

## Tangent Bundle of the Hypersurfaces in a Euclidean Space

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**ABSTRACT.** Let  $M$  be an orientable hypersurface in the Euclidean space  $R^{2n}$  with induced metric  $g$  and  $TM$  be its tangent bundle. It is known that the tangent bundle  $TM$  has induced metric  $\bar{g}$  as submanifold of the Euclidean space  $R^{4n}$  which is not a natural metric in the sense that the submersion  $\pi : (TM, \bar{g}) \rightarrow (M, g)$  is not the Riemannian submersion. In this paper, we use the fact that  $R^{4n}$  is the tangent bundle of the Euclidean space  $R^{2n}$  to define a special complex structure  $\bar{J}$  on the tangent bundle  $R^{4n}$  so that  $(R^{4n}, \bar{J}, \langle, \rangle)$  is a Kaehler manifold, where  $\langle, \rangle$  is the Euclidean metric which is also the Sasaki metric of the tangent bundle  $R^{4n}$ . We study the structure induced on the tangent bundle  $(TM, \bar{g})$  of the hypersurface  $M$ , which is a submanifold of the Kaehler manifold  $(R^{4n}, \bar{J}, \langle, \rangle)$ . We show that the tangent bundle  $TM$  is a CR-submanifold of the Kaehler manifold  $(R^{4n}, \bar{J}, \langle, \rangle)$ . We find conditions under which certain special vector fields on the tangent bundle  $(TM, \bar{g})$  are Killing vector fields. It is also shown that the tangent bundle  $TS^{2n-1}$  of the unit sphere  $S^{2n-1}$  admits a Riemannian metric  $\bar{g}$  and that there exists a nontrivial Killing vector field on the tangent bundle  $(TS^{2n-1}, \bar{g})$ .

**Keywords:** Tangent bundle, Hypersurface, Kaehler manifold, Almost contact structure, Killing vector field, CR-Submanifold, Second fundamental form, Wiengarten map.

**2000 Mathematics subject classification:** 53C42, 53C56, 53D10.

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## 1. INTRODUCTION

Recently efforts are made to study the geometry of the tangent bundle of a hypersurface  $M$  in the Euclidean space  $R^{n+1}$  (cf. [3]), where the authors have shown that the induced metric on its tangent bundle  $TM$  as submanifold of the Euclidean space  $R^{2n+2}$  is not a natural metric. In [4], we have extended the study initiated in [3] on the geometry of the tangent bundle  $TM$  of an immersed orientable hypersurface  $M$  in the Euclidean space  $R^{n+1}$ . It is well known that Killing vector fields play an important role in shaping the geometry of a Riemannian manifold, for instance the presence of nonzero Killing vector field on a compact Riemannian manifold forces its Ricci curvature to be non-negative and this in particular implies that on a compact Riemannian manifolds of negative Ricci curvature there does not exist a nonzero Killing vector field. The study of Killing vector fields becomes more interesting on the tangent bundle  $TM$  of a Riemannian manifold  $(M, g)$  as the tangent bundle  $TM$  is noncompact. It is known that if the tangent bundle  $TM$  of a Riemannian manifold  $(M, g)$  is equipped with Sasaki metric, then the vertical lift of a parallel vector field on  $M$  is a Killing vector field (cf. [15]). However if the Sasaki metric is replaced by the Cheeger-Gromoll metric, then the vertical lift of any nonzero vector field on  $M$  is never Killing (cf. [14]). Note that both Sasaki metric as well as Cheeger-Gromoll metrics are natural metrics. We consider an orientable real hypersurface  $M$  of the Euclidean space  $R^{2n}$  with the induced metric  $g$ . Then as the tangent bundle  $TM$  of  $M$  is a submanifold of codimension two in  $R^{4n}$ , it has induced metric  $\bar{g}$  and this metric  $\bar{g}$  on  $TM$  is not a natural metric as the submersion  $\pi : (TM, \bar{g}) \rightarrow (M, g)$  is not the Riemannian submersion (cf. [3]). Let  $N$  be the unit normal vector field to the hypersurface  $M$  and  $J$  be the natural complex structure on the Euclidean space  $R^{2n}$ . Then we have a globally defined unit vector field  $\xi$  on the hypersurface given by  $\xi = -JN$  called the characteristic vector field of the real hypersurface (cf. [1, 2, 5, 6, 7, 8, 9]), and this vector field  $\xi$  gives rise to two vector fields  $\xi^h$  (the horizontal lift) and  $\xi^v$  (the vertical lift) on the tangent bundle  $(TM, \bar{g})$ . In this paper, we use the fact that  $R^{4n}$  is the tangent bundle of the Euclidean space  $R^{2n}$  and that the projection  $\bar{\pi} : R^{4n} \rightarrow R^{2n}$  is a Riemannian submersion, to define a special almost complex structure  $\bar{J}$  on the tangent bundle  $R^{4n}$  which is different from the canonical complex structure of the Euclidean space  $R^{4n}$  and show that  $(R^{4n}, \bar{J}, \langle, \rangle)$  is a Kaehler manifold, where  $\langle, \rangle$  is the Euclidean metric on  $R^{4n}$ . It is shown that the codimension two submanifold  $(TM, \bar{g})$  of the Kaehler manifold  $(R^{4n}, \bar{J}, \langle, \rangle)$  is a CR-submanifold (cf. [10]) and it naturally inherits certain special vector fields other than  $\xi^h$  and  $\xi^v$ , and in this paper we are interested in finding conditions under which these special vector fields are Killing vector fields on  $(TM, \bar{g})$ . One of the interesting outcome of this study is, we have shown that the tangent bundle  $TS^{2n-1}$  of the unit sphere  $S^{2n-1}$  as

submanifold of  $R^{4n}$  admits a nontrivial Killing vector field. It is worth pointing out that on the tangent bundle  $TS^{2n-1}$  with Sasakian metric no vertical or horizontal lift of a vector field is Killing as this will require the corresponding vector field on  $S^{2n-1}$  is parallel which is impossible as  $S^{2n-1}$  is space of constant curvature 1. Note that on even dimensional Riemannian manifolds which are irreducible, it is difficult to find Killing vector fields, where as on products like  $S^{2k-1} \times S^{2l-1}$ ,  $S^{2k-1} \times R^{2l-1}$ ,  $R^{2k-1} \times R^{2l-1}$  one can easily find Killing vector fields. Since the tangent bundle  $TS^{2n-1}$  is trivial for  $n = 1, 2, 4$ , finding Killing vector fields is easy in these dimensions, but for  $n \geq 5$ , it is not trivial.

