Killing as this will require the corresponding vector field on \( S^{2n-1} \) is parallel which is impossible as \( S^{2n-1} \) is space of constant curvature 1. Note that on even dimensional Riemannian manifolds which are irreducible, it is difficult to find Killing vector fields, where as on products like \( S^{2k-1} \times S^{2l-1}, S^{2k-1} \times R^{2l-1}, R^{2k-1} \times R^{2l-1} \) one can easily find Killing vector fields. Since the tangent bundle \( TS^{2n-1} \) is trivial for \( n = 1, 2, 4 \), finding Killing vector fields is easy in these dimensions, but for \( n \geq 5 \), it is not trivial.

2. Preliminaries

Let \((M,g)\) be a Riemannian manifold and \( TM \) be its tangent bundle with projection map \( \pi : TM \rightarrow M \). Then for each \((p,u) \in TM\), the tangent space \( T_{(p,u)}TM = \mathcal{H}_{(p,u)} \oplus \mathcal{V}_{(p,u)} \), where \( \mathcal{H}_{(p,u)} \) is the kernel of \( d\pi_{(p,u)} : T_{(p,u)}(TM) \rightarrow T_pM \) and \( \mathcal{V}_{(p,u)} \) is the kernel of the connection map \( K_{(p,u)} : T_{(p,u)}(TM) \rightarrow T_pM \) with respect to the Riemannian connection on \((M,g)\). The subspaces \( \mathcal{H}_{(p,u)}, \mathcal{V}_{(p,u)} \) are called the horizontal and vertical subspaces respectively. Consequently, the Lie algebra of smooth vector fields \( \mathfrak{X}(TM) \) on the tangent bundle \( TM \) admits the decomposition \( \mathfrak{X}(TM) = \mathcal{H} \oplus \mathcal{V} \) where \( \mathcal{H} \) is called the horizontal distribution and \( \mathcal{V} \) is called the vertical distribution on the tangent bundle \( TM \). For each \( X_p \in T_pM \), the horizontal lift of \( X_p \) to a point \( z = (p,u) \in TM \) is the unique vector \( X^h_z \in \mathcal{H}_z \) such that \( d\pi(X^h_z) = X_p \circ \pi \) and the vertical lift of \( X_p \) to a point \( z = (p,u) \in TM \) is the unique vector \( X^v_z \in \mathcal{V}_z \) such that \( X^v_z(df) = X_p(f) \) for all functions \( f \in C^\infty(M) \), where \( df \) is the function defined by \( (df)(p,u) = u(f) \). Also for a vector field \( X \in \mathfrak{X}(M) \), the horizontal lift of \( X \) is a vector field \( X^h \in \mathfrak{X}(TM) \) whose value at a point \((p,u)\) is the horizontal lift of \( X(p) \) to \((p,u)\), the vertical lift \( X^v \) of \( X \) is defined similarly. For \( X \in \mathfrak{X}(M) \) the horizontal and vertical lifts \( X^h, X^v \) of \( X \) are uniquely determined vector fields on \( TM \) satisfying

\[
d\pi(X^h_z) = X_{\pi(z)}, K(X^h_z) = 0, d\pi(X^v_z) = 0, K(X^v_z) = X_{\pi(z)}
\]

Also, we have for a smooth function \( f \in C^\infty(M) \) and vector fields \( X,Y \in \mathfrak{X}(M) \), that \((fX)^h = (f \circ \pi)X^h, (fX)^v = (f \circ \pi)X^v, (X + Y)^h = X^h + Y^h \) and \((X + Y)^v = X^v + Y^v \). If \( \dim M = m \) and \((U,\varphi)\) is a chart on \( M \) with local coordinates \( x^1, x^2, \ldots, x^m \), then \((\pi^{-1}(U), \varphi)\) is a chart on \( TM \) with local coordinates \( x^1, x^2, \ldots, x^m, y^1, y^2, \ldots, y^m \), where \( x^i = x^i \circ \pi \) and \( y^i = dx^i \), \( i = 1, 2, \ldots, m \).

