Abstract. Let $M$ be an orientable hypersurface in the Euclidean space $\mathbb{R}^{2n}$ with induced metric $g$ and $TM$ be its tangent bundle. It is known that the tangent bundle $TM$ has induced metric $\mathcal{F}$ as submanifold of the Euclidean space $\mathbb{R}^{4n}$ which is not a natural metric in the sense that the submersion $\pi : (TM, \mathcal{F}) \to (M, g)$ is not the Riemannian submersion. In this paper, we use the fact that $\mathbb{R}^{4n}$ is the tangent bundle of the Euclidean space $\mathbb{R}^{2n}$ to define a special complex structure $\mathcal{J}$ on the tangent bundle $\mathbb{R}^{4n}$ so that $(\mathbb{R}^{4n}, \mathcal{J}, \langle \cdot, \cdot \rangle)$ is a Kaehler manifold, where $\langle \cdot, \cdot \rangle$ is the Euclidean metric which is also the Sasaki metric of the tangent bundle $\mathbb{R}^{4n}$. We study the structure induced on the tangent bundle $(TM, \mathcal{F})$ of the hypersurface $M$, which is a submanifold of the Kaehler manifold $(\mathbb{R}^{4n}, \mathcal{J}, \langle \cdot, \cdot \rangle)$. We show that the tangent bundle $TM$ is a CR-submanifold of the Kaehler manifold $(\mathbb{R}^{4n}, \mathcal{J}, \langle \cdot, \cdot \rangle)$. We find conditions under which certain special vector fields on the tangent bundle $(TM, \mathcal{F})$ are Killing vector fields. It is also shown that the tangent bundle $TS^{2n-1}$ of the unit sphere $S^{2n-1}$ admits a Riemannian metric $\mathcal{F}$ and that there exists a nontrivial Killing vector field on the tangent bundle $(TS^{2n-1}, \mathcal{F})$.

Keywords: Tangent bundle, Hypersurface, Kaehler manifold, Almost contact structure, Killing vector field, CR-Submanifold, Second fundamental form, Wiengarten map.

1. Introduction

Recently efforts are made to study the geometry of the tangent bundle of a hypersurface $M$ in the Euclidean space $\mathbb{R}^{n+1}$ (cf. [3]), where the authors have shown that the induced metric on its tangent bundle $TM$ as submanifold of the Euclidean space $\mathbb{R}^{2n+2}$ is not a natural metric. In [4], we have extended the study initiated in [3] on the geometry of the tangent bundle $TM$ of an immersed orientable hypersurface $M$ in the Euclidean space $\mathbb{R}^{n+1}$. It is well known that Killing vector fields play an important role in shaping the geometry of a Riemannian manifold, for instance the presence of nonzero Killing vector field on a compact Riemannian manifold forces its Ricci curvature to be non-negative and this in particular implies that on a compact Riemannian manifolds of negative Ricci curvature there does not exist a nonzero Killing vector field.