## 2. PRELIMINARIES

Let  $(M, g)$  be a Riemannian manifold and  $TM$  be its tangent bundle with projection map  $\pi : TM \rightarrow M$ . Then for each  $(p, u) \in TM$ , the tangent space  $T_{(p,u)}TM = \mathfrak{H}_{(p,u)} \oplus \mathfrak{V}_{(p,u)}$ , where  $\mathfrak{V}_{(p,u)}$  is the kernel of  $d\pi_{(p,u)} : T_{(p,u)}(TM) \rightarrow T_pM$  and  $\mathfrak{H}_{(p,u)}$  is the kernel of the connection map  $K_{(p,u)} : T_{(p,u)}(TM) \rightarrow T_pM$  with respect to the Riemannian connection on  $(M, g)$ . The subspaces  $\mathfrak{H}_{(p,u)}$ ,  $\mathfrak{V}_{(p,u)}$  are called the horizontal and vertical subspaces respectively. Consequently, the Lie algebra of smooth vector fields  $\mathfrak{X}(TM)$  on the tangent bundle  $TM$  admits the decomposition  $\mathfrak{X}(TM) = \mathfrak{H} \oplus \mathfrak{V}$  where  $\mathfrak{H}$  is called the horizontal distribution and  $\mathfrak{V}$  is called the vertical distribution on the tangent bundle  $TM$ . For each  $X_p \in T_pM$ , the horizontal lift of  $X_p$  to a point  $z = (p, u) \in TM$  is the unique vector  $X_z^h \in \mathfrak{H}_z$  such that  $d\pi(X_z^h) = X_p \circ \pi$  and the vertical lift of  $X_p$  to a point  $z = (p, u) \in TM$  is the unique vector  $X_z^v \in \mathfrak{V}_z$  such that  $X_z^v(df) = X_p(f)$  for all functions  $f \in C^\infty(M)$ , where  $df$  is the function defined by  $(df)(p, u) = u(f)$ . Also for a vector field  $X \in \mathfrak{X}(M)$ , the horizontal lift of  $X$  is a vector field  $X^h \in \mathfrak{X}(TM)$  whose value at a point  $(p, u)$  is the horizontal lift of  $X(p)$  to  $(p, u)$ , the vertical lift  $X^v$  of  $X$  is defined similarly. For  $X \in \mathfrak{X}(M)$  the horizontal and vertical lifts  $X^h, X^v$  of  $X$  are uniquely determined vector fields on  $TM$  satisfying

$$d\pi(X_z^h) = X_{\pi(z)}, K(X_z^h) = 0, d\pi(X_z^v) = 0, K(X_z^v) = X_{\pi(z)}$$

Also, we have for a smooth function  $f \in C^\infty(M)$  and vector fields  $X, Y \in \mathfrak{X}(M)$ , that  $(fX)^h = (f \circ \pi)X^h$ ,  $(fX)^v = (f \circ \pi)X^v$ ,  $(X + Y)^h = X^h + Y^h$  and  $(X + Y)^v = X^v + Y^v$ . If  $\dim M = m$  and  $(U, \varphi)$  is a chart on  $M$  with local coordinates  $x^1, x^2, \dots, x^m$ , then  $(\pi^{-1}(U), \varphi)$  is a chart on  $TM$  with local coordinates  $x^1, x^2, \dots, x^m, y^1, y^2, \dots, y^m$ , where  $x^i = x^i \circ \pi$  and  $y^i = dx^i$ ,  $i = 1, 2, \dots, m$ .

A Riemannian metric  $\bar{g}$  on the tangent bundle  $TM$  is said to be natural metric with respect to  $g$  on  $M$  if  $\bar{g}_{(p,u)}(X^h, Y^h) = g_p(X, Y)$  and  $\bar{g}_{(p,u)}(X^h, Y^v) = 0$ , for all vectors fields  $X, Y \in \mathfrak{X}(M)$  and  $(p, u) \in TM$ , that is the projection map  $\pi : TM \rightarrow M$  is a Riemannian submersion.

Let  $M$  be an orientable hypersurface of the Euclidean space  $R^{2n}$  with immersion  $f : M \rightarrow R^{2n}$  and  $TM$  be its tangent bundle. Then as  $F = df : TM \rightarrow R^{4n} = TR^{2n}$  is also an immersion,  $TM$  is an immersed submanifold of the Euclidean space  $R^{4n}$ . We denote the induced metrics on  $M, TM$  by  $g, \bar{g}$  respectively and the Euclidean metric on  $R^{2n}$  as well as on  $R^{4n}$  by  $\langle, \rangle$ . Also, we denote by  $\bar{\nabla}, \bar{\nabla}, D$  and  $\bar{D}$  the Riemannian connections on  $M, TM, R^{2n}$ , and  $R^{4n}$  respectively. Let  $N$  and  $S$  be the unit normal vector field and the shape operator of the hypersurface  $M$ . For the hypersurface  $M$  of the Euclidean space  $R^{2n}$  we have the following Gauss and Weingarten formulae

$$D_X Y = \bar{\nabla}_X Y + \langle S(X), Y \rangle N, \quad D_X N = -S(X), \quad X, Y \in \mathfrak{X}(M) \quad (2.1)$$

where  $S$  is the shape operator (Weingarten map). Similarly for the submanifold  $TM$  of the Euclidean space  $R^{4n}$  we have the Gauss and Weingarten formulae

$$\bar{D}_E F = \bar{\nabla}_E F + h(E, F), \quad \bar{D}_E \bar{N} = -\bar{S}_{\bar{N}}(E) + \bar{\nabla}_E^\perp \bar{N} \quad (2.2)$$

where  $E, F \in \mathfrak{X}(TM)$ ,  $\bar{\nabla}^\perp$  is the connection in the normal bundle of  $TM$  and  $\bar{S}_{\bar{N}}$  denotes the Weingarten map in the direction of the normal  $\bar{N}$  and is related to the second fundamental form  $h$  by

$$\langle h(X, Y), \bar{N} \rangle = \bar{g}(\bar{S}_{\bar{N}}(X), Y) \quad (2.3)$$

Also we observe that for  $X \in \mathfrak{X}(M)$  the vertical lift  $X^v$  of  $X$  to  $TM$ , as  $X^v \in \ker d\pi$ , where  $\pi : TM \rightarrow M$  is the natural submersion, we have  $d\pi(X^v) = 0$  that is  $df(d\pi(X^v)) = 0$  or equivalently we get  $d(f \circ \pi)(X^v) = 0$ , that is  $d(\bar{\pi} \circ F)(X^v) = 0$  ( $\bar{\pi} : TR^{2n} \rightarrow R^{2n}$ ), which gives  $dF(X^v) \in \ker d\bar{\pi} = \bar{\mathfrak{V}}$ .

