Tangent Bundle of the Hypersurfaces in a Euclidean Space

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Abstract. Let \( M \) be an orientable hypersurface in the Euclidean space \( \mathbb{R}^{2n} \) with induced metric \( g \) and \( TM \) be its tangent bundle. It is known that the tangent bundle \( TM \) has induced metric \( \overline{g} \) as submanifold of the Euclidean space \( \mathbb{R}^{4n} \) which is not a natural metric in the sense that the submersion \( \pi : (TM, \overline{g}) \to (M, g) \) is not the Riemannian submersion. In this paper, we use the fact that \( \mathbb{R}^{4n} \) is the tangent bundle of the Euclidean space \( \mathbb{R}^{2n} \) to define a special complex structure \( J \) on the tangent bundle \( \mathbb{R}^{4n} \) so that \( (\mathbb{R}^{4n}, J, \langle \cdot, \cdot \rangle) \) is a Kaehler manifold, where \( \langle \cdot, \cdot \rangle \) is the Euclidean metric which is also the Sasaki metric of the tangent bundle \( \mathbb{R}^{4n} \). We study the structure induced on the tangent bundle \( (TM, \overline{g}) \) of the hypersurface \( M \), which is a submanifold of the Kaehler manifold \( (R^{4n}, J, \langle \cdot, \cdot \rangle) \). We show that the tangent bundle \( TM \) is a CR-submanifold of the Kaehler manifold \( (R^{4n}, J, \langle \cdot, \cdot \rangle) \). We find conditions under which certain special vector fields on the tangent bundle \( (TM, \overline{g}) \) are Killing vector fields. It is also shown that the tangent bundle \( TS^{2n-1} \) of the unit sphere \( S^{2n-1} \) admits a Riemannian metric \( \overline{g} \) and that there exists a nontrivial Killing vector field on the tangent bundle \( (TS^{2n-1}, \overline{g}) \).

Keywords: Tangent bundle, Hypersurface, Kaehler manifold, Almost contact structure, Killing vector field, CR-Submanifold, Second fundamental form, Wiengarten map.


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1. Introduction

Recently efforts are made to study the geometry of the tangent bundle of a hypersurface $M$ in the Euclidean space $R^{n+1}$ (cf. [3]), where the authors have shown that the induced metric on its tangent bundle $TM$ as submanifold of the Euclidean space $R^{2n+2}$ is not a natural metric. In [4], we have extended the study initiated in [3] on the geometry of the tangent bundle $TM$ of an immersed orientable hypersurface $M$ in the Euclidean space $R^{n+1}$. It is well known that Killing vector fields play an important role in shaping the geometry of a Riemannian manifold, for instance the presence of nonzero Killing vector field on a compact Riemannian manifold forces its Ricci curvature to be non-negative and this in particular implies that on a compact Riemannian manifolds of negative Ricci curvature there does not exist a nonzero Killing vector field. The study of Killing vector fields becomes more interesting on the tangent bundle $TM$ of a Riemannian manifold $(M, g)$ as the tangent bundle $TM$ is noncompact. It is known that if the tangent bundle $TM$ of a Riemannian manifold $(M, g)$ is equipped with Sasaki metric, then the vertical lift of a parallel vector field on $M$ is a Killing vector field (cf. [15]). However if the Sasaki metric is replaced by the Cheeger-Gromoll metric, then the vertical lift of any nonzero vector field on $M$ is never Killing (cf. [14]). Note that both Sasaki metric as well as Cheeger-Gromoll metrics are natural metrics. We consider an orientable real hypersurface $M$ of the Euclidean space $R^{2n}$ with the induced metric $g$. Then as the tangent bundle $TM$ of $M$ is a submanifold of codimension two in $R^{4n}$, it has induced metric $\overline{g}$ and this metric $\overline{g}$ on $TM$ is not a natural metric as the submersion $\pi : (TM, \overline{g}) \rightarrow (M, g)$ is not the Riemannian submersion (cf. [3]). Let $N$ be the unit normal vector field to the hypersurface $M$ and $J$ be the natural complex structure on the Euclidean space $R^{4n}$. Then we have a globally defined unit vector field $\xi$ on the hypersurface given by $\xi = -JN$ called the characteristic vector field of the real hypersurface (cf. [1, 2, 5, 6, 7, 8, 9]), and this vector field $\xi$ gives rise to two vector fields $\xi^h$ (the horizontal lift) and $\xi^v$ (the vertical lift) on the tangent bundle $(TM, \overline{g})$. In this paper, we use the fact that $R^{4n}$ is the tangent bundle of the Euclidean space $R^{2n}$ and that the projection $\pi : R^{4n} \rightarrow R^{2n}$ is a Riemannian submersion, to define a special almost complex structure $\overline{J}$ on the tangent bundle $R^{4n}$ which is different from the canonical complex structure of the Euclidean space $R^{4n}$ and show that $(R^{4n}, \overline{J}, \langle \cdot, \cdot \rangle)$ is a Kaehler manifold, where $\langle \cdot, \cdot \rangle$ is the Euclidean metric on $R^{4n}$. It is shown that the codimension two submanifold $(TM, \overline{g})$ of the Kaehler manifold $(R^{4n}, \overline{J}, \langle \cdot, \cdot \rangle)$ is a CR-submanifold (cf. [10]) and it naturally inherits certain special vector fields other than $\xi^h$ and $\xi^v$, and in this paper we are interested in finding conditions under which these special vector fields are Killing vector fields on $(TM, \overline{g})$. One of the interesting outcome of this study is, we have shown that the tangent bundle $TS^{2n-1}$ of the unit sphere $S^{2n-1}$ as
submanifold of \( \mathbb{R}^{4n} \) admits a nontrivial Killing vector field. It is worth pointing out that on the tangent bundle \( TS^{2n-1} \) with Sasakian metric no vertical or horizontal lift of a vector field is Killing as this will require the corresponding vector field on \( S^{2n-1} \) is parallel which is impossible as \( S^{2n-1} \) is space of constant curvature 1. Note that on even dimensional Riemannian manifolds which are irreducible, it is difficult to find Killing vector fields, whereas on products like \( S^{2k-1} \times \mathbb{R}^{2l-1}, R^{2k-1} \times R^{2l-1} \) one can easily find Killing vector fields. Since the tangent bundle \( TS^{2n-1} \) is trivial for \( n = 1, 2, 4 \), finding Killing vector fields is easy in these dimensions, but for \( n \geq 5 \), it is not trivial.