The study of Killing vector fields becomes more interesting on the tangent bundle $TM$ of a Riemannian manifold $(M,g)$ as the tangent bundle is noncompact. It is known that if the tangent bundle $TM$ of a Riemannian manifold $(M,g)$ is equipped with Sasaki metric, then the verticle lift of a parallel vector field on $M$ is a Killing vector field (cf. [15]). However if the Sasaki metric is replaced by the Cheeger-Gromoll metric, then the vertical lift of any nonzero vector field on $M$ is never Killing (cf. [14]). Note that both Sasaki metric as well as Cheeger-Gromoll metrics are natural metrics. We consider an orientable real hypersurface $M$ of the Euclidean space $\mathbb{R}^{2n}$ with the induced metric $g$. Then as the tangent bundle $TM$ of $M$ is a submanifold of codimension two in $\mathbb{R}^{4n}$, it has induced metric $\overline{g}$ and this metric on $TM$ is not a natural metric as the submersion $\pi : (TM,\overline{g}) \to (M,g)$ is not the Riemannian submersion (cf. [3]). Let $N$ be the unit normal vector field to the hypersurface $M$ and $J$ be the natural complex structure on the Euclidean space $\mathbb{R}^{4n}$. Then we have a globally defined unit vector field $\xi$ on the hypersurface given by $\xi = -JN$ called the characteristic vector field of the real hypersurface (cf. [1, 2, 5, 6, 7, 8, 9]), and this vector field $\xi$ gives rise to two vector fields $\xi^h$ (the horizontal lift) and $\xi^v$ (the vertical lift) on the tangent bundle $(TM,\overline{g})$. In this paper, we use the fact that $R^{4n}$ is the tangent bundle of the Euclidean space $\mathbb{R}^{2n}$ and that the projection $\pi : R^{4n} \to R^{2n}$ is a Riemannian submersion, to define a special almost complex structure $\overline{J}$ on the tangent bundle $R^{4n}$ which is different from the canonical complex structure of the Euclidean space $R^{4n}$ and show that $(R^{4n},\overline{J},\langle\cdot,\cdot\rangle)$ is a Kaehler manifold, where $\langle\cdot,\cdot\rangle$ is the Euclidean metric on $R^{4n}$. It is shown that the codimension two submanifold $(TM,\overline{g})$ of the Kaehler manifold $(R^{4n},\overline{J},\langle\cdot,\cdot\rangle)$ is a CR-submanifold (cf. [10]) and it naturally inherits certain special vector fields other than $\xi^h$ and $\xi^v$, and in this paper we are interested in finding conditions under which these special vector fields are Killing vector fields on $(TM,\overline{g})$. One of the interesting outcome of this study is, we have shown that the tangent bundle $TS^{2n-1}$ of the unit sphere $S^{2n-1}$ as
submanifold of $\mathbb{R}^{4n}$ admits a nontrivial Killing vector field. It is worth pointing out that on the tangent bundle $TS^{2n-1}$ with Sasakian metric no vertical or horizontal lift of a vector field is Killing as this will require the corresponding vector field on $S^{2n-1}$ is parallel which is impossible as $S^{2n-1}$ is space of constant curvature 1. Note that on even dimensional Riemannian manifolds which are irreducible, it is difficult to find Killing vector fields, whereas on products like $S^{2k-1} \times S^{2l-1}$, $S^{2k-1} \times R^{2l-1}$, $R^{2k-1} \times R^{2l-1}$ one can easily find Killing vector fields. Since the tangent bundle $TS^{2n-1}$ is trivial for $n = 1, 2, 4$, finding Killing vector fields is easy in these dimensions, but for $n \geq 5$, it is not trivial.

2. Preliminaries

Let $(M, g)$ be a Riemannian manifold and $TM$ be its tangent bundle with projection map $\pi : TM \rightarrow M$. Then for each $(p, u) \in TM$, the tangent space $T_{(p,u)}TM = \mathfrak{h}_{(p,u)} \oplus \mathfrak{v}_{(p,u)}$, where $\mathfrak{h}_{(p,u)}$ is the kernel of $d\pi_{(p,u)} : T_{(p,u)}TM \rightarrow T_pM$ and $\mathfrak{v}_{(p,u)}$ is the kernel of the connection map $K_{(p,u)} : T_{(p,u)}TM \rightarrow T_pM$ with respect to the Riemannian connection on $(M, g)$. The subspaces $\mathfrak{h}_{(p,u)}$, $\mathfrak{v}_{(p,u)}$ are called the horizontal and vertical subspaces respectively. Consequently, the Lie algebra of smooth vector fields $\mathfrak{x}(TM)$ on the tangent bundle $TM$ admits the decomposition $\mathfrak{x}(TM) = \mathfrak{h} \oplus \mathfrak{v}$ where $\mathfrak{h}$ is called the horizontal distribution and $\mathfrak{v}$ is called the vertical distribution on the tangent bundle $TM$. For each $X_p \in T_pM$, the horizontal lift of $X_p$ to a point $z = (p, u) \in TM$ is the unique vector $X^h_p \in \mathfrak{h}_z$ such that $d\pi(X^h_p) = X_p \circ \pi$ and the vertical lift of $X_p$ to a point $z = (p, u) \in TM$ is the unique vector $X^v_p \in \mathfrak{v}_z$ such that $X^v_p(df) = X_p(f)$ for all functions $f \in C^\infty(M)$, where $df$ is the function defined by $(df)(p, u) = u(f)$. Also for a vector field $X \in \mathfrak{x}(M)$, the horizontal lift of $X$ is a vector field $X^h \in \mathfrak{x}(TM)$ whose value at a point $(p, u)$ is the horizontal lift of $X(p)$ to $(p, u)$, the vertical lift $X^v$ of $X$ is defined similarly. For $X \in \mathfrak{x}(M)$ the horizontal and vertical lifts $X^h, X^v$ of $X$ are uniquely determined vector fields on $TM$ satisfying

$$d\pi(X^h_p) = X^h_{\pi(z)}, K(X^h_p) = 0, d\pi(X^v_p) = 0, K(X^v_p) = X^v_{\pi(z)}$$