Now we state the following results which are needed in our work.

**Lemma 2.1.** [3] *Let  $N$  be the unit normal vector field to the hypersurface  $M$  of  $R^{2n}$  and  $P = (p, X_p) \in TM$ . Then the horizontal and vertical lifts  $Y_P^h, Y_P^v$  of  $Y_p \in T_p M$  satisfy*

$$dF_P(Y_P^h) = (df_p(Y_p))^h + V_P, \quad dF_P(Y_P^v) = (df_p(Y_p))^v$$

where  $V_P \in \mathfrak{V}_P$  is given by  $V_P = \langle S_p(X_p), Y_p \rangle N_P^v$ ,  $N_P^v$  being the vertical lift of the unit normal  $N$  to with respect to the tangent bundle  $\bar{\pi} : R^{4n} \rightarrow R^{2n}$ .

**Lemma 2.2.** [3] *If  $(M, g)$  is an orientable hypersurface of  $R^{2n}$ , and  $(TM, \bar{g})$  is its tangent bundle as submanifold of  $R^{4n}$ , then the metric  $\bar{g}$  on  $TM$  for  $P = (p, u) \in TM$ , satisfies:*

- (i)  $\bar{g}_P(X_P^h, Y_P^h) = g_p(X_p, Y_p) + g_p(S_p(X_p), u)g_p(S_p(Y_p), u)$ .
- (ii)  $\bar{g}_P(X_P^h, Y_P^v) = 0$ .
- (ii)  $\bar{g}(X^v, Y^v) = g_p(X_p, Y_p)$ .

*Remark 2.3.* It is well known that a metric  $\bar{g}$  defined on  $TM$  using the Riemannian metric  $g$  of  $M$  (such as Sasaki metric, Cheeger-Gromoll metric) are

natural metrics in the sense that the submersion  $\pi : (TM, \bar{g}) \rightarrow (M, g)$  becomes a Riemannian submersion with respect to these metrics. However, as seen from above Lemmas, the induced metric on the tangent bundle  $TM$  of a hypersurface  $M$  of the Euclidean space  $R^{2n}$ , as a submanifold of  $R^{4n}$  is not a natural metric because of the present of the term  $g_p(S_p(X_p), u)g_p(S_p(Y_p), u)$  in the inner product of horizontal vectors on  $TM$ . Moreover, note that the for an orientable hypersurface  $M$  of the Euclidean space  $R^{2n}$ , the vertical lift  $N^v$  of the unit normal is tangential to the submanifold  $TM$  of  $R^{4n}$  as seen in 2.1

In what follows, we drop the suffixes like in  $g_p(S_p(X_p), u)$  and and it will be understood from the context of the entities appearing in the equations.

**Theorem 2.4.** [3] *Let  $(M, g)$  be an orientable hypersurface of  $R^{2n}$ , and  $(TM, \bar{g})$  be its tangent bundle as submanifold of  $R^{4n}$ . If  $\nabla$  and  $\bar{\nabla}$  denote the Riemannian connections on  $(M, g)$  and  $(TM, \bar{g})$  respectively, then*

$$\begin{aligned} (i) \quad \bar{\nabla}_{X^h} Y^h &= (\bar{\nabla}_X Y)^h - \frac{1}{2}(R(X, Y)u)^v, \\ (ii) \quad \bar{\nabla}_{X^v} Y^h &= g(S(X), Y) \circ \pi N^v \\ (iii) \quad \bar{\nabla}_{X^v} Y^v &= 0, \quad (iv) \quad \bar{\nabla}_{X^h} Y^v = (\bar{\nabla}_X Y)^v + g(S(X), Y) \circ \pi N^v. \end{aligned}$$

**Lemma 2.5.** [4] *Let  $TM$  be the tangent bundle of an orientable hypersurface  $M$  of  $R^{2n}$ . Then for  $X, Y \in \mathfrak{X}(M)$ ,*

$$\begin{aligned} (i) \quad h(X^v, Y^v) &= 0, \\ (ii) \quad h(X^v, Y^h) &= 0, \\ (iii) \quad h(X^h, Y^h) &= g(S(X), Y) \circ \pi N^h. \end{aligned}$$

**Lemma 2.6.** [4] *For the tangent bundle  $TM$  of an orientable hypersurface  $M$  of  $R^{2n}$  and  $X \in \mathfrak{X}(M)$ , we have*

$$\begin{aligned} (i) \quad \bar{D}_{X^v} N^v &= 0, \\ (ii) \quad \bar{D}_{X^v} N^h &= 0, \\ (iii) \quad \bar{D}_{X^h} N^v &= -(S(X))^v, \quad (iv) \quad \bar{D}_{X^h} N^h = -(S(X))^h. \end{aligned}$$

Let  $J$  be the natural complex structure on the Euclidean space  $R^{2n}$ , which makes  $(R^{2n}, J, \langle, \rangle)$  a Kaehler manifold. Then on an orientable real hypersurface  $M$  of  $R^{2n}$  with unit normal  $N$ , we define a unit vector field  $\xi \in \mathfrak{X}(M)$  by  $\xi = -JN$ , with its dual 1-form  $\eta(X) = g(X, \xi)$ , where  $g$  is the induced metric on  $M$ . For  $X \in \mathfrak{X}(M)$ , we express  $JX = \varphi(X) + \eta(X)N$ , where  $\varphi(X)$  is the tangential component of  $JX$ , and it follows that  $\varphi$  is a  $(1, 1)$  tensor field on  $M$ , and that  $(\varphi, \xi, \eta, g)$  defines an almost contact metric structure on  $M$  (cf. [5], [8], [9]), that is

$$\varphi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \eta \circ \varphi = 0, \quad \varphi(\xi) = 0$$

and

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad X, Y \in \mathfrak{X}(M)$$

Moreover, we have the following.