2. Preliminaries

Let \( (M, g) \) be a Riemannian manifold and \( TM \) be its tangent bundle with projection map \( \pi : TM \rightarrow M \). Then for each \( (p, u) \in TM \), the tangent space \( T_{(p, u)}TM = \mathfrak{H}_{(p, u)} \oplus \mathfrak{V}_{(p, u)} \), where \( \mathfrak{H}_{(p, u)} \) is the kernel of \( d\pi_{(p, u)} : T_{(p, u)}(TM) \rightarrow T_{p}M \) and \( \mathfrak{H}_{(p, u)} \) is the kernel of the connection map \( K_{(p, u)} : T_{(p, u)}(TM) \rightarrow T_{p}M \) with respect to the Riemannian connection on \( (M, g) \). The subspaces \( \mathfrak{H}_{(p, u)}, \mathfrak{V}_{(p, u)} \) are called the horizontal and vertical subspaces respectively. Consequently, the Lie algebra of smooth vector fields \( \mathfrak{X}(TM) \) on the tangent bundle \( TM \) admits the decomposition \( \mathfrak{X}(TM) = \mathfrak{H} \oplus \mathfrak{V} \) where \( \mathfrak{H} \) is called the horizontal distribution and \( \mathfrak{V} \) is called the vertical distribution on the tangent bundle \( TM \). For each \( X_{p} \in T_{p}M \), the horizontal lift of \( X_{p} \) to a point \( z = (p, u) \in TM \) is the unique vector \( X_{z}^{h} \in \mathfrak{H}_{z} \) such that \( d\pi(X_{z}^{h}) = X_{p} \circ \pi \) and the vertical lift of \( X_{p} \) to a point \( z = (p, u) \in TM \) is the unique vector \( X_{z}^{v} \in \mathfrak{V}_{z} \) such that \( X_{z}^{v}(df) = X_{p}(f) \) for all functions \( f \in C^\infty(M) \), where \( df \) is the function defined by \( (df)(p, u) = u(f) \). Also for a vector field \( X \in \mathfrak{X}(M) \), the horizontal lift of \( X \) is a vector field \( X^{h} \in \mathfrak{X}(TM) \) whose value at a point \( (p, u) \) is the horizontal lift of \( X(p) \) to \( (p, u) \), the vertical lift \( X^{v} \) of \( X \) is defined similarly. For \( X \in \mathfrak{X}(M) \) the horizontal and vertical lifts \( X^{h}, X^{v} \) of \( X \) are uniquely determined vector fields on \( TM \) satisfying

\[
d\pi(X_{z}^{h}) = X_{\pi(z)}, K(X_{z}^{h}) = 0, d\pi(X_{z}^{v}) = 0, K(X_{z}^{v}) = X_{\pi(z)}
\]

Also, for a smooth function \( f \in C^\infty(M) \) and vector fields \( X, Y \in \mathfrak{X}(M) \), that \( (fX)^{h} = (f \circ \pi)X^{h}, (fX)^{v} = (f \circ \pi)X^{v}, (X + Y)^{h} = X^{h} + Y^{h} \) and \( (X + Y)^{v} = X^{v} + Y^{v} \). If \( \dim M = m \) and \( (U, \varphi) \) is a chart on \( M \) with local coordinates \( x^{1}, x^{2}, \ldots, x^{m} \), then \( (\pi^{-1}(U), \varphi) \) is a chart on \( TM \) with local coordinates \( x^{1}, x^{2}, \ldots, x^{m}, y^{1}, y^{2}, \ldots, y^{m} \), where \( x^{i} = x^{i} \circ \pi \) and \( y^{i} = dx^{i} \), \( i = 1, 2, \ldots, m \).

A Riemannian metric \( \overline{g} \) on the tangent bundle \( TM \) is said to be natural metric with respect to \( g \) on \( M \) if \( \overline{g}_{(p, u)}(X^{h}, Y^{h}) = g_{p}(X, Y) \) and \( \overline{g}_{(p, u)}(X^{h}, Y^{v}) = 0 \), for all vector fields \( X, Y \in \mathfrak{X}(M) \) and \( (p, u) \in TM \), that is the projection map \( \pi : TM \rightarrow M \) is a Riemannian submersion.
Let $M$ be an orientable hypersurface of the Euclidean space $\mathbb{R}^{2n}$ with immersion $f : M \to \mathbb{R}^{2n}$ and $TM$ be its tangent bundle. Then as $F = df : TM \to \mathbb{R}^{4n} = TR^{2n}$ is also an immersion, $TM$ is an immersed submanifold of the Euclidean space $\mathbb{R}^{4n}$. We denote the induced metrics on $M, TM$ by $g, \bar{g}$ respectively and the Euclidean metric on $\mathbb{R}^{2n}$ as well as on $\mathbb{R}^{4n}$ by $\langle , \rangle$. Also, we denote by $\nabla, \bar{\nabla}, D$ and $\bar{D}$ the Riemannian connections on $M, TM, R^{2n}$, and $R^{4n}$ respectively. Let $N$ and $S$ be the unit normal vector field and the shape operator of the hypersurface $M$. For the hypersurface $M$ of the Euclidean space $\mathbb{R}^{2n}$ we have the following Gauss and Weingarten formulae