A Riemannian metric \( g \) on the tangent bundle \( TM \) is said to be natural metric with respect to \( g \) on \( M \) if \( \overline{g}_{(p,u)}(X^h, Y^h) = g_p(X, Y) \) and \( \overline{g}_{(p,u)}(X^h, Y^v) = 0 \), for all vector fields \( X,Y \in \mathfrak{X}(M) \) and \((p,u) \in TM \), that is the projection map \( \pi : TM \rightarrow M \) is a Riemannian submersion.
Let $M$ be an orientable hypersurface of the Euclidean space $\mathbb{R}^{2n}$ with immersion $f : M \rightarrow R^{2n}$ and $TM$ be its tangent bundle. Then as $F = df : TM \rightarrow R^{4n} = TR^{2n}$ is also an immersion, $TM$ is an immersed submanifold of the Euclidean space $\mathbb{R}^{4n}$. We denote the induced metrics on $M, TM$ by $g, \overline{g}$ respectively and the Euclidean metric on $R^{2n}$ as well as on $R^{4n}$ by $\langle \cdot, \cdot \rangle$. Also, we denote by $\nabla, \overline{\nabla}, D$ and $\overline{D}$ the Riemannian connections on $M, TM, R^{2n}$, and $R^{4n}$ respectively. Let $N$ and $S$ be the unit normal vector field and the shape operator of the hypersurface $M$. For the hypersurface $M$ of the Euclidean space $R^{2n}$ we have the following Gauss and Weingarten formulae
\[
D_X Y = \nabla_X Y + \langle S(X), Y \rangle N, \quad D_X N = -S(X), \quad X, Y \in \mathfrak{X}(M) \tag{2.1}
\]
where $S$ is the shape operator (Weingarten map). Similarly for the submanifold $TM$ of the Euclidean space $R^{4n}$ we have the Gauss and Weingarten formulae
\[
\overline{D}_E F = \overline{\nabla}_E F + h(E, F), \quad \overline{D}_E \overline{N} = -\overline{S}_N(E) + \overline{\nabla}_E^\perp \overline{N} \tag{2.2}
\]
where $E, F \in \mathfrak{X}(TM)$, $\overline{\nabla}^\perp$ is the connection in the normal bundle of $TM$ and $\overline{S}_N$ denotes the Weingarten map in the direction of the normal $\overline{N}$ and is related to the second fundamental form $h$ by
\[
\langle h(X, Y), N \rangle = \overline{g}(\overline{S}_N(X), Y) \tag{2.3}
\]
Also we observe that for $X \in \mathfrak{X}(M)$ the vertical lift $X^v$ of $X$ to $TM$, as $X^v \in \ker d\pi$, where $\pi : TM \rightarrow M$ is the natural submersion, we have $d\pi(X^v) = 0$ that is $df( d\pi(X^v)) = 0$ or equivalently we get $d(f \circ \pi)(X^v) = 0$, that is $d(\overline{\pi} \circ F)(X^v) = 0$ ($\pi : TR^{2n} \rightarrow R^{2n}$), which gives $dF(X^v) \in \ker d\overline{\pi} = \mathfrak{X}$.

Now we state the following results which are needed in our work.

**Lemma 2.1.** [3] Let $N$ be the unit normal vector field to the hypersurface $M$ of $R^{2n}$ and $P = (p, X_p) \in TM$. Then the horizontal and vertical lifts $Y_P^h, Y_P^v$ of $Y_p \in T_p M$ satisfy
\[
dF_P(Y_P^h) = (df_P(Y_p))^h + V_P, \quad dF_P(Y_P^v) = (df_P(Y_p))^v
\]
where $V_P \in \mathfrak{X}_P$ is given by $V_P = \langle S_p(X_p), Y_p \rangle N_p^v$, $N_p^v$ being the vertical lift of the unit normal $N$ to with respect to the tangent bundle $\pi : R^{4n} \rightarrow R^{2n}$.

**Lemma 2.2.** [3] If $(M, g)$ is an orientable hypersurface of $R^{2n}$, and $(TM, \overline{g})$ is its tangent bundle as submanifold of $R^{4n}$, then the metric $\overline{g}$ on $TM$ for $P = (p, u) \in TM$, satisfies:
\[
\begin{align*}
(i) \quad & \overline{g}_P(X_p^h, Y_p^h) = g_p(X_p, Y_p) + g_p(S_p(X_p), u)g_p(S_p(Y_p), u). \\
(ii) \quad & \overline{g}_P(X_p^h, Y_p^v) = 0. \\
(iii) \quad & \overline{g}(X^v, Y^v) = g_p(X_p, Y_p).
\end{align*}
\]

**Remark 2.3.** It is well known that a metric $\overline{g}$ defined on $TM$ using the Riemannian metric $g$ of $M$ (such as Sasaki metric, Cheeger-Gromoll metric) are
natural metrics in the sense that the submersion $\pi : (TM, \mathcal{F}) \rightarrow (M, g)$ becomes a Riemannian submersion with respect to these metrics. However, as seen from above Lemmas, the induced metric on the tangent bundle $TM$ of a hypersurface $M$ of the Euclidean space $R^{2n}$, as a submanifold of $R^{4n}$ is not a natural metric because of the present of the term $g_p(S_p(X_p), u)g_p(S_p(Y_p), u)$ in the inner product of horizontal vectors on $TM$. Moreover, note that for an orientable hypersurface $M$ of the Euclidean space $R^{2n}$, the vertical lift $N^v$ of the unit normal is tangential to the submanifold $TM$ of $R^{4n}$ as seen in 2.1.

In what follows, we drop the suffixes like in $g_p(S_p(X_p), u)$ and and it will be understood from the context of the entities appearing in the equations.