**Lemma 2.7.** [8] *Let  $M$  be an orientable real hypersurface of  $R^{2n}$ . Then the structure  $(\varphi, \xi, \eta, g)$  on  $M$  satisfies*

- (i)  $(\bar{\nabla}_X \varphi)(Y) = \eta(Y)SX - g(SX, Y)\xi,$
- (ii)  $\bar{\nabla}_X \xi = \varphi SX, X, Y \in \mathfrak{X}(M).$

### 3. A STRUCTURE ON $(TM, \bar{g})$

We know that the Euclidean space  $R^{4n}$  has many complex structures, however in this section we treat  $R^{4n}$  as the tangent bundle of  $R^{2n}$  and consider a specific complex structure on the Euclidean space  $R^{4n}$ . Let  $\bar{\pi} : R^{4n} = TR^{2n} \rightarrow R^{2n}$  be the submersion of the tangent bundle of  $R^{2n}$ . Then it is easy to show that the Euclidean metric  $\langle, \rangle$  on the tangent bundle  $R^{4n}$  is Sasaki metric and using the canonical almost complex structure  $J$  of  $R^{2n}$ , we define  $\bar{J} : \mathfrak{X}(R^{4n}) \rightarrow \mathfrak{X}(R^{4n})$  by

$$\bar{J}(E^h) = (JE)^h, \quad \bar{J}(E^v) = (JE)^v, \quad E \in \mathfrak{X}(R^{2n})$$

and it easily follows that  $\bar{J}$  is an almost complex structure, satisfying  $\langle \bar{J}E, \bar{J}F \rangle = \langle E, F \rangle$  with respect to the Euclidean metric  $\langle, \rangle$  on  $R^{4n}$  and that  $(\bar{D}_E \bar{J})(F) = 0, E, F \in \mathfrak{X}(R^{4n})$  that is  $(R^{4n}, \bar{J}, \langle, \rangle)$  is a Kaehler manifold. Regarding the complex structure  $\bar{J}$  defined above, we have the following

**Lemma 3.1.** *Let  $\bar{\pi} : R^{4n} \rightarrow R^{2n}$  be the submersion of the tangent bundle  $R^{4n} = TR^{2n}$ . Then complex structure  $\bar{J}$  on  $R^{4n}$  satisfies*

$$J \circ d\bar{\pi} = d\bar{\pi} \circ \bar{J}$$

*Proof.* Take  $X \in \mathfrak{X}(R^{2n})$ , then for the horizontal lift  $X^h$ , we have:

$$J \circ d\bar{\pi}(X^h) = J(d\bar{\pi}(X^h)) = JX \circ \bar{\pi}$$

and

$$d\bar{\pi} \circ \bar{J}(X^h) = d\bar{\pi}(JX)^h = JX \circ \bar{\pi}$$

which proves

$$J \circ d\bar{\pi}(X^h) = d\bar{\pi} \circ \bar{J}(X^h)$$

Similarly for the vertical lift  $X^v$  we have

$$J \circ d\bar{\pi}(X^v) = J(d\bar{\pi}(X^v)) = 0$$

and

$$d\bar{\pi} \circ \bar{J}(X^v) = d\bar{\pi}(JX)^v = 0$$

This proves the Lemma. □

*Remark 3.2.* If  $M$  is an orientable real hypersurface of the Euclidean space  $R^{2n}$  with immersion  $f$ , then  $F = df$  is the immersion of the tangent bundle  $TM$  into the Euclidean space  $R^{4n}$  and as immersions are local embeddings, in general, we identify the local quantities on submanifold with those of the ambient space for instance we identify  $df(X)$  with  $X$  for  $X \in \mathfrak{X}(M)$ . However,

while dealing with the immersion  $F$  of  $TM$  in  $R^{4n}$  one need to be cautious specially while dealing with the horizontal lifts (cf. 2.1). Therefore in what follows, we shall bring  $dF$  in to play whenever it is needed specially in the case of horizontal lifts.

Observe that if  $M$  is an orientable real hypersurface of the Euclidean space  $R^{2n}$  with unit normal vector field  $N$ , then we know that horizontal lift  $N^h$  is a unit normal vector field to the submanifold  $TM$  of  $R^{4n}$  and that the vertical lift  $N^v \in \mathfrak{X}(TM)$  (cf.[1]). We have

$$\bar{J}N^h = (JN)^h = -(df(\xi))^h = -dF(\xi^h) + g(S(\xi), u)N^v \in \mathfrak{X}(TM) \quad (3.1)$$

and

$$\bar{J}N^v = (JN)^v = -\xi^v \in \mathfrak{X}(TM) \quad (3.2)$$

Let  $M$  be an orientable real hypersurface of the Kaehler manifold  $(R^{2n}, J, \langle, \rangle)$ . Then as  $TM$  is submanifold of the Kaehler manifold  $(R^{4n}, \bar{J}, \langle, \rangle)$ , we denote by  $\Gamma(T^\perp TM)$  the space of smooth normal vector fields to  $TM$ . The restriction of the complex structure  $\bar{J}$  on  $R^{4n}$  to  $\mathfrak{X}(TM)$  and  $\Gamma(T^\perp TM)$  can be expressed as

$$\bar{J}(E) = \bar{\varphi}(E) + \bar{\psi}(E), \quad \bar{J}(\bar{N}) = \bar{G}(\bar{N}) + \bar{\chi}(\bar{N}), \quad E \in \mathfrak{X}(TM), \quad \bar{N} \in \Gamma(T^\perp TM)$$

where  $\bar{\varphi}(E), \bar{G}(\bar{N})$  are the tangential and  $\bar{\psi}(E), \bar{\chi}(\bar{N})$  are the normal components of  $\bar{J}E$ , and  $\bar{J}(\bar{N})$  respectively. Note that the horizontal lift  $N^h$  of the unit normal  $N$  to the hypersurface  $M$  is normal to  $TM$  that is  $N^h \in \Gamma(T^\perp TM)$ , where as the vertical lift  $N^v \in \mathfrak{X}(TM)$ .