Also, we have for a smooth function $f \in C^\infty(M)$ and vector fields $X, Y \in \mathfrak{x}(M)$, that $(fX)^h = (f \circ \pi)X^h$, $(fX)^v = (f \circ \pi)X^v$, $(X + Y)^h = X^h + Y^h$ and $(X + Y)^v = X^v + Y^v$. If $\dim M = m$ and $(U, \varphi)$ is a chart on $M$ with local coordinates $x^1, x^2, \ldots, x^m$, then $(\pi^{-1}(U), \varphi)$ is a chart on $TM$ with local coordinates $x^1, x^2, \ldots, x^m, y^1, y^2, \ldots, y^m$, where $x^i = x^i \circ \pi$ and $y^i = dx^i$, $i = 1, 2, \ldots, m$.

A Riemannian metric $\overline{g}$ on the tangent bundle $TM$ is said to be natural metric with respect to $g$ on $M$ if $\overline{g}_{(p,u)}(X^h, Y^h) = g_p(X, Y)$ and $\overline{g}_{(p,u)}(X^h, Y^v) = 0$, for all vector fields $X, Y \in \mathfrak{x}(M)$ and $(p, u) \in TM$, that is the projection map $\pi : TM \rightarrow M$ is a Riemannian submersion.
Let $M$ be an orientable hypersurface of the Euclidean space $\mathbb{R}^{2n}$ with immersion $f : M \rightarrow \mathbb{R}^{2n}$ and $TM$ be its tangent bundle. Then as $F = df : TM \rightarrow R^{4n} = TR^{2n}$ is also an immersion, $TM$ is an immersed submanifold of the Euclidean space $R^{2n}$. We denote the induced metrics on $M$, $TM$ by $g, \bar{g}$ respectively and the Euclidean metric on $\mathbb{R}^{2n}$ as well as on $R^{4n}$ by $\langle , \rangle$. Also, we denote by $\nabla, \nabla^*, D$ and $\bar{\nabla}$ the Riemannian connections on $M$, $TM$, $\mathbb{R}^{2n}$, and $R^{4n}$ respectively. Let $N$ and $S$ be the unit normal vector field and the shape operator of the hypersurface $M$. For the hypersurface $M$ of the Euclidean space $\mathbb{R}^{2n}$ we have the following Gauss and Weingarten formulæ

$$D_XY = \nabla_XY + \langle \mathbf{s}(X), Y \rangle N, D_XN = -\mathbf{s}(X), \quad X, Y \in \mathfrak{X}(M) \tag{2.1}$$

where $\mathbf{s}$ is the shape operator (Weingarten map). Similarly for the submanifold $TM$ of the Euclidean space $\mathbb{R}^{4n}$ we have the Gauss and Weingarten formulæ

$$\bar{D}_E F = \bar{\nabla}_E F + h(E, F), \quad \bar{D}_E N = -\bar{\mathbf{s}}(E) + \bar{\nabla}^* E \nabla N \tag{2.2}$$

where $E, F \in \mathfrak{X}(TM)$, $\bar{\nabla}^*$ is the connection in the normal bundle of $TM$ and $\bar{\mathbf{s}}$ denotes the Weingarten map in the direction of the normal $\nabla$ and is related to the second fundamental form $h$ by

$$\langle h(X, Y), N \rangle = \bar{g}(\bar{\mathbf{s}}(X), Y) \tag{2.3}$$

Also we observe that for $X \in \mathfrak{X}(M)$ the vertical lift $X^v$ of $X$ to $TM$, as $X^v \in \ker d\pi$, where $\pi : TM \rightarrow M$ is the natural submersion, we have $d\pi(X^v) = 0$ that is $df(\ d\pi(X^v)) = 0$ or equivalently we get $d(f \circ \pi)(X^v) = 0$, that is $d(\bar{\pi} \circ F)(X^v) = 0$ ($\bar{\pi} : TR^{2n} \rightarrow R^{2n}$), which gives $dF(X^v) \in \ker d\bar{\pi} = \mathfrak{X}$.