**Theorem 2.4.** [3] Let $(M, g)$ be an orientable hypersurface of $R^{2n}$, and $(TM, \mathcal{F})$ be its tangent bundle as submanifold of $R^{4n}$. If $\nabla$ and $\nabla$ denote the Riemannian connections on $(M, g)$ and $(TM, \mathcal{F})$ respectively, then

(i) $\nabla_X Y^h = (\nabla_X Y)^h - \frac{1}{2}(R(X, Y)u)^v$,

(ii) $\nabla_X Y^v = g(S(X), Y) \circ \pi N^v$,

(iii) $\nabla_X Y^v = 0$, (iv) $\nabla_X Y^v = (\nabla_X Y)^v + g(S(X), Y) \circ \pi N^v$.

**Lemma 2.5.** [4] Let $TM$ be the tangent bundle of an orientable hypersurface $M$ of $R^{2n}$. Then for $X, Y \in \mathfrak{X}(M)$,

(i) $h(X^v, Y^v) = 0$,

(ii) $h(X^v, Y^h) = 0$,

(iii) $h(X^h, Y^h) = g(S(X), Y) \circ \pi N^h$.

**Lemma 2.6.** [4] For the tangent bundle $TM$ of an orientable hypersurface $M$ of $R^{2n}$ and $X \in \mathfrak{X}(M)$, we have

(i) $\mathcal{D}_X N^v = 0$,

(ii) $\mathcal{D}_X N^h = 0$,

(iii) $\mathcal{D}_X N^v = -(S(X))^v$, (iv) $\mathcal{D}_X N^h = -(S(X))^h$.

Let $J$ be the natural complex structure on the Euclidean space $R^{2n}$, which makes $(R^{2n}, J, \langle , \rangle)$ a Kaeckler manifold. Then on an orientable real hypersurface $M$ of $R^{2n}$ with unit normal $N$, we define a unit vector field $\xi \in \mathfrak{X}(M)$ by $\xi = -JN$, with its dual 1-form $\eta(X) = g(X, \xi)$, where $g$ is the induced metric on $M$. For $X \in \mathfrak{X}(M)$, we express $JX = \varphi(X) + \eta(X)N$, where $\varphi(X)$ is the tangential component of $JX$, and it follows that $\varphi$ is a $(1, 1)$ tensor field on $M$, and that $(\varphi, \xi, \eta, g)$ defines an almost contact metric structure on $M$ (cf. [5], [8], [9]), that is

$$\varphi^2 X = -X + \eta(X)\xi, \eta(\xi) = 1, \eta \circ \varphi = 0, \varphi(\xi) = 0$$

and

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),\ X, Y \in \mathfrak{X}(M)$$

Moreover, we have the following.
Lemma 2.7. [8] Let $M$ be an orientable real hypersurface of $R^{2n}$. Then the structure $(\varphi, \xi, \eta, g)$ on $M$ satisfies

(i) $(\nabla_X \varphi)(Y) = \eta(Y)SX - g(SX, Y)\xi,$
(ii) $\nabla_X \xi = \varphi SX, \; X, Y \in \mathfrak{X}(M).$

3. A Structure on $(TM, \widetilde{g})$

We know that the Euclidean space $R^{4n}$ has many complex structures, however in this section we treat $R^{4n}$ as the tangent bundle of $R^{2n}$ and consider a specific complex structure on the Euclidean space $R^{4n}$. Let $\pi : R^{4n} = TR^{2n} \to R^{2n}$ be the submersion of the tangent bundle of $R^{2n}$. Then it is easy to show that the Euclidean metric $\langle \cdot, \cdot \rangle$ on the tangent bundle $R^{4n}$ is Sasaki metric and using the canonical almost complex structure $J$ of $R^{2n}$, we define $J : \mathfrak{X}(R^{4n}) \to \mathfrak{X}(R^{4n})$ by

$$J(E^h) = (JE)^h, \quad J(E^v) = (JE)^v, \quad E \in \mathfrak{X}(R^{2n})$$

and it is easily follows that $J$ is an almost complex structure, satisfying $\langle JE, JF \rangle = \langle E, F \rangle$ with respect to the Euclidean metric $\langle \cdot, \cdot \rangle$ on $R^{4n}$ and that $(\overline{D}_E J)(F) = 0, E, F \in \mathfrak{X}(R^{4n})$ that is $(R^{4n}, J, \langle \cdot, \cdot \rangle)$ is a Kaehler manifold. Regarding the complex structure $\overline{J}$ defined above, we have the following

Lemma 3.1. Let $\pi : R^{4n} \to R^{2n}$ be the submersion of the tangent bundle $R^{4n} = TR^{2n}$. Then complex structure $\overline{J}$ on $R^{4n}$ satisfies

$$J \circ d\pi = d\pi \circ \overline{J}$$

Proof. Take $X \in \mathfrak{X}(R^{2n})$, then for the horizontal lift $X^h$, we have:

$$J \circ d\pi(X^h) = J(d\pi(X^h)) = JX \circ \pi$$

and

$$d\pi \circ J(X^h) = d\pi(JX)^h = JX \circ \pi$$

which proves

$$J \circ d\pi(X^h) = d\pi \circ J(X^h)$$

Similarly for the vertical lift $X^v$ we have

$$J \circ d\pi(X^v) = J(d\pi(X^v)) = 0$$

and

$$d\pi \circ J(X^v) = d\pi(JX)^v = 0$$

This proves the Lemma.