**Lemma 3.3.** *Let  $TM$  be the tangent bundle of an orientable real hypersurface of  $R^{2n}$ . Then for  $X \in \mathfrak{X}(M)$ ,*

$$\begin{aligned} \bar{\varphi}(X^h) &= (\varphi(X))^h - g(S(X), u)\xi^v, & \bar{\varphi}(X^v) &= (\varphi(X))^v + \eta(X) \circ \pi N^v \\ \bar{\psi}(X^h) &= \eta(X) \circ \pi N^h, & \bar{\psi}(X^v) &= 0 \end{aligned}$$

*Proof.* Note that for the horizontal lift  $X^h$  we have

$$\begin{aligned} \bar{J}X^h &= \bar{J}dF(X^h) = \bar{J}((df(X))^h + g(SX, u) \circ \pi N^v) \\ &= (Jdf(X))^h + g(SX, u) \circ \pi (JN)^v \\ &= (\varphi X + \eta(X)N)^h - g(SX, u) \circ \pi \xi^v \\ &= (\varphi(X))^h - g(SX, u) \circ \pi \xi^v + \eta(X) \circ \pi N^h \end{aligned}$$

which together with the definition  $\bar{J}X^h = \bar{\varphi}(X^h) + \bar{\psi}(X^h)$ , on equating tangential and normal components give

$$\bar{\varphi}(X^h) = (\varphi(X))^h - g(S(X), u)\xi^v \text{ and } \bar{\psi}(X^h) = \eta(X) \circ \pi N^h$$

Similarly for the vertical lift  $X^v$ , we have

$$\bar{J}X^v = \bar{\varphi}(X^v) + \bar{\psi}(X^v) = (JX)^v = (\varphi X + \eta(X)N)^v$$

which gives

$$(\overline{\varphi}(X^v)) + \overline{\psi}(X^v) = (\varphi X)^v + \eta(X) \circ \pi N^v$$

Comparing the tangential and normal components we conclude

$$\overline{\varphi}(X^v) = (\varphi(X))^v + \eta(X) \circ \pi N^v, \quad \text{and} \quad \overline{\psi}(X^v) = 0.$$

□

We choose a unit normal vector field  $N^* \in \Gamma(T^\perp TM)$  such that  $\{N^*, N^h\}$  is a local orthonormal frame of normals for the submanifold  $TM$ . It is known that  $N^*$  is vertical vector field on the tangent bundle  $R^{4n}$  (cf. [1]). Since,  $\langle \overline{J}N^*, N^* \rangle = 0$ ,  $\langle \overline{J}N^*, N^h \rangle = \langle N^*, \xi^h \rangle = 0$ , it follows that  $\overline{J}N^* \in \mathfrak{X}(TM)$  and we define unit vector field  $\zeta \in \mathfrak{X}(TM)$  by

$$\zeta = -\overline{J}N^* \tag{3.3}$$

Now, for any normal vector field  $\overline{N} \in \Gamma(T^\perp TM)$ , we have

$$\overline{N} = \langle \overline{N}, N^* \rangle N^* + \langle \overline{N}, N^h \rangle N^h$$

which together with equations (3.1), (3.2) and (3.3) gives  $\overline{\chi}(\overline{N}) = 0$  and that  $\overline{J}(\overline{N}) \in \mathfrak{X}(TM)$ , is given by

$$\overline{J}(\overline{N}) = \langle \overline{J}(\overline{N}), \zeta \rangle \zeta + \langle \overline{J}(\overline{N}), T \rangle T \tag{3.4}$$

where  $T \in \mathfrak{X}(TM)$ , is given by

$$T = \xi^h - g(S(\xi), u)N^v = -\overline{J}N^h \tag{3.5}$$

Also, using equation (3.2), we have

$$-\xi^v = \overline{J}N^v = \overline{\varphi}(N^v) + \overline{\psi}(N^v)$$

which gives

$$\overline{\varphi}(N^v) = -\xi^v \quad \text{and} \quad \overline{\psi}(N^v) = 0 \tag{3.6}$$

Moreover, we have

$$\overline{\varphi}(\zeta) = 0 \quad \text{and} \quad \overline{\psi}(\zeta) = N^*, \quad \overline{\psi}(\xi^h) = N^h \tag{3.7}$$

If we denote by  $\alpha, \beta$  the smooth 1-forms on  $TM$  dual to the vector field  $\zeta$  and  $T$  respectively, then for  $E \in \mathfrak{X}(TM)$ , it follows that

$$\overline{J}(\overline{\psi}(E)) = -\alpha(E)\zeta - \beta(E)T$$

and consequently, operating  $\overline{J}$  on  $\overline{J}(E) = \overline{\varphi}(E) + \overline{\psi}(E)$ ,  $E \in \mathfrak{X}(TM)$ , we get

$$\overline{\varphi}^2 = -I + \alpha \otimes \zeta + \beta \otimes T \quad \text{and} \quad \overline{\psi} \circ \overline{\varphi} = 0 \tag{3.8}$$

Using Lemma 2.1 and equations (3.3), (3.5), (3.6), (3.8), we see that the vector fields  $\zeta, T$  and 1-forms  $\alpha, \beta$  satisfy

$$\overline{\varphi}(\zeta) = 0, \quad \overline{\varphi}(T) = 0, \quad \overline{g}(\zeta, T) = 0, \quad \alpha \circ \overline{\varphi} = 0, \quad \beta \circ \overline{\varphi} = 0 \tag{3.9}$$



Also, as  $\bar{g}$  is the induced metric on the submanifold  $TM$  and  $\bar{J}$  is skew symmetric with respect to the Hermitian metric  $\langle, \rangle$ , we have

$$\bar{g}(\bar{\varphi}(E), F) = -\bar{g}(E, \bar{\varphi}(F)), \quad E, F \in \mathfrak{X}(TM) \quad (3.10)$$

Then using equations (3.8), (3.9) and (3.10), we have

$$\bar{g}(\bar{\varphi}(E), \bar{\varphi}(F)) = \bar{g}(E, F) - \alpha(E)\alpha(F) - \beta(E)\beta(F), \quad E, F \in \mathfrak{X}(TM) \quad (3.11)$$