$$D_X Y = \nabla_X Y + \langle S(X), Y \rangle N, \quad D_X N = -S(X), \quad X, Y \in \mathfrak{X}(M) \quad (2.1)$$

where $S$ is the shape operator (Weingarten map). Similarly for the submanifold $TM$ of the Euclidean space $\mathbb{R}^{4n}$ we have the Gauss and Weingarten formulae

$$\bar{D}_E F = \bar{\nabla}_E F + h(E, F), \quad \bar{D}_E N = -\bar{S}_N(E) + \bar{\nabla}_E N \quad (2.2)$$

where $E, F \in \mathfrak{X}(TM)$, $\bar{\nabla}_E$ is the connection in the normal bundle of $TM$ and $\bar{S}_N$ denotes the Weingarten map in the direction of the normal $N$ and is related to the second fundamental form $h$ by

$$\langle h(X, Y), N \rangle = \bar{g}(\bar{S}_N(X), Y) \quad (2.3)$$

Also we observe that for $X \in \mathfrak{X}(M)$ the vertical lift $X^v$ of $X$ to $TM$, as $X^v \in \ker d\pi$, where $\pi : TM \to M$ is the natural submersion, we have $d\pi(X^v) = 0$ that is $df(d\pi(X^v)) = 0$ or equivalently we get $d(f \circ \pi)(X^v) = 0$, that is $d(\tilde{\pi} \circ F)(X^v) = 0 (\pi : TR^{2n} \to R^{2n})$, which gives $dF(X^v) \in \ker d\tilde{\pi} = \mathfrak{N}$.

Now we state the following results which are needed in our work.

**Lemma 2.1.** [3] Let $N$ be the unit normal vector field to the hypersurface $M$ of $\mathbb{R}^{2n}$ and $P = (p, X_p) \in TM$. Then the horizontal and vertical lifts $Y^h_p, Y^v_p$ of $Y_p \in T_pM$ satisfy

$$dF_p(Y^h_p) = (df_p(Y_p))^h + V_p, \quad dF_p(Y^v_p) = (df_p(Y_p))^v$$

where $V_p \in \mathfrak{N}_P$ is given by $V_p = \langle S_p(X_p), Y_p \rangle N^v_p$, $N^v_p$ being the vertical lift of the unit normal $N$ to with respect to the tangent bundle $\pi : R^{4n} \to R^{2n}$.

**Lemma 2.2.** [3] If $(M, g)$ is an orientable hypersurface of $\mathbb{R}^{2n}$, and $(TM, \bar{g})$ is its tangent bundle as submanifold of $\mathbb{R}^{4n}$, then the metric $\bar{g}$ on $TM$ for $P = (p, u) \in TM$, satisfies:

(i) $\bar{g}_p(X^h_p, Y^h_p) = g_p(X_p, Y_p) + g_p(S_p(X_p), u)g_p(S_p(Y_p), u)$.

(ii) $\bar{g}_p(X^h_p, Y^v_p) = 0$.

(ii) $\bar{g}(X^v, Y^v) = g_p(X_p, Y_p)$.

**Remark 2.3.** It is well known that a metric $\bar{g}$ defined on $TM$ using the Riemannian metric $g$ of $M$ (such as Sasaki metric, Cheeger-Gromoll metric) are
natural metrics in the sense that the submersion \( \pi : (TM, \mathcal{G}) \rightarrow (M, g) \) becomes a Riemannian submersion with respect to these metrics. However, as seen from above Lemmas, the induced metric on the tangent bundle \( TM \) of a hypersurface \( M \) of the Euclidean space \( \mathbb{R}^{2n} \), as a submanifold of \( \mathbb{R}^{4n} \) is not a natural metric because of the present of the term \( g_p(S_p(X_p), u)g_p(S_p(Y_p), u) \) in the inner product of horizontal vectors on \( TM \). Moreover, note that the for an orientable hypersurface \( M \) of the Euclidean space \( \mathbb{R}^{2n} \), the vertical lift \( N^v \) of the unit normal is tangential to the submanifold \( TM \) of \( \mathbb{R}^{4n} \) as seen in 2.1

In what follows, we drop the suffixes like in \( g_p(S_p(X_p), u) \) and and it will be understood from the context of the entities appearing in the equations.

**Theorem 2.4.** [3] Let \((M, g)\) be an orientable hypersurface of \( \mathbb{R}^{2n} \), and \((TM, \mathcal{G})\) be its tangent bundle as submanifold of \( \mathbb{R}^{4n} \). If \( \nabla \) and \( \nabla \) denote the Riemannian connections on \((M, g)\) and \((TM, \mathcal{G})\) respectively, then

(i) \( \nabla_X Y^h = (\nabla_X Y)^h - \frac{1}{2}(R(X, Y)u)^v \),

(ii) \( \nabla_X Y^v = g(S(X), Y) \circ \pi N^v \),

(iii) \( \nabla_X Y^v = 0 \), (iv) \( \nabla_X Y^v = (\nabla_X Y)^v + g(S(X), Y) \circ \pi N^v \).