Remark 3.2. If $M$ is an orientable real hypersurface of the Euclidean space $R^{2n}$ with immersion $f$, then $F = df$ is the immersion of the tangent bundle $TM$ into the Euclidean space $R^{4n}$ and as immersions are local embeddings, in general, we identify the local quantities on submanifold with those of the ambient space for instance we identify $df(X)$ with $X$ for $X \in \mathfrak{X}(M)$. However,
while dealing with the immersion $F$ of $TM$ in $R^{4n}$ one need to be cautious specially while dealing with the horizontal lifts (cf. 2.1). Therefore in what follows, we shall bring $dF$ in to play whenever it is needed specially in the case of horizontal lifts.

Observe that if $M$ is an orientable real hypersurface of the Euclidean space $R^{2n}$ with unit normal vector field $N$, then we know that horizontal lift $N^h$ is a unit normal vector field to the submanifold $TM$ of $R^{4n}$ and that the vertical lift $N^v \in \mathfrak{X}(TM)$ (cf.[1]). We have

\[ JN^h = (JN)^h = -(df(\xi))^h = -dF(\xi^h) + g(S(\xi), u)N^v \in \mathfrak{X}(TM) \]  

(3.1)

and

\[ JN^v = (JN)^v = -\xi^v \in \mathfrak{X}(TM) \]  

(3.2)

Let $M$ be an orientable real hypersurface of the Kaehler manifold $(R^{2n}, J, \langle \cdot , \cdot \rangle )$. Then as $TM$ is submanifold of the Kaehler manifold $(R^{4n}, \bar{J}, \langle \cdot , \cdot \rangle )$, we denote by $\Gamma(T^\perp TM)$ the space of smooth normal vector fields to $TM$. The restriction of the complex structure $\bar{J}$ on $R^{4n}$ to $\mathfrak{X}(TM)$ and $\Gamma(T^\perp TM)$ can be expressed as

\[ \bar{J}(E) = \varphi(E) + \bar{\psi}(E), \quad \bar{J}(N) = \bar{G}(N) + \bar{\chi}(N), \quad E \in \mathfrak{X}(TM), \quad N \in \Gamma(T^\perp TM) \]

where $\varphi(E), \bar{G}(N)$ are the tangential and $\bar{\psi}(E), \bar{\chi}(N)$ are the normal components of $\bar{J}E$, and $\bar{J}(\bar{N})$ respectively. Note that the horizontal lift $N^h$ of the unit normal $N$ to the hypersurface $M$ is normal to $TM$ that is $N^h \in \Gamma(T^\perp TM)$, where as the vertical lift $N^v \in \mathfrak{X}(TM)$.

**Lemma 3.3.** Let $TM$ be the tangent bundle of an orientable real hypersurface of $R^{2n}$. Then for $X \in \mathfrak{X}(M)$,

\[ \varphi(X^h) = (\varphi(X))^h - g(S(X), u)\xi^v, \quad \varphi(X^v) = (\varphi(X))^v + \eta(X) \circ \pi N^v \]
\[ \bar{\psi}(X^h) = \eta(X) \circ \pi N^h, \quad \bar{\psi}(X^v) = 0 \]

**Proof.** Note that for the horizontal lift $X^h$ we have

\[ JX^h = JdF(X^h) = J((df(X))^h + g(SX, u) \circ \pi N^v) \]
\[ = (JdF(X))^h + g(SX, u) \circ (JN)^v \]
\[ = (\varphi X + \eta(X)N)^h - g(SX, u) \circ \pi \xi^v \]
\[ = (\varphi(X))^h - g(SX, u) \circ \pi \xi^v + \eta(X) \circ \pi N^h \]

which together with the definition $JX^h = \varphi(X^h) + \bar{\psi}(X^h)$, on equating tangential and normal components give

\[ \varphi(X^h) = (\varphi(X))^h - g(S(X), u)\xi^v \quad \text{and} \quad \bar{\psi}(X^h) = \eta(X) \circ \pi N^h \]

Similarly for the vertical lift $X^v$, we have

\[ JX^v = \varphi(X^v) + \bar{\psi}(X^v) = (JX)^v = (\varphi X + \eta(X)N)^v \]
which gives

\[(\varphi(X^v)) + \psi(X^v) = (\varphi X)^v + \eta(X) \circ \pi N^v\]