Thus we have proved the following

**Lemma 3.4.** *Let  $TM$  be the tangent bundle of an orientable real hypersurface of  $R^{2n}$ . Then there is a structure  $(\bar{\varphi}, \zeta, T, \alpha, \beta, \bar{g})$  similar to contact metric structure on  $TM$ , where  $\bar{\varphi}$  is a tensor field of type  $(1, 1)$ ,  $\zeta, T$  are smooth vector fields and  $\alpha, \beta$  are smooth 1-forms dual to  $\zeta, T$  with respect to the Riemannian metric  $\bar{g}$  satisfying*

$$\begin{aligned} \bar{\varphi}^2 &= -I + \alpha \otimes \zeta + \beta \otimes T, \quad \bar{\varphi}(\zeta) = 0, \quad \bar{\varphi}(T) = 0, \quad \alpha \circ \bar{\varphi} = 0, \quad \beta \circ \bar{\varphi} = 0, \quad \bar{g}(\zeta, T) = 0 \\ \bar{g}(\bar{\varphi}(E), \bar{\varphi}(F)) &= \bar{g}(E, F) - \alpha(E)\alpha(F) - \beta(E)\beta(F), \quad E, F \in \mathfrak{X}(TM). \end{aligned}$$

In the next Lemma, we compute the co-variant derivatives of the tensor  $\bar{\varphi}$ .

**Lemma 3.5.** *Let  $(\bar{\varphi}, \zeta, T, \alpha, \beta, \bar{g})$  be the structure on the tangent bundle  $TM$  of an orientable real hypersurface  $M$  of the Euclidean space  $R^{2n}$ . Then*

- (i)  $(\bar{\nabla}_{X^h} \bar{\varphi})(Y^h) = \{(\nabla_X \varphi)(Y)\}^h - \{X(g(SY, u) + g(SY, u)JSX)\}^v$
- (ii)  $(\bar{\nabla}_{X^h} \bar{\varphi})(Y^v) = 0,$
- (iii)  $(\bar{\nabla}_{X^v} \bar{\varphi})(Y^v) = 0, (\bar{\nabla}_{X^v} \bar{\varphi})(Y^h) = g(SX, \varphi Y) \circ \pi N^v + g(SX, Y) \circ \pi \xi^v.$

*Proof.* Using the definition of  $\bar{J}$ , Lemma 2.1 and Lemma 3.3 together with equation (3.1), we get for  $X, Y \in \mathfrak{X}(M)$

$$\begin{aligned} \bar{J}Y^h &= \bar{J}dF(Y^h) = \bar{J}\left((df(Y))^h + g(SY, u) \circ \pi N^v\right) \\ &= (\varphi Y + \eta(Y)N)^h - g(SY, u) \circ \pi \xi^v \\ &= \bar{\varphi}(Y^h) + \eta(Y) \circ \pi N^h \end{aligned}$$

which gives

$$\begin{aligned} \bar{D}_{X^h} \bar{J}Y^h &= \bar{D}_{(df(X))^h + g(SX, u) \circ \pi N^v} (\bar{\varphi}(Y^h) + \eta(Y) \circ \pi N^h) \\ &= \bar{D}_{(df(X))^h} \bar{\varphi}(Y^h) + X(\eta(Y)) \circ \pi N^h + \eta(Y) \circ \pi \bar{D}_{(df(X))^h} N^h \\ &\quad + g(SX, u) \circ \pi \bar{D}_{N^v} ((\varphi(Y))^h - g(S(Y), u) \circ \pi \xi^v) + 0 \\ &\quad + g(SX, u) \circ \pi \eta(Y) \circ \pi \bar{D}_{N^v} N^h \end{aligned}$$

Note that the tangent bundle  $TR^{2n} = R^{4n}$  has Sasaki metric and thus using Lemma 7.2 of [10] (keeping in view that  $R^{2n}$  is flat), in the above equation, we get

$$\bar{D}_{X^h} \bar{J}Y^h = \bar{\nabla}_{X^h} \bar{\varphi}(Y^h) + h(X^h, \bar{\varphi}(Y^h)) + X(\eta(Y)) \circ \pi N^h - \eta(Y) \circ \pi (SX)^h \quad (3.12)$$

Similarly we have

$$\begin{aligned}
 \overline{J}\overline{D}_{X^h}Y^h &= \overline{J}\left(\overline{D}_{((df(X))^h+g(SX,u)\circ\pi N^v)}\left((df(Y))^h+g(SY,u)\circ\pi N^v\right)\right) \\
 &= \overline{J}\left\{\overline{\nabla}_{X^h}Y^h+h(X^h,Y^h)+X(g(SY,u)\circ\pi N^v+g(SY,u)\circ\pi(D_XN)^v\right. \\
 &\quad \left.+g(SX,u)\circ\pi\overline{D}_{N^v}(dfY)^h+0+0\right\} \\
 &= \overline{\varphi}\left(\overline{\nabla}_{X^h}Y^h\right)+\overline{\psi}\left(\overline{\nabla}_{X^h}Y^h\right)+\overline{J}h(X^h,Y^h)-X(g(SY,u)\circ\pi\xi^v \\
 &\quad -g(SY,u)\circ\pi\overline{J}(SX)^v) \\
 &= \overline{\varphi}\left(\overline{\nabla}_{X^h}Y^h\right)+\overline{\psi}\left(\overline{\nabla}_{X^h}Y^h\right)-g(SX,Y)\circ\pi\xi^h-X(g(SY,u)\circ\pi\xi^v \\
 &\quad -g(SY,u)\circ\pi(\varphi SX)^v-g(SY,u)\circ\pi\eta(SX)N^v) \quad (3.13)
 \end{aligned}$$

where we used Lemmas 2.3, 2.4 and Lemma 7.2 in [10]. Now as  $(R^{4n}, \overline{J}, \langle, \rangle)$  is a Kaehler manifold, the equations (3.12) and (3.13) on comparing tangential we get

$$(\overline{\nabla}_{X^h}\overline{\varphi})(Y^h) = \{(\nabla_X\varphi)(Y)\}^h - \{X(g(SY,u) + g(SY,u)JSX)\}^v$$

which proves (i).