Now we state the following results which are needed in our work.

**Lemma 2.1.** [3] Let $N$ be the unit normal vector field to the hypersurface $M$ of $\mathbb{R}^{2n}$ and $P = (p, X_p) \in TM$. Then the horizontal and vertical lifts $Y^h_p, Y^v_p$ of $Y_p \in \mathcal{T}_p M$ satisfy

$$dF_P(Y^h_p) = (df_P(Y_p))^h + V_p, \quad dF_P(Y^v_p) = (df_P(Y_p))^v$$

where $V_p \in \mathfrak{X}_P$ is given by $V_p = \langle S_P(X_p), Y_p \rangle N^v_p$, $N^v_p$ being the vertical lift of the unit normal $N$ with respect to the tangent bundle $\pi : R^{4n} \rightarrow \mathbb{R}^{2n}$.

**Lemma 2.2.** [3] If $(M, g)$ is an orientable hypersurface of $\mathbb{R}^{2n}$, and $(TM, \bar{g})$ is its tangent bundle as submanifold of $R^{4n}$, then the metric $\bar{g}$ on $TM$ for $P = (p, u) \in TM$, satisfies:

(i) $\bar{g}_P(X^h_p, Y^h_p) = g_p(X_p, Y_p) + g_p(S_p(X_p), u)g_p(S_p(Y_p), u).

(ii) $\bar{g}_P(X^h_p, Y^v_p) = 0.$

(ii) $\bar{g}(X^v, Y^v) = g_p(X_p, Y_p).

**Remark 2.3.** It is well known that a metric $\bar{g}$ defined on $TM$ using the Riemannian metric $g$ of $M$ (such as Sasaki metric, Cheeger-Gromoll metric) are
natural metrics in the sense that the submersion \( \pi : (TM, \mathcal{G}) \rightarrow (M, g) \) becomes a Riemannian submersion with respect to these metrics. However, as seen from above Lemmas, the induced metric on the tangent bundle \( TM \) of a hypersurface \( M \) of the Euclidean space \( R^{2n} \), as a submanifold of \( R^{4n} \) is not a natural metric because of the present of the term \( g_p(S_p(X_p), u)g_p(S_p(Y_p), u) \) in the inner product of horizontal vectors on \( TM \). Moreover, note that for an orientable hypersurface \( M \) of the Euclidean space \( R^{2n} \), the vertical lift \( N^v \) of the unit normal is tangential to the submanifold \( TM \) of \( R^{4n} \) as seen in 2.1

In what follows, we drop the suffixes like in \( g_p(S_p(X_p), u) \) and and it will be understood from the context of the entities appearing in the equations.

**Theorem 2.4.** [3] Let \( (M, g) \) be an orientable hypersurface of \( R^{2n} \), and \( (TM, \mathcal{G}) \) be its tangent bundle as submanifold of \( R^{4n} \). If \( \nabla \) and \( \nabla \) denote the Riemannian connections on \( (M, g) \) and \( (TM, \mathcal{G}) \) respectively, then

1. \( \nabla_X Y^h = (\nabla_X Y)^h - \frac{1}{2}(\nabla X Y)u^v \),
2. \( \nabla_X Y^v = g(S(X), Y) \circ \pi N^v \)
3. \( \nabla_X Y^v = 0, \) (iv) \( \nabla_X Y^v = (\nabla_X Y)^v + g(S(X), Y) \circ \pi N^v \).

**Lemma 2.5.** [4] Let \( TM \) be the tangent bundle of an orientable hypersurface \( M \) of \( R^{2n} \). Then for \( X, Y \in \mathfrak{X}(M) \),

1. \( h(X^v, Y^v) = 0 \),
2. \( h(X^v, Y^h) = 0 \),
3. \( h(X^h, Y^h) = g(S(X), Y) \circ \pi N^h \).

**Lemma 2.6.** [4] For the tangent bundle \( TM \) of an orientable hypersurface \( M \) of \( R^{2n} \) and \( X \in \mathfrak{X}(M) \), we have

1. \( \mathcal{D}_X N^v = 0 \),
2. \( \mathcal{D}_X N^h = 0 \),
3. \( \mathcal{D}_X N^v = -(S(X))^v \), (iv) \( \mathcal{D}_X N^h = -(S(X))^h \).