Comparing the tangential and normal components we conclude

\[\varphi(X^v) = (\varphi(X))^v + \eta(X) \circ \pi N^v, \quad \text{and} \quad \psi(X^v) = 0.\]

\[\square\]

We choose a unit normal vector field \(N^* \in \Gamma(T^*T^*M)\) such that \(\{N^*, N^h\}\) is a local orthonormal frame of normals for the submanifold \(TM\). It is known that \(N^*\) is vertical vector field on the tangent bundle \(R^4\) (cf. [1]). Since,

\[\langle JN^*, N^* \rangle = 0, \quad \langle JN^*, N^h \rangle = \langle N^*, \xi^h \rangle = 0,\]

it follows that \(JN^* \in \mathfrak{X}(TM)\) and we define unit vector field \(\zeta \in \mathfrak{X}(TM)\) by

\[\zeta = -JN^* \quad (3.3)\]

Now, for any normal vector field \(N \in \Gamma(T^*T^*M)\), we have

\[N = \langle N, N^* \rangle N^* + \langle N, N^h \rangle N^h\]

which together with equations (3.1), (3.2) and (3.3) gives \(\chi(N) = 0\) and that \(\mathcal{J}(N) \in \mathfrak{X}(TM)\), is given by

\[\mathcal{J}(N) = \langle \mathcal{J}(N), \zeta \rangle \zeta + \langle \mathcal{J}(N), T \rangle T \quad (3.4)\]

where \(T \in \mathfrak{X}(TM)\), is given by

\[T = \xi^h - g(S(\xi), u)N^v = -JN^h \quad (3.5)\]

Also, using equation (3.2), we have

\[\xi^v = \mathcal{J}N^v = \varphi(N^v) + \psi(N^v)\]

which gives

\[\varphi(N^v) = -\xi^v \quad \text{and} \quad \psi(N^v) = 0 \quad (3.6)\]

Moreover, we have

\[\varphi(\zeta) = 0 \quad \text{and} \quad \psi(\zeta) = N^*, \quad \psi(\xi^h) = N^h \quad (3.7)\]

If we denote by \(\alpha, \beta\) the smooth 1-forms on \(TM\) dual to the vector field \(\zeta\) and \(T\) respectively, then for \(E \in \mathfrak{X}(TM)\), it follows that

\[\mathcal{J}(\psi(E)) = -\alpha(E)\zeta - \beta(E)T\]

and consequently, operating \(\mathcal{J}\) on \(\mathcal{J}(E) = \varphi(E) + \psi(E), \ E \in \mathfrak{X}(TM)\), we get

\[\varphi^2 = -I + \alpha \otimes \zeta + \beta \otimes T \quad \text{and} \quad \psi \circ \varphi = 0 \quad (3.8)\]

Using Lemma 2.1 and equations (3.3), (3.5), (3.6), (3.8), we see that the vector fields \(\zeta, T\) and 1-forms \(\alpha, \beta\) satisfy

\[\varphi(\zeta) = 0, \ \varphi(T) = 0, \ \varphi(\zeta, T) = 0, \ \alpha \circ \varphi = 0, \ \beta \circ \varphi = 0 \quad (3.9)\]
Also, as $\overline{g}$ is the induced metric on the submanifold $TM$ and $\mathcal{J}$ is skew-symmetric with respect to the Hermitian metric $\langle \cdot, \cdot \rangle$, we have

$$\overline{g}(\overline{\varphi}(E), F) = -\overline{g}(E, \overline{\varphi}(F)), \quad E, F \in \mathfrak{X}(TM) \quad (3.10)$$

Then using equations (3.8), (3.9) and (3.10), we have

$$\overline{g}(\overline{\varphi}(E), \overline{\varphi}(F)) = \overline{g}(E, F) - \alpha(E)\alpha(F) - \beta(E)\beta(F), \quad E, F \in \mathfrak{X}(TM) \quad (3.11)$$

Thus we have proved the following

**Lemma 3.4.** Let $TM$ be the tangent bundle of an orientable real hypersurface of $R^{2n}$. Then there is a structure $(\overline{\varphi}, \zeta, T, \alpha, \beta, \overline{g})$ similar to contact metric structure on $TM$, where $\overline{\varphi}$ is a tensor field of type $(1, 1)$, $\zeta, T$ are smooth vector fields and $\alpha, \beta$ are smooth 1-forms dual to $\zeta, T$ with respect to the Riemannian metric $\overline{g}$ satisfying

$$\overline{g}^2 = -I + \alpha \otimes \zeta + \beta \otimes T, \quad \overline{\varphi}(\zeta) = 0, \quad \overline{\varphi}(T) = 0, \quad \alpha \circ \overline{\varphi} = 0, \quad \beta \circ \overline{\varphi} = 0, \quad \overline{g}(\zeta, T) = 0$$

In the next Lemma, we compute the co-variant derivatives of the tensor $\overline{\varphi}$.