Now, using  $h(X^v, Y^v) = 0$  and  $\overline{S}_{\overline{\psi}(Y^v)}X^v = 0$  together with  $\overline{D}_{X^v}\overline{J}Y^v = \overline{J}\overline{D}_{X^v}Y^v$ , and comparing tangential components, we immediately arrive at

$$(\overline{\nabla}_{X^v}\overline{\varphi})(Y^v) = 0$$

Next, we have  $\overline{\nabla}_{X^v}\overline{\varphi}(Y^h) = \overline{\nabla}_{X^v}\left((\varphi Y)^h - g(SX, u)\circ\pi\xi^v\right) = \overline{\nabla}_{X^v}(\varphi Y)^h = g(SX, \varphi Y)\circ\pi N^v$  and  $\overline{\varphi}(\overline{\nabla}_{X^v}Y^h) = g(SX, \varphi Y)\circ\pi\overline{\varphi}(N^v) = -g(SX, Y)\circ\pi\xi^v$ . Thus, we get

$$(\overline{\nabla}_{X^v}\overline{\varphi})(Y^h) = g(SX, \varphi Y)\circ\pi N^v + g(SX, Y)\circ\pi\xi^v$$

Finally, using  $h(X^h, Y^v) = 0$  and  $\overline{S}_{\overline{\psi}(Y^v)}X^h = 0$  together with  $\overline{D}_{X^h}\overline{J}Y^v = \overline{J}\overline{D}_{X^h}Y^v$ , and comparing tangential components, we immediately arrive at

$$(\overline{\nabla}_{X^h}\overline{\varphi})(Y^v) = 0$$

□

**Lemma 3.6.** *Let  $(\overline{\varphi}, \zeta, T, \alpha, \beta, \overline{g})$  be the structure on the tangent bundle  $TM$  of an orientable real hypersurface  $M$  of the Euclidean space  $R^{2n}$ . Then for  $E \in \mathfrak{X}(TM)$ ,*

$$\overline{\nabla}_E\zeta = \overline{\varphi}(\overline{S}_{N^*}(E)) - \overline{J}\left(\overline{\nabla}_E^\perp N^*\right), \quad h(E, \zeta) = \overline{\psi}(\overline{S}_{N^*}(E))$$

$$\overline{\nabla}_ET = \overline{\varphi}(\overline{S}_{N^h}(E)) - \overline{J}\left(\overline{\nabla}_E^\perp N^h\right), \quad h(E, T) = \overline{\psi}(\overline{S}_{N^h}(E))$$

*Proof.* Using equation (2.2), we have

$$\begin{aligned}\bar{\nabla}_E \zeta &= \bar{D}_E \zeta - h(E, \zeta) \\ &= -\bar{J} \bar{D}_E N^* - h(E, \zeta) \\ &= \bar{J} (\bar{S}_{N^*}(E)) - \bar{J} (\bar{\nabla}_E^\perp N^*) - h(E, \zeta) \\ &= \bar{\varphi} (\bar{S}_{N^*}(E)) + \bar{\psi} (\bar{S}_{N^*}(E)) - \bar{J} (\bar{\nabla}_E^\perp N^*) - h(E, \zeta)\end{aligned}$$

Since  $\bar{J}(\bar{N}) \in \mathfrak{X}(TM)$  for each normal  $\bar{N} \in \Gamma(T^\perp TM)$ , equation tangential and normal components in above equation, we get the first part. The second part follows similarly using  $T = -\bar{J}N^h$ .  $\square$

Now, we prove the following:

**Theorem 3.7.** *The tangent bundle  $TM$  of an orientable real hypersurface  $M$  of the Euclidean space  $R^{2n}$  is a CR-submanifold of the Kaehler manifold  $(R^{4n}, \bar{J}, \langle, \rangle)$ .*

*Proof.* Use the structure  $(\bar{\varphi}, \zeta, T, \alpha, \beta, \bar{g})$  on the submanifold  $TM$  of  $R^{4n}$  to define the distribution  $D$  by

$$D = \{E \in \mathfrak{X}(TM) : \alpha(E) = \beta(E) = 0\}$$

and  $D^\perp$  be the distribution spanned by the orthogonal vector fields  $\zeta$  and  $T$ . Note that  $\zeta$  is unit vector field on  $TM$  and the length of the vector field  $T$  satisfies

$$\|T\|^2 = 1 + 2g(S(\xi), u)^2 \geq 1$$

which shows that  $D^\perp$  is 2-dimensional distribution on  $TM$  and that  $\bar{J}D^\perp = \Gamma(T^\perp TM)$ . It is easy to see that  $D$  and  $D^\perp$  are orthogonal complementary distributions and that  $\dim D = 4(n-1)$ . Note that for  $E \in \mathfrak{X}(TM)$ , we have

$$\bar{\psi}(E) = \langle \bar{\psi}(E), N^* \rangle N^* + \langle \bar{\psi}(E), N^h \rangle N^h = \alpha(E)N^* + \beta(E)N^h$$

and consequently if  $E \in D$ , then above equation gives  $\bar{J}E = \bar{\varphi}E$  which is orthogonal to both  $\zeta$  and  $T$  and that  $\bar{J}E \in D$ , which implies  $\bar{J}D = D$ . This proves that  $TM$  is a CR-submanifold of the Kaehler manifold  $(R^{4n}, \bar{J}, \langle, \rangle)$  (cf. [8]).  $\square$

#### 4. KILLING VECTOR FIELDS ON $TM$

Let  $TM$  be the tangent bundle of an orientable real hypersurface  $M$  of the Euclidean space  $R^{2n}$ . Recall that a vector field  $\varsigma \in \mathfrak{X}(TM)$  on the Riemannian manifold  $(TM, \bar{g})$  is said to be Killing if

$$(\mathcal{L}_\varsigma \bar{g})(E, F) = 0, \quad E, F \in \mathfrak{X}(TM)$$

where  $\mathcal{L}_\varsigma$  is the Lie derivative with respect to the vector field  $\varsigma$ . We have seen in previous section that the tangent bundle  $(TM, \bar{g})$  admits a structure  $(\bar{\varphi}, \zeta, T, \alpha, \beta, \bar{g})$ , that is similar to the almost contact structure. In this section

we are interested in finding conditions under which the special vector fields  $\zeta$  and  $T$  are Killing vector fields and as a particular case we get that the tangent bundle  $(TS^{2n-1}, \bar{g})$  of the unit sphere  $S^{2n-1}$  in the Euclidean space  $R^{2n}$  admits a nontrivial Killing vector field.