Let \( J \) be the natural complex structure on the Euclidean space \( R^{2n} \), which makes \( (R^{2n}, J, \langle \cdot, \cdot \rangle) \) a Koechler manifold. Then on an orientable real hypersurface \( M \) of \( R^{2n} \) with unit normal \( N \), we define a unit vector field \( \xi \in \mathfrak{X}(M) \) by \( \xi = -JN \), with its dual 1-form \( \eta(X) = g(X, \xi) \), where \( g \) is the induced metric on \( M \). For \( X \in \mathfrak{X}(M) \), we express \( JX = \varphi(X) + \eta(X)N \), where \( \varphi(X) \) is the tangential component of \( JX \), and it follows that \( \varphi \) is a \( (1, 1) \) tensor field on \( M \), and that \( (\varphi, \xi, \eta, g) \) defines an almost contact metric structure on \( M \) (cf. [5], [8], [9]), that is

\[ \varphi^2 X = -X + \eta(X)\xi, \eta(\xi) = 1, \eta \circ \varphi = 0, \varphi(\xi) = 0 \]

and

\[ g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \]  \( X, Y \in \mathfrak{X}(M) \)

Moreover, we have the following.
Lemma 2.7. [8] Let $M$ be an orientable real hypersurface of $R^{2n}$. Then the structure $(\varphi, \xi, \eta, g)$ on $M$ satisfies

(i) $(\nabla_X \varphi)(Y) = \eta(Y)SX - g(SX, Y)\xi,$
(ii) $\nabla_X \xi = \varphi SX, \; X, Y \in \mathfrak{X}(M).$

3. A Structure on $(TM, g)$

We know that the Euclidean space $R^{4n}$ has many complex structures, however in this section we treat $R^{4n}$ as the tangent bundle of $R^{2n}$ and consider a specific complex structure on the Euclidean space $R^{4n}$. Let $\pi : R^{4n} = TR^{2n} \to R^{2n}$ be the submersion of the tangent bundle of $R^{2n}$. Then it is easy to show that the Euclidean metric $(\cdot)$ on the tangent bundle $R^{4n}$ is Sasaki metric and using the canonical almost complex structure $J$ of $R^{2n}$, we define $J : \mathfrak{X}(R^{4n}) \to \mathfrak{X}(R^{4n})$ by

$J(E^h) = (JE)^h, \; J(E^v) = (JE)^v, \; E \in \mathfrak{X}(R^{2n})$

and it is easily follows that $J$ is an almost complex structure, satisfying $\langle JE, JF \rangle = \langle E, F \rangle$ with respect to the Euclidean metric $(\cdot)$ on $R^{4n}$ and that $(\overline{D}_E J)(F) = 0, E, F \in \mathfrak{X}(R^{4n})$ that is $(R^{4n}, J, (\cdot))$ is a Kaehler manifold. Regarding the complex structure $J$ defined above, we have the following

Lemma 3.1. Let $\pi : R^{4n} \to R^{2n}$ be the submersion of the tangent bundle $R^{4n} = TR^{2n}$. Then complex structure $\overline{J}$ on $R^{4n}$ satisfies

$J \circ d\pi = d\pi \circ \overline{J}$

Proof. Take $X \in \mathfrak{X}(R^{2n})$, then for the horizontal lift $X^h$, we have:

$J \circ d\pi(X^h) = J(d\pi(X^h)) = JX \circ \pi$

and

$d\pi \circ J(X^h) = d\pi(JX)^h = JX \circ \pi$

which proves

$J \circ d\pi(X^h) = d\pi \circ J(X^h)$

Similarly for the vertical lift $X^v$we have

$J \circ d\pi(X^v) = J(d\pi(X^v)) = 0$

and

$d\pi \circ J(X^v) = d\pi(JX)^v = 0$

This proves the Lemma. □

Remark 3.2. If $M$ is an orientable real hypersurface of the Euclidean space $R^{2n}$ with immersion $f$, then $F = df$ is the immersion of the tangent bundle $TM$ into the Euclidean space $R^{4n}$ and as immersions are local embeddings, in general, we identify the local quantities on submanifold with those of the ambient space for instance we identify $df(X)$ with $X$ for $X \in \mathfrak{X}(M)$. However,
while dealing with the immersion $F$ of $TM$ in $R^{4n}$ one need to be cautious specially while dealing with the horizontal lifts (cf. 2.1). Therefore in what follows, we shall bring $dF$ in to play whenever it is needed specially in the case of horizontal lifts.