**Lemma 3.5.** Let $(\overline{\varphi}, \zeta, T, \alpha, \beta, \overline{g})$ be the structure on the tangent bundle $TM$ of an orientable real hypersurface $M$ of the Euclidean space $R^{2n}$. Then

(i) $$(\nabla_X \overline{\varphi})(Y^h) = \left\{ (\nabla_X \overline{\varphi})(Y) \right\}^h - \{ X (g(SY, u) + g(SY, u)JSX) \}^\nu$$

(ii) $$(\nabla_X \overline{\varphi})(Y^\nu) = 0$$

(iii) $$(\nabla_X \overline{\varphi})(Y^h) = 0, \quad (\nabla_X \overline{\varphi})(Y^h) = g(SX, \varphi Y) \circ \pi N^\nu + g(SX, Y) \circ \pi \xi^\nu.$$  

**Proof.** Using the definition of $\mathcal{J}$, Lemma 2.1 and Lemma 3.3 together with equation (3.1), we get for $X, Y \in \mathfrak{X}(M)$

$$\mathcal{J}Y^h = \mathcal{J}dF(Y^h) = \mathcal{J} \left( (df(Y))^h + g(SX, u) \circ \pi N^\nu \right)$$

$$= (\varphi Y + \eta(Y))^h - g(SY, u) \circ \pi \xi^\nu$$

$$= \overline{\varphi}(Y^h) + \eta(Y) \circ \pi N^h$$

which gives

$$\overline{\nabla}_X \mathcal{J}Y^h = \overline{\nabla}_X (df(X))^h + g(SX, u) \circ \pi N^\nu \left( \overline{\varphi}(Y^h) + \eta(Y) \circ \pi N^h \right)$$

$$= \overline{\nabla}_X (df(X))^h \overline{\varphi}(Y^h) + X(\eta(Y)) \circ \pi N^h + \eta(Y) \circ \pi \overline{\nabla}_X (df(X))^h N^h$$

$$+ g(SX, u) \circ \pi \eta(Y) \circ \pi \overline{\nabla}_X (df(X))^h N^h$$

Note that the tangent bundle $\mathcal{T}R^{2n} = R^{4n}$ has Sasaki metric and thus using Lemma 7.2 of [10] (keeping in view that $R^{2n}$ is flat), in the above equation, we get

$$\overline{\nabla}_X \mathcal{J}Y^h = \nabla_X \overline{\varphi}(Y^h) + h(X^h, \overline{\varphi}(Y^h) + X(\eta(Y)) \circ \pi N^h - \eta(Y) \circ \pi (SX)^h$$

(3.12)
Similarly we have
\[
\mathcal{J}\mathcal{D}_{X^h}Y^h = \mathcal{J}\left(\mathcal{D}_{(df(X)^h + g(SX,u)\circ\pi N^v)} \left((df(Y)^h + g(SY,u) \circ \pi N^v)\right)\right)
\]
\[
= \mathcal{J}\left\{\nabla_{X^h}Y^h + h(X^h, Y^h) + X(g(SY,u) \circ \pi N^v + g(SY,u) \circ \pi (DXN)^v)
\right. \\
+ g(SX,u) \circ \pi\mathcal{D}_{N^v}(dfY)^h + 0 + 0\left.\right\}
\]
\[
= \varphi\left(\nabla_{X^h}Y^h + \overline{\varphi}\left(\nabla_{X^h}Y^h\right) + \mathcal{J}h(X^h, Y^h) - X(g(SY,u) \circ \pi \xi^v)
\right. \\
- g(SY,u) \circ \pi J(SX)^v \\
\left.\right\}
\]
\[
= \varphi\left(\nabla_{X^h}Y^h + \overline{\varphi}\left(\nabla_{X^h}Y^h\right) - g(SX,Y) \circ \pi \xi^h - X(g(SY,u) \circ \pi \xi^v)
\right. \\
- g(SY,u) \circ \pi (\varphi SX)^v - g(SY,u) \circ \pi \eta(SX)^N\right) \tag{3.13}
\]

where we used Lemmas 2.3, 2.4 and Lemma 7.2 in [10]. Now as \((R^{4n},\mathcal{J},\langle,\rangle)\) is a Kaehler manifold, the equations (3.12) and (3.13) on comparing tangential components, we immediately arrive at
\[
(\nabla_{X}^h\varphi)(Y^h) = \{((\nabla_{X}^h\varphi)(Y))\}^h - \{X(g(SY,u) + g(SY,u)JSX)\}^v
\]
which proves (i).