**Lemma 2.5.** [4] Let \( TM \) be the tangent bundle of an orientable hypersurface \( M \) of \( \mathbb{R}^{2n} \). Then for \( X, Y \in \mathfrak{X}(M) \),

(i) \( h(X^v, Y^v) = 0 \),

(ii) \( h(X^v, Y^h) = 0 \),

(iii) \( h(X^h, Y^h) = g(S(X), Y) \circ \pi N^h \).

**Lemma 2.6.** [4] For the tangent bundle \( TM \) of an orientable hypersurface \( M \) of \( \mathbb{R}^{2n} \) and \( X \in \mathfrak{X}(M) \), we have

(i) \( \mathcal{D}_X N^v = 0 \),

(ii) \( \mathcal{D}_X N^h = 0 \),

(iii) \( \mathcal{D}_X S^v = - (S(X))^v \), (iv) \( \mathcal{D}_X N^h = - (S(X))^h \).

Let \( J \) be the natural complex structure on the Euclidean space \( \mathbb{R}^{2n} \), which makes \( (\mathbb{R}^{2n}, J, \langle , \rangle) \) a Kähler manifold. Then on an orientable real hypersurface \( M \) of \( \mathbb{R}^{2n} \) with unit normal \( N \), we define a unit vector field \( \xi \in \mathfrak{X}(M) \) by \( \xi = -JN \), with its dual 1-form \( \eta(X) = g(X, \xi) \), where \( g \) is the induced metric on \( M \). For \( X \in \mathfrak{X}(M) \), we express \( JX = \varphi(X) + \eta(X)N \), where \( \varphi(X) \) is the tangential component of \( JX \), and it follows that \( \varphi \) is a \((1, 1)\) tensor field on \( M \), and that \((\varphi, \xi, \eta, g)\) defines an almost contact metric structure on \( M \) (cf. [5], [8], [9]), that is

\[ \varphi^2 X = -X + \eta(X)\xi, \eta(\xi) = 1, \eta \circ \varphi = 0, \varphi(\xi) = 0 \]

and

\[ g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad X, Y \in \mathfrak{X}(M) \]

Moreover, we have the following.
Lemma 2.7. [8] Let $M$ be an orientable real hypersurface of $R^{2n}$. Then the structure $(\varphi, \xi, \eta, g)$ on $M$ satisfies

$(i)$ $(\nabla_X \varphi)(Y) = \eta(Y)SX - g(SX, Y)\xi,$
$(ii)$ $\nabla_X \xi = \varphi SX, \; X, Y \in \mathfrak{X}(M).$

3. A Structure on $(TM, g)$

We know that the Euclidean space $R^{4n}$ has many complex structures, however in this section we treat $R^{4n}$ as the tangent bundle of $R^{2n}$ and consider a specific complex structure on the Euclidean space $R^{4n}$. Let $\pi : R^{4n} = TR^{2n} \to R^{2n}$ be the submersion of the tangent bundle of $R^{2n}$. Then it is easy to show that the Euclidean metric $\langle , \rangle$ on the tangent bundle $R^{4n}$ is Sasaki metric and using the canonical almost complex structure $J$ of $R^{2n}$, we define $J : \mathfrak{X}(R^{4n}) \to \mathfrak{X}(R^{4n})$ by

$J(E^h) = (JE)^h, \quad J(E^v) = (JE)^v, \quad E \in \mathfrak{X}(R^{2n})$

and it is easily follows that $J$ is an almost complex structure, satisfying $\langle JE, JF \rangle = \langle E, F \rangle$ with respect to the Euclidean metric $\langle , \rangle$ on $R^{4n}$ and that $(\bar{D}_E J)(F) = 0, \; E, F \in \mathfrak{X}(R^{4n})$ that is $(R^{4n}, J, \langle , \rangle)$ is a Kaehler manifold. Regarding the complex structure $J$ defined above, we have the following

Lemma 3.1. Let $\pi : R^{4n} \to R^{2n}$ be the submersion of the tangent bundle $R^{4n} = TR^{2n}$. Then complex structure $\bar{J}$ on $R^{4n}$ satisfies

$J \circ d\pi = d\pi \circ \bar{J}$

Proof. Take $X \in \mathfrak{X}(R^{2n})$, then for the horizontal lift $X^h$, we have:

$J \circ d\pi(X^h) = J(d\pi(X^h)) = JX \circ \pi$

and

$d\pi \circ \bar{J}(X^h) = d\pi(JX)^h = JX \circ \pi$

which proves

$J \circ d\pi(X^h) = d\pi \circ \bar{J}(X^h)$

Similarly for the vertical lift $X^v$ we have

$J \circ d\pi(X^v) = J(d\pi(X^v)) = 0$

and

$d\pi \circ \bar{J}(X^v) = d\pi(JX)^v = 0$

This proves the Lemma.

Remark 3.2. If $M$ is an orientable real hypersurface of the Euclidean space $R^{2n}$ with immersion $f$, then $F = df$ is the immersion of the tangent bundle $TM$ into the Euclidean space $R^{4n}$ and as immersions are local embeddings, in general, we identify the local quantities on submanifold with those of the ambient space for instance we identify $df(X)$ with $X$ for $X \in \mathfrak{X}(M)$. However,
while dealing with the immersion $F$ of $TM$ in $R^{4n}$ one need to be cautious specially while dealing with the horizontal lifts (cf. 2.1). Therefore in what follows, we shall bring $dF$ in to play whenever it is needed specially in the case of horizontal lifts.