**Theorem 4.1.** *Let  $(\bar{\varphi}, \zeta, T, \alpha, \beta, \bar{g})$  be the structure on the tangent bundle  $TM$  of an orientable real hypersurface  $M$  of the Euclidean space  $R^{2n}$ . Then the vector field  $\zeta$  is Killing.*

*Proof.* First note that on taking inner product with  $N^*$  in each part of Lemma 2.5, we conclude that  $\bar{S}_{N^*}(X^h) = 0, \bar{S}_{N^*}(X^v) = 0, X \in \mathfrak{X}(M)$  and consequently,

$$\bar{S}_{N^*}(E) = 0, \quad E \in \mathfrak{X}(TM) \tag{4.1}$$

Also using second part of equation (2.2) in (ii) and (iv) of Lemma 2.4, we conclude that  $\bar{\nabla}_E^\perp N^h = 0, E \in \mathfrak{X}(TM)$ , that is  $N^h$  is parallel on the normal bundle of  $TM$ . Moreover, we have

$$\bar{\nabla}_E^\perp N^* = \langle \bar{\nabla}_E^\perp N^*, N^h \rangle N^h = - \langle N^*, \bar{\nabla}_E^\perp N^h \rangle N^h = 0$$

that is  $N^*$  is parallel in the normal bundle of  $TM$ . Thus using equation (4.1) in Lemma 3.5, it follows that  $\zeta$  is a parallel vector field and consequently, it is a Killing vector field. □

**Theorem 4.2.** *Let  $(\bar{\varphi}, \zeta, T, \alpha, \beta, \bar{g})$  be the structure on the tangent bundle  $TM$  of an orientable real hypersurface  $M$  of the Euclidean space  $R^{2n}$ . Then the vector field  $T$  is Killing if and only if the following condition holds*

$$\bar{g}((\bar{\varphi} \circ \bar{S}_{N^h} - \bar{S}_{N^h} \circ \bar{\varphi})(X^h), Y^h) = 0, \quad X, Y \in \mathfrak{X}(M)$$

*Proof.* Since  $N^h$  is parallel in the normal bundle of  $TM$ , by Lemma 3.5, we have

$$\bar{\nabla}_E T = \bar{\varphi}(\bar{S}_{N^h}(E)), \quad E \in \mathfrak{X}(TM) \tag{4.2}$$

Also using Lemma 2.4, we conclude that

$$\bar{S}_{N^h}(X^v) = 0, \bar{S}_{N^h}(X^h) = (S(X))^h, \quad X \in \mathfrak{X}(M) \tag{4.3}$$

Then using skew-symmetry of the tensor  $\bar{\varphi}$ , and equations (4.2) and (4.3) together with Lemma 3.3, we immediately arrive at

$$(\mathcal{L}_T \bar{g})(X^v, Y^v) = 0 \tag{4.4}$$

$$\begin{aligned} (\mathcal{L}_T \bar{g})(X^h, Y^v) &= \bar{g}(\bar{\varphi} \circ \bar{S}_{N^h}(X^h), Y^v) = -\bar{g}(\bar{S}_{N^h}(X^h), \bar{\varphi}(Y^v)) \\ &= -\bar{g}(\bar{S}_{N^h}(X^h), (\varphi(Y))^v + \eta(X) \circ \pi N^v) = 0 \end{aligned} \tag{4.5}$$

$$(\mathcal{L}_T \bar{g})(X^h, Y^h) = \bar{g}((\bar{\varphi} \circ \bar{S}_{N^h} - \bar{S}_{N^h} \circ \bar{\varphi})(X^h), Y^h) \tag{4.6}$$

and the equations (4.4)-(4.6) prove the Theorem. □

Consider the unit sphere  $S^{2n-1}$  in the Euclidean space  $R^{2n}$ , whose shape operator is given by  $S = -I$ . Using Lemma 2.4, we get on the tangent bundle  $TS^{2n-1}$  that

$$\bar{S}_{N^h}(X^h) = (S(X))^h = -X^h, \quad \bar{S}_{N^h}(X^v) = 0$$

Then the Lemma 3.3 together with above equation, gives

$$\begin{aligned} (\bar{\varphi} \circ \bar{S}_{N^h} - \bar{S}_{N^h} \circ \bar{\varphi})(X^h) &= -\bar{\varphi}(X^h) - \bar{S}_{N^h}((\varphi(X))^h - g(S(X), u) \circ \pi \xi^v) \\ &= -g(X, u) \circ \pi \xi^v, \quad X \in \mathfrak{X}(S^{2n-1}) \end{aligned}$$

and consequently,

$$\bar{g}((\bar{\varphi} \circ \bar{S}_{N^h} - \bar{S}_{N^h} \circ \bar{\varphi})(X^h), Y^h) = 0, \quad X, Y \in \mathfrak{X}(S^{2n-1})$$

Thus as a particular case of the Theorem 4.2, we have

**Corollary 4.3.** *Let  $(\bar{\varphi}, \zeta, T, \alpha, \beta, \bar{g})$  be the structure on the tangent bundle  $TS^{2n-1}$  of the unit sphere  $S^{2n-1}$  in the Euclidean space  $R^{2n}$ ,  $n > 1$ . Then the vector field  $T$  is a nontrivial Killing vector field.*

*Proof.* It remains to be shown that  $T$  is nontrivial. Since,  $N^h$  is parallel in the normal bundle of  $TS^{2n-1}$ , by Lemmas 2.4 and 3.5, we have

$$\bar{\nabla}_{X^h} T = -\bar{\varphi}(X^h), \quad X \in \mathfrak{X}(S^{2n-1}) \quad (4.7)$$

where we used the fact that the shape operator  $S$  of the unit sphere  $S^{2n-1}$  is given by  $S = -I$ . The Lemma 3.4 gives the rank of  $\bar{\varphi}$  is  $4(n-1)$  and consequently, equation (4.7) gives that the Killing vector field  $T$  is not parallel, that is  $T$  is a nontrivial Killing vector field.  $\square$

#### ACKNOWLEDGMENTS

This Work is supported by King Saud University, Deanship of Scientific Research, College of Science Research Center.

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