Now, using \(h(X^v, Y^v) = 0\) and \(\overline{S}_{\varphi(Y^v),X^v} = 0\) together with \(\mathcal{D}_{X^v}\overline{J}Y^v = \overline{J}\mathcal{D}_{X^v}Y^v\), and comparing tangential components, we immediately arrive at
\[
(\nabla_{X^v}^h\varphi)(Y^v) = 0
\]

Next, we have \(\nabla_{X^v}^h\varphi(Y^v) = \overline{\nabla}_{X^v}^h\left((\varphi Y)^h - g(SX,u) \circ \pi N^v\right) = \nabla_{X^v}^h(\varphi Y)^h = g(SX, \varphi Y) \circ \pi N^v\) and \(\varphi(\nabla_{X^v}^hY^h) = g(SX, \varphi Y) \circ \pi \varphi(N^v) = -g(SX,Y) \circ \pi \xi^v\).
Thus, we get
\[
(\nabla_{X^v}^h\varphi)(Y^v) = g(SX, \varphi Y) \circ \pi N^v + g(SX,Y) \circ \pi \xi^v
\]

Finally, using \(h(X^h, Y^v) = 0\) and \(\overline{S}_{\varphi(Y^v),X^h} = 0\) together with \(\mathcal{D}_{X^h}\overline{J}Y^v = \overline{J}\mathcal{D}_{X^h}Y^v\), and comparing tangential components, we immediately arrive at
\[
(\nabla_{X^h}^h\varphi)(Y^v) = 0
\]

\[\square\]

**Lemma 3.6.** Let \((\varphi, \xi, T, \alpha, \beta, g)\) be the structure on the tangent bundle \(TM\) of an orientable real hypersurface \(M\) of the Euclidean space \(R^{2n}\). Then for \(E \in \mathfrak{X}(TM)\),
\[
\nabla_{E\xi}^h = \varphi(SN^\ast(E)) - J\left(\nabla_{E}^hN^\ast\right), \quad h(E, \xi) = \overline{\varphi}(SN^\ast(E))
\]
\[
\nabla_{ET} = \varphi(SN^\ast(E)) - J\left(\nabla_{E}^hN^\ast\right), \quad h(E, T) = \overline{\psi}(SN^\ast(E))
\]
Proof. Using equation (2.2), we have

\[ \nabla_E \zeta = D_E \zeta - h(E, \zeta) = -\mathcal{J} D_E N^* - h(E, \zeta) = \mathcal{J} (SN^*(E)) - \mathcal{J} (\nabla_E N^*) - h(E, \zeta) \]

Since \( \mathcal{J}(N) \in \mathcal{X}(TM) \) for each normal \( N \in \Gamma(T^\perp TM) \), equation tangential and normal components in above equation, we get the first part. The second part follows similarly using \( T = -\mathcal{J}N^h \).

Now, we prove the following:

**Theorem 3.7.** The tangent bundle \( TM \) of an orientable real hypersurface \( M \) of the Euclidean space \( \mathbb{R}^{2n} \) is a CR-submanifold of the Kaehler manifold \( (\mathbb{R}^{4n}, \mathcal{J}, \langle \cdot, \cdot \rangle) \).

Proof. Use the structure \((\varphi, \zeta, T, \alpha, \beta, g)\) on the submanifold \( TM \) of \( \mathbb{R}^{4n} \) to define the distribution \( D \) by

\[ D = \{ E \in \mathcal{X}(TM) : \alpha(E) = \beta(E) = 0 \} \]

and \( D^\perp \) be the distribution spanned by the orthogonal vector fields \( \zeta \) and \( T \). Note that \( \zeta \) is unit vector field on \( TM \) and the length of the vector field \( T \) satisfies

\[ ||T||^2 = 1 + 2g(S(\xi), u)^2 \geq 1 \]

which shows that \( D^\perp \) is 2-dimensional distribution on \( TM \) and that \( \mathcal{J}D^\perp = \Gamma(T^\perp TM) \). It is easy to see that \( D \) and \( D^\perp \) are orthogonal complementary distributions and that \( \text{dim} D = 4(n - 1) \). Note that for \( E \in \mathcal{X}(TM) \), we have

\[ \varphi(E) = \langle \varphi(E), N^* \rangle N^* + \langle \varphi(E), N^h \rangle N^h = \alpha(E) N^* + \beta(E) N^h \]

and consequently if \( E \in D \), then above equation gives \( \mathcal{J}E = \varphi E \) which is orthogonal to both \( \zeta \) and \( T \) and that \( \mathcal{J}E \in D \), which implies \( \mathcal{J}D = D \). This proves that \( TM \) is a CR-submanifold of the Kaehler manifold \( (\mathbb{R}^{4n}, \mathcal{J}, \langle \cdot, \cdot \rangle) \) (cf. [8]).

\[ \square \]