Observe that if $M$ is an orientable real hypersurface of the Euclidean space $R^{2n}$ with unit normal vector field $N$, then we know that horizontal lift $N^h$ is a unit normal vector field to the submanifold $TM$ of $R^{4n}$ and that the vertical lift $N^v \in \mathfrak{X}(TM)$ (cf.[1]). We have
\[ JN^h = (JN)^h = -(df(\xi))^h = -dF(\xi^h) + g(S(\xi), u)N^v \in \mathfrak{X}(TM) \tag{3.1} \]
and
\[ JN^v = (JN)^v = -\xi^v \in \mathfrak{X}(TM) \tag{3.2} \]

Let $M$ be an orientable real hypersurface of the Kaehler manifold $(R^{2n}, J, \langle \cdot, \cdot \rangle)$. Then as $TM$ is submanifold of the Kaehler manifold $(R^{4n}, J, \langle \cdot, \cdot \rangle)$, we denote by $\Gamma(T^\perp TM)$ the space of smooth normal vector fields to $TM$. The restriction of the complex structure $J$ on $R^{4n}$ to $\mathfrak{X}(TM)$ and $\Gamma(T^\perp TM)$ can be expressed as
\[ J(E) = \varphi(E) + \overline{\psi}(E), \quad J(N) = \overline{G}(N) + \overline{\mathfrak{N}}, \quad E \in \mathfrak{X}(TM), \quad N \in \Gamma(T^\perp TM) \]
where $\varphi(E), \overline{G}(N)$ are the tangential and $\overline{\psi}(E), \overline{\mathfrak{N}}$ are the normal components of $JE$, and $J(N)$ respectively. Note that the horizontal lift $N^h$ of the unit normal $N$ to the hypersurface $M$ is normal to $TM$ that is $N^h \in \Gamma(T^\perp TM)$, where as the vertical lift $N^v \in \mathfrak{X}(TM)$.

**Lemma 3.3.** Let $TM$ be the tangent bundle of an orientable real hypersurface of $R^{2n}$. Then for $X \in \mathfrak{X}(M)$,
\[ \varphi(X^h) = (\varphi(X))^h - g(SX, u)\xi^v, \quad \varphi(X^v) = (\varphi(X))^v + \eta(X) \circ \pi N^v \]
\[ \overline{\psi}(X^h) = \eta(X) \circ \pi N^h, \quad \overline{\psi}(X^v) = 0 \]

**Proof.** Note that for the horizontal lift $X^h$ we have
\[ JX^h = JdF(X^h) = J((df(X))^h + g(SX, u) \circ \pi N^v) \]
\[ = (Jdf(X))^h + g(SX, u) \circ (JN)^v \]
\[ = (\varphi X + \eta(X)N)^h - g(SX, u) \circ \pi \xi^v \]
\[ = (\varphi(X))^h - g(SX, u) \circ \pi \xi^v + \eta(X) \circ \pi N^h \]
which together with the definition $JX^h = \varphi(X^h) + \overline{\psi}(X^h)$, on equating tangential and normal components give
\[ \varphi(X^h) = (\varphi(X))^h - g(S(X), u)\xi^v \quad \text{and} \quad \overline{\psi}(X^h) = \eta(X) \circ \pi N^h \]
Similarly for the vertical lift $X^v$, we have
\[ JX^v = \varphi(X^v) + \overline{\psi}(X^v) = (JX)^v = (\varphi X + \eta(X)N)^v \]
which gives
\[(\varphi(X^v)) + \psi(X^v) = (\varphi X)^v + \eta(\pi N^v)\]
Comparing the tangential and normal components we conclude
\[\varphi(X^v) = (\varphi X)^v + \eta(\pi N^v), \quad \text{and} \quad \psi(X^v) = 0.
\]
\[\square\]
We choose a unit normal vector field \(N^* \in \Gamma(T^\bot TM)\) such that \(\{N^*, N^h\}\) is a local orthonormal frame of normals for the submanifold \(TM\). It is known that \(N^*\) is vertical vector field on the tangent bundle \(\mathbb{R}^n\) (cf. [1]). Since,
\[\langle JN^*, N^* \rangle = 0, \quad \langle JN^*, N^h \rangle = \langle N^*, \xi^h \rangle = 0, \]
it follows that \(JN^* \in \mathfrak{X}(TM)\) and we define unit vector field \(\zeta \in \mathfrak{X}(TM)\) by
\[\zeta = -JN^*\]
(3.3)

Now, for any normal vector field \(N \in \Gamma(T^\bot TM)\), we have
\[N = \langle N, N^* \rangle N^* + \langle N, N^h \rangle N^h\]
which together with equations (3.1), (3.2) and (3.3) gives \(\chi(N) = 0\) and that \(\mathcal{J}(N) \in \mathfrak{X}(TM)\), is given by
\[\mathcal{J}(N) = \langle \mathcal{J}(N), \zeta \rangle \zeta + \langle \mathcal{J}(N), T \rangle T\]
(3.4)
where \(T \in \mathfrak{X}(TM)\), is given by
\[T = \xi^h - g(S(\xi, u)N^v = -JN^h\]
(3.5)
Also, using equation (3.2), we have
\[-\xi^v = JN^v = \varphi(N^v) + \psi(N^v)\]
which gives
\[\varphi(N^v) = -\xi^v \text{ and } \psi(N^v) = 0\]
(3.6)
Moreover, we have
\[\varphi(\zeta) = 0 \text{ and } \psi(\zeta) = N^*, \quad \psi(\xi^h) = N^h\]
(3.7)
If we denote by \(\alpha, \beta\) the smooth 1-forms on \(TM\) dual to the vector field \(\zeta\) and \(T\) respectively, then for \(E \in \mathfrak{X}(TM)\), it follows that
\[\mathcal{J}(\overline{\psi}(E)) = -\alpha(E)\zeta - \beta(E)T\]
and consequently, operating \(\mathcal{J}\) on \(\mathcal{J}(E) = \varphi(E) + \psi(E), \quad E \in \mathfrak{X}(TM)\), we get
\[\overline{\varphi}^2 = -I + \alpha \otimes \zeta + \beta \otimes T \quad \text{and} \quad \psi \circ \varphi = 0\]
(3.8)
Using Lemma 2.1 and equations (3.3), (3.5), (3.6), (3.8), we see that the vector fields \(\zeta, T\) and 1-forms \(\alpha, \beta\) satisfy
\[\varphi(\zeta) = 0, \quad \varphi(T) = 0, \quad \mathcal{J}(\zeta, T) = 0, \quad \alpha \circ \varphi = 0, \quad \beta \circ \varphi = 0\]
(3.9)
Also, as \( \varphi \) is the induced metric on the submanifold \( TM \) and \( J \) is skew symmetric with respect to the Hermitian metric \( \langle \cdot, \cdot \rangle \), we have