Observe that if $M$ is an orientable real hypersurface of the Euclidean space $R^{2n}$ with unit normal vector field $N$, then we know that horizontal lift $N^h$ is a unit normal vector field to the submanifold $TM$ of $R^{4n}$ and that the vertical lift $N^v \in \mathfrak{X}(TM)$ (cf.[1]). We have

$$\overline{J}N^h = (JN)^h = -(df(\xi))^h = -dF(\xi^h) + g(S(\xi),u)N^v \in \mathfrak{X}(TM) \quad (3.1)$$

and

$$\overline{J}N^v = (JN)^v = -\xi^v \in \mathfrak{X}(TM) \quad (3.2)$$

Let $M$ be an orientable real hypersurface of the Kaehler manifold $(R^{2n},J,\langle \cdot,\cdot \rangle)$. Then as $TM$ is submanifold of the Kaehler manifold $(R^{4n},\overline{J},\langle \cdot,\cdot \rangle)$, we denote by $\Gamma(T^\perp TM)$ the space of smooth normal vector fields to $TM$. The restriction of the complex structure $\overline{J}$ on $R^{4n}$ to $\mathfrak{X}(TM)$ and $\Gamma(T^\perp TM)$ can be expressed as

$$(\overline{J}E) = \overline{\varphi(E)} + \overline{\psi(E)}, \quad \overline{J}(\overline{N}) = \overline{G(N)} + \overline{\chi(N)}, \quad E \in \mathfrak{X}(TM), \quad \overline{N} \in \Gamma(T^\perp TM)$$

where $\overline{\varphi(E)}$, $\overline{G(N)}$ are the tangential and $\overline{\psi(E)}$, $\overline{\chi(N)}$ are the normal components of $\overline{J}E$, and $\overline{J}(\overline{N})$ respectively. Note that the horizontal lift $N^h$ of the unit normal $N$ to the hypersurface $M$ is normal to $TM$ that is $N^h \in \Gamma(T^\perp TM)$, whereas the vertical lift $N^v \in \mathfrak{X}(TM)$.

**Lemma 3.3.** Let $TM$ be the tangent bundle of an orientable real hypersurface of $R^{2n}$. Then for $X \in \mathfrak{X}(M)$,

$$\varphi(X^h) = (\varphi(X))^h - g(S(X),u)\xi^v, \quad \varphi(X^v) = (\varphi(X))^v + \eta(X) \circ \pi N^v$$

$$\overline{\psi}(X^h) = \eta(X) \circ \pi N^h, \quad \overline{\psi}(X^v) = 0$$

**Proof.** Note that for the horizontal lift $X^h$ we have

$$\overline{J}X^h = \overline{J}dF(X^h) = \overline{J}((df(X))^h + g(SX,u) \circ \pi N^v)$$

$$= (Jdf(X))^h + g(SX,u) \circ (JN)^v$$

$$= (\varphi X + \eta(X)N)^h - g(SX,u) \circ \pi \xi^v$$

$$= (\varphi(X))^h - g(SX,u) \circ \pi \xi^v + \eta(X) \circ \pi N^h$$

which together with the definition $\overline{J}X^h = \varphi(X^h) + \overline{\psi}(X^h)$, on equating tangential and normal components give

$$\varphi(X^h) = (\varphi(X))^h - g(S(X),u)\xi^v \text{ and } \overline{\psi}(X^h) = \eta(X) \circ \pi N^h$$

Similarly for the vertical lift $X^v$, we have

$$\overline{J}X^v = \varphi(X^v) + \overline{\psi}(X^v) = (JX)^v = (\varphi X + \eta(X)N)^v$$
which gives
\[(\varphi(X^v)) + \psi(X^v) = (\varphi X)^v + \eta(X) \circ \pi N^v\]
Comparing the tangential and normal components we conclude
\[\varphi(X^v) = (\varphi X)^v + \eta(X) \circ \pi N^v, \text{ and } \psi(X^v) = 0.\]