4. Killing Vector Fields on \( TM \)

Let \( TM \) be the tangent bundle of an orientable real hypersurface \( M \) of the Euclidean space \( \mathbb{R}^{2n} \). Recall that a vector field \( \zeta \in \mathcal{X}(TM) \) on the Riemannian manifold \( (TM, g) \) is said to be Killing if

\[ (L_\zeta g)(E, F) = 0, \quad E, F \in \mathcal{X}(TM) \]

where \( L_\zeta \) is the Lie derivative with respect to the vector field \( \zeta \). We have seen in previous section that the tangent bundle \( (TM, g) \) admits a structure \((\varphi, \zeta, T, \alpha, \beta, g)\), that is similar to the almost contact structure. In this section
we are interested in finding conditions under which the special vector fields $\zeta$ and $T$ are Killing vector fields and as a particular case we get that the tangent bundle $(TS^{2n-1}, \bar{g})$ of the unit sphere $S^{2n-1}$ in the Euclidean space $R^{2n}$ admits a nontrivial Killing vector field.

**Theorem 4.1.** Let $(\varphi, \zeta, T, \alpha, \beta, \bar{g})$ be the structure on the tangent bundle $TM$ of an orientable real hypersurface $M$ of the Euclidean space $R^{2n}$. Then the vector field $\zeta$ is Killing.

*Proof.* First note that on taking inner product with $N^*$ in each part of Lemma 2.5, we conclude that $\bar{S}_N^*(X^h) = 0$, $\bar{S}_N^*(X^v) = 0$, $X \in \mathfrak{X}(M)$ and consequently,

$$\bar{S}_N^*(E) = 0, \quad E \in \mathfrak{X}(TM) \quad (4.1)$$

Also using second part of equation (2.2) in (ii) and (iv) of Lemma 2.4, we conclude that $\nabla_E N^h = 0$, $E \in \mathfrak{X}(TM)$, that is $N^h$ is parallel on the normal bundle of $TM$. Moreover, we have

$$\nabla_E N^* = \left( \nabla_E N^*, N^h \right) N^h = - \left( N^*, \nabla_E N^h \right) N^h = 0$$

that is $N^*$ is parallel in the normal bundle of $TM$. Thus using equation (4.1) in Lemma 3.5, it follows that $\zeta$ is a parallel vector field and consequently, it is a Killing vector field. $\square$

**Theorem 4.2.** Let $(\varphi, \zeta, T, \alpha, \beta, \bar{g})$ be the structure on the tangent bundle $TM$ of an orientable real hypersurface $M$ of the Euclidean space $R^{2n}$. Then the vector field $T$ is Killing if and only if the following condition holds

$$\bar{g} \left( (\varphi \circ \bar{S}_{N^h}^* - \bar{S}_{N^h} \circ \varphi)(X^h), Y^h \right) = 0, \quad X, Y \in \mathfrak{X}(M) \quad (4.6)$$

and the equations (4.4)-(4.6) prove the Theorem. $\square$
Consider the unit sphere $S^{2n-1}$ in the Euclidean space $R^{2n}$, whose shape operator is given by $S = -I$. Using Lemma 2.4, we get on the tangent bundle $TS^{2n-1}$ that

$$\bar{S}_{N^h}(X^h) = (S(X))^h = -X^h, \quad \bar{S}_{N^h}(X^v) = 0$$

Then the Lemma 3.3 together with above equation, gives

$$(\varphi \circ \bar{S}_{N^h} - \bar{S}_{N^h} \circ \varphi)(X^h) = -\varphi(X^h) - \bar{S}_{N^h}(\varphi(X)^h - g(S(X), u) \circ \pi v^u)$$

and consequently,

$$\bar{g}((\varphi \circ \bar{S}_{N^h} - \bar{S}_{N^h} \circ \varphi)(X^h), Y^h) = 0, \quad X, Y \in \mathcal{X}(S^{2n-1})$$

Thus as a particular case of the Theorem 4.2, we have

**Corollary 4.3.** Let $(\varphi, \zeta, T, \alpha, \beta, \bar{g})$ be the structure on the tangent bundle $TS^{2n-1}$ of the unit sphere $S^{2n-1}$ in the Euclidean space $R^{2n}$, $n > 1$. Then the vector field $T$ is a nontrivial Killing vector field.

**Proof.** It remains to be shown that $T$ is nontrivial. Since, $N^h$ is parallel in the normal bundle of $TS^{2n-1}$, by Lemmas 2.4 and 3.5, we have

$$\nabla X^h T = -\varphi(X^h), \quad X \in \mathcal{X}(S^{2n-1}) \quad (4.7)$$

where we used the fact that the shape operator $S$ of the unit sphere $S^{2n-1}$ is given by $S = -I$. The Lemma 3.4 gives the rank of $\varphi$ is $4(n - 1)$ and consequently, equation (4.7) gives that the Killing vector field $T$ is not parallel, that is $T$ is a nontrivial Killing vector field. \[\square\]

**ACKNOWLEDGMENTS**

This Work is supported by King Saud University, Deanship of Scientific Research, College of Science Research Center.

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