\[
\varphi (\varphi (E), F) = -\varphi (E, \varphi (F)), \quad E, F \in \mathfrak{X}(TM)
\]

Then using equations (3.8), (3.9) and (3.10), we have

\[
\varphi (\varphi (E), \varphi (F)) = \varphi (E, F) - \alpha (E)\alpha (F) - \beta (E)\beta (F), \quad E, F \in \mathfrak{X}(TM)
\]

Thus we have proved the following

**Lemma 3.4.** Let \( TM \) be the tangent bundle of an orientable real hypersurface of \( R^{2n} \). Then there is a structure \( (\varphi, \zeta, T, \alpha, \beta, \varphi) \) similar to contact metric structure on \( TM \), where \( \varphi \) is a tensor field of type \( (1, 1) \), \( \zeta, T \) are smooth vector fields and \( \alpha, \beta \) are smooth 1-forms dual to \( \zeta, T \) with respect to the Riemannian metric \( \varphi \) satisfying

\[
\varphi^2 = -I + \alpha \otimes \zeta + \beta \otimes T, \quad \varphi(\zeta) = 0, \quad \varphi(T) = 0, \quad \alpha \circ \varphi = 0, \quad \beta \circ \varphi = 0, \quad \varphi(\zeta, T) = 0
\]

In the next Lemma, we compute the co-variant derivatives of the tensor \( \varphi \).

**Lemma 3.5.** Let \( (\varphi, \zeta, T, \alpha, \beta, \varphi) \) be the structure on the tangent bundle \( TM \) of an orientable real hypersurface \( M \) of the Euclidean space \( R^{2n} \). Then

(i) \( \nabla_{X^h} \varphi(Y^h) = \{ \nabla_X \varphi(Y) \}^h - \{ X(g(SY, u) + g(SY, u)JSX) \}^v \)

(ii) \( \nabla_{X^h} \varphi(Y^v) = 0 \).

(iii) \( \nabla_{X^h} \varphi(Y^v) = 0, \quad (\nabla_X \varphi(Y^h)) = g(SX, \varphi Y) \circ \pi N^v + g(SX, Y) \circ \pi \xi^v \).

**Proof.** Using the definition of \( J \), Lemma 2.1 and Lemma 3.3 together with equation (3.1), we get for \( X, Y \in \mathfrak{X}(M) \)

\[
JY^h = JdF(Y^h) = J \left( (df(Y))^h + g(SY, u) \circ \pi N^v \right) = (\varphi Y + \eta(Y)N)^h - g(SY, u) \circ \pi \xi^v = \varphi (Y^h) + \eta(Y) \circ \pi N^h
\]

which gives

\[
\overline{\nabla}_{X^h} JY^h = \overline{\nabla} \left( (df(X))^h + g(SX, u) \circ \pi N^v \right) (\varphi (Y^h) + \eta(Y) \circ \pi N^h)
\]

\[
= \overline{\nabla} \left( (df(X))^h \varphi (Y^h) + X(\eta(Y)) \circ \pi N^h + \eta(Y) \circ \pi \overline{\nabla} \left( (df(X))^h \right) N^h \right) + g(SX, u) \circ \pi \eta(Y) \circ \pi \overline{\nabla} N^h + g(SX, u) \circ \pi \eta(X) \circ \pi \overline{\nabla} N^h
\]

Note that the tangent bundle \( TR^{2n} = R^{4n} \) has Sasaki metric and thus using Lemma 7.2 of [10] (keeping in view that \( R^{2n} \) is flat), in the above equation, we get

\[
\overline{\nabla}_{X^h} JY^h = \nabla_{X^h} \varphi (Y^h) + h(X^h, \varphi (Y^h) + X(\eta(Y)) \circ \pi N^h - \eta(Y) \circ \pi (SX)^h
\]

(3.12)
Similarly we have
\[
\nabla h X h Y h = J \left( \nabla ((df(X)) h + g(SX,u) \circ \pi N v) \right) (df(Y)) h + g(SY,u) \circ \pi N v + g(SY,u) \circ \pi (D X) v
+ g(SX,u) \circ \pi \nabla ((df(Y)) h + 0 + 0)
\]
\[
= \nabla (\nabla X h Y h + \nabla (\nabla X h Y h) + \nabla h(X^h,Y^h) - X(g(SY,u) \circ \pi \eta v - \nabla (\nabla X h Y h) + \nabla (\nabla X h Y h) - g(SX,Y) \circ \pi \xi v
- g(SY,u) \circ \pi (\phi SX)^v - g(SY,u) \circ \pi \eta (SX) N v (3.13)
\]
where we used Lemmas 2.3, 2.4 and Lemma 7.2 in [10]. Now as \( (R^{4n},J,\langle , \rangle) \)

is a Kaehler manifold, the equations (3.12) and (3.13) on comparing tangential
we get
\[
(\nabla X h \varphi) (Y h) = ((\nabla X \varphi) (Y)) h - X (g(SY,u) + g(SY,u) J SX) \]

which proves (i).

Now, using \( h(X^v,Y^v) = 0 \) and \( \nabla (\nabla X h Y h) = 0 \) together with \( \nabla X h Y v = \nabla X h Y v \), and comparing tangential components, we immediately arrive at
\[
(\nabla X h \varphi) (Y v) = 0
\]

Next, we have \( \nabla X h \varphi (Y h) = \nabla X h (\varphi Y h - g(SX,u) \circ \pi \xi v) = \nabla (\varphi Y h = g(SX,\varphi Y) \circ \pi \xi v
\]
\[
= \nabla (\varphi Y h = g(SX,\varphi Y) \circ \pi \xi v)
\]

Thus, we get
\[
(\nabla X h \varphi) (Y h) = g(SX,\varphi Y) \circ \pi \xi v + g(SX,Y) \circ \pi \xi v
\]

Finally, using \( h(X^h,Y^v) = 0 \) and \( \nabla (\nabla X h Y h) = 0 \) together with \( \nabla X h Y v = \nabla X h Y v \), and comparing tangential components, we immediately arrive at
\[
(\nabla X h \varphi) (Y v) = 0
\]

\[\square\]

Lemma 3.6. Let \( (\varphi,\zeta,T,\alpha,\beta,\gamma) \) be the structure on the tangent bundle \( TM \)
of an orientable real hypersurface \( M \) of the Euclidean space \( R^{2n} \). Then for
\( E \in T(M), \)
\[
\nabla E \zeta = \varphi (\nabla h N v) - J \left( \nabla E h N^v \right), \quad h(E,\zeta) = \varphi (\nabla h N \zeta)
\]
\[
\nabla E T = \varphi (\nabla h N E) - J \left( \nabla E h N^h \right), \quad h(E,T) = \varphi (\nabla h N h)
\]
Proof. Using equation (2.2), we have
\[
\nabla E \zeta = \overline{D} E \zeta - h(E, \zeta)
\]
\[
= -\mathcal{J} \overline{D} E \mathcal{N}^* - h(E, \zeta)
\]
\[
= \mathcal{J} (S N^*(E)) - \mathcal{J} \left( \nabla_L^* E \mathcal{N}^* \right) - h(E, \zeta)
\]
\[
= \varphi (S N^*(E)) - \psi (S N^*(E)) - \mathcal{J} \left( \nabla_L^* E \mathcal{N}^* \right) - h(E, \zeta)
\]
Since \( \mathcal{J}(N) \in \mathcal{X}(TM) \) for each normal \( N \in \Gamma(T^\perp TM) \), equation tangential and normal components in above equation, we get the first part. The second part follows similarly using \( T = -\mathcal{J} \mathcal{N} \).

Now, we prove the following:

**Theorem 3.7.** The tangent bundle \( TM \) of an orientable real hypersurface \( M \) of the Euclidean space \( R^{2n} \) is a CR-submanifold of the Kaehler manifold \( (R^{4n}, J, \langle \cdot, \cdot \rangle) \).

**Proof.** Use the structure \((\varphi, \zeta, T, \alpha, \beta, g)\) on the submanifold \( TM \) of \( R^{4n} \) to define the distribution \( D \) by
\[
D = \{ E \in \mathcal{X}(TM) : \alpha(E) = \beta(E) = 0 \}
\]
and \( D^\perp \) be the distribution spanned by the orthogonal vector fields \( \zeta \) and \( T \). Note that \( \zeta \) is unit vector field on \( TM \) and the length of the vector field \( T \) satisfies
\[
||T||^2 = 1 + 2 g(S(\xi), u)^2 \geq 1
\]
which shows that \( D^\perp \) is 2-dimensional distribution on \( TM \) and that \( J D^\perp = \Gamma(T^\perp TM) \). It is easy to see that \( D \) and \( D^\perp \) are orthogonal complementary distributions and that \( \dim D = 4(n - 1) \). Note that for \( E \in \mathcal{X}(TM) \), we have
\[
\overline{\psi}(E) = \langle \overline{\psi}(E), N^* \rangle N^* + \langle \overline{\psi}(E), N^h \rangle N^h = \alpha(E) N^* + \beta(E) N^h
\]
and consequently if \( E \in D \), then above equation gives \( \overline{J} E = \varphi E \) which is orthogonal to both \( \zeta \) and \( T \) and that \( \overline{J} E \in D \), which implies \( JD = D \). This proves that \( TM \) is a CR-submanifold of the Kaehler manifold \( (R^{4n}, J, \langle \cdot, \cdot \rangle) \) (cf. [8]).

**4. Killing Vector Fields on \( TM \)**

Let \( TM \) be the tangent bundle of an orientable real hypersurface \( M \) of the Euclidean space \( R^{2n} \). Recall that a vector field \( \varsigma \in \mathcal{X}(TM) \) on the Riemannian manifold \((TM, g)\) is said to be Killing if
\[
(L_\varsigma g)(E, F) = 0, \quad E, F \in \mathcal{X}(TM)
\]
where \( L_\varsigma \) is the Lie derivative with respect to the vector field \( \varsigma \). We have seen in previous section that the tangent bundle \((TM, \overline{g})\) admits a structure \((\overline{\varphi}, \zeta, T, \alpha, \beta, \overline{g})\), that is similar to the almost contact structure. In this section...
we are interested in finding conditions under which the special vector fields \( \zeta \) and \( T \) are Killing vector fields and as a particular case we get that the tangent bundle \((TS^{2n-1}, g)\) of the unit sphere \( S^{2n-1} \) in the Euclidean space \( R^{2n} \) admits a nontrivial Killing vector field.

**Theorem 4.1.** Let \((\varphi, \zeta, T, \alpha, \beta, g)\) be the structure on the tangent bundle \( TM \) of an orientable real hypersurface \( M \) of the Euclidean space \( R^{2n} \). Then the vector field \( \zeta \) is Killing.

**Proof.** First note that on taking inner product with \( N^* \) in each part of Lemma 2.5, we conclude that
\[
S_{N^*}(X^h) = 0, \quad S_{N^*}(X^v) = 0, \quad X \in \mathfrak{X}(M)
\]
and consequently,
\[
S_{N^*}(E) = 0, \quad E \in \mathfrak{X}(TM) \quad (4.1)
\]
Also using second part of equation (2.2) in (ii) and (iv) of Lemma 2.4, we conclude that
\[
\nabla_E^h N^h = 0, \quad E \in \mathfrak{X}(TM),
\]
that is \( N^* \) is parallel in the normal bundle of \( TM \). Thus using equation (4.1) in Lemma 3.5, it follows that \( \zeta \) is a parallel vector field and consequently, it is a Killing vector field. \( \square \)

**Theorem 4.2.** Let \((\varphi, \zeta, T, \alpha, \beta, g)\) be the structure on the tangent bundle \( TM \) of an orientable real hypersurface \( M \) of the Euclidean space \( R^{2n} \). Then the vector field \( T \) is Killing if and only if the following condition holds
\[
g\left( (\varphi \circ S_{N^h} - S_{N^h} \circ \varphi)(X^h), Y^h \right) = 0 \quad X, Y \in \mathfrak{X}(M)
\]
**Proof.** Since \( N^h \) is parallel in the normal bundle of \( TM \), by Lemma 3.5, we have
\[
\nabla_E T = \varphi(S_{N^h}(E)), \quad E \in \mathfrak{X}(TM) \quad (4.2)
\]
Also using Lemma 2.4, we conclude that
\[
S_{N^h}(X^v) = 0, \quad S_{N^h}(X^h) = (S(X))^h, \quad X \in \mathfrak{X}(M) \quad (4.3)
\]
Then using skew-symmetry of the tensor \( \varphi \), and equations (4.2) and (4.3) together with Lemma 3.3, we immediately arrive at
\[
(\mathcal{L}_T g)(X^v, Y^v) = 0 \quad (4.4)
\]
\[
(\mathcal{L}_T g)(X^h, Y^v) = g\left( (\varphi \circ S_{N^h} - S_{N^h} \circ \varphi)(X^h), Y^v \right) = -g\left( S_{N^h}(X^h), \varphi(Y^v) \right) = 0
\]
\[
(\mathcal{L}_T g)(X^h, Y^h) = g\left( (\varphi \circ S_{N^h} - S_{N^h} \circ \varphi)(X^h), Y^h \right) = 0 \quad (4.5)
\]
and the equations (4.4)-(4.6) prove the Theorem. \( \square \)
Consider the unit sphere $S^{2n-1}$ in the Euclidean space $\mathbb{R}^{2n}$, whose shape operator is given by $S = -I$. Using Lemma 2.4, we get on the tangent bundle $TS^{2n-1}$ that

$$\overline{S}_{N^h}(X^h) = (S(X))^h = -X^h, \quad \overline{S}_{N^h}(X^v) = 0$$

Then the Lemma 3.3 together with above equation, gives

$$(\varphi \circ \overline{S}_{N^h} - \overline{S}_{N^h} \circ \varphi)(X^h) = -\varphi(X^h) - \overline{S}_{N^h} \left( (\varphi(X))^h - g(S(X), u) \circ \pi \xi^v \right)$$

and consequently,

$$\overline{g} \left( (\varphi \circ \overline{S}_{N^h} - \overline{S}_{N^h} \circ \varphi)(X^h), Y^h \right) = 0, \quad X, Y \in \mathfrak{X}(S^{2n-1})$$

Thus as a particular case of the Theorem 4.2, we have

**Corollary 4.3.** Let $(\varphi, \zeta, T, \alpha, \beta, \overline{g})$ be the structure on the tangent bundle $TS^{2n-1}$ of the unit sphere $S^{2n-1}$ in the Euclidean space $\mathbb{R}^{2n}$, $n > 1$. Then the vector field $T$ is a nontrivial Killing vector field.

*Proof.* It remains to be shown that $T$ is nontrivial. Since, $N^h$ is parallel in the normal bundle of $TS^{2n-1}$, by Lemmas 2.4 and 3.5, we have

$$\nabla_{X^h} T = -\overline{\varphi}(X^h), \quad X \in \mathfrak{X}(S^{2n-1}) \quad (4.7)$$

where we used the fact that the shape operator $S$ of the unit sphere $S^{2n-1}$ is given by $S = -I$. The Lemma 3.4 gives the rank of $\varphi$ is $4(n-1)$ and consequently, equation (4.7) gives that the Killing vector field $T$ is not parallel, that is $T$ is a nontrivial Killing vector field. \(\square\)

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**References**