□

We choose a unit normal vector field \(N^* \in \Gamma(T^\perp TM)\) such that \(\{N^*, N^h\}\) is a local orthonormal frame of normals for the submanifold \(TM\). It is known that \(N^*\) is vertical vector field on the tangent bundle \(R^4\) (cf. [1]). Since,
\[
\langle JN^*, N^* \rangle = 0, \quad \langle JN^*, N^h \rangle = \langle N^*, \xi^h \rangle = 0,
\]
it follows that \(JN^* \in \mathfrak{X}(TM)\) and we define unit vector field \(\zeta \in \mathfrak{X}(TM)\) by
\[
\zeta = -JN^*
\]
Now, for any normal vector field \(N \in \Gamma(T^\perp TM)\), we have
\[
N = \langle N, N^* \rangle N^* + \langle N, N^h \rangle N^h
\]
which together with equations (3.1), (3.2) and (3.3) gives \(\chi(N) = 0\) and that \(J(N) \in \mathfrak{X}(TM)\), is given by
\[
J(N) = \langle J(N), \zeta \rangle \zeta + \langle J(N), T \rangle T
\]
where \(T \in \mathfrak{X}(TM)\), is given by
\[
T = \xi^h - g(S(\xi), u)N^v = -JN^h
\]
Also, using equation (3.2), we have
\[
-\xi^v = JN^v = \varphi(N^v) + \psi(N^v)
\]
which gives
\[
\varphi(N^v) = -\xi^v \text{ and } \psi(N^v) = 0
\]
Moreover, we have
\[
\varphi(\zeta) = 0 \text{ and } \psi(\zeta) = N^*, \quad \psi(\xi^h) = N^h
\]
If we denote by \(\alpha, \beta\) the smooth 1-forms on \(TM\) dual to the vector field \(\zeta\) and \(T\) respectively, then for \(E \in \mathfrak{X}(TM)\), it follows that
\[
J(\psi(E)) = -\alpha(E)\zeta - \beta(E)T
\]
and consequently, operating \(J\) on \(J(E) = \varphi(E) + \psi(E), E \in \mathfrak{X}(TM)\), we get
\[
\varphi^2 = -I + \alpha \otimes \zeta + \beta \otimes T \text{ and } \psi \circ \varphi = 0
\]
Using Lemma 2.1 and equations (3.3), (3.5), (3.6), (3.8), we see that the vector fields \(\zeta, T\) and 1-forms \(\alpha, \beta\) satisfy
\[
\varphi(\zeta) = 0, \varphi(T) = 0, \quad \varphi(\zeta, T) = 0, \quad \alpha \circ \varphi = 0, \beta \circ \varphi = 0
\]
Also, as $\overline{g}$ is the induced metric on the submanifold $TM$ and $\mathcal{J}$ is skew-symmetric with respect to the Hermitian metric $\langle \cdot , \cdot \rangle$, we have
\[ \overline{g}(\mathcal{J}(E), F) = -\overline{g}(E, \mathcal{J}(F)), \quad E, F \in \mathfrak{X}(TM) \] (3.10)

Then using equations (3.8), (3.9) and (3.10), we have
\[ \overline{g}(\mathcal{J}(E), \mathcal{J}(F)) = \overline{g}(E, F) - \alpha(E)\alpha(F) - \beta(E)\beta(F), \quad E, F \in \mathfrak{X}(TM) \] (3.11)

Thus we have proved the following

**Lemma 3.4.** Let $TM$ be the tangent bundle of an orientable real hypersurface of $R^{2n}$. Then there is a structure $(\mathcal{F}, \zeta, T, \alpha, \beta, \overline{g})$ similar to contact metric structure on $TM$, where $\mathcal{F}$ is a tensor field of type $(1,1)$, $\zeta, T$ are smooth vector fields and $\alpha, \beta$ are smooth 1-forms dual to $\zeta, T$ with respect to the Riemannian metric $\overline{g}$ satisfying
\[
\mathcal{F}^2 = -I + \alpha \otimes \zeta + \beta \otimes T, \quad \mathcal{F}(\zeta) = 0, \quad \mathcal{F}(T) = 0, \quad \alpha \circ \mathcal{F} = 0, \quad \beta \circ \mathcal{F} = 0, \quad \overline{g}(\zeta, T) = 0
\]

Thus, as $g$ is the induced metric on the submanifold $TM$ and $\mathcal{J}$ is skew-symmetric with respect to the Hermitian metric $\langle \cdot , \cdot \rangle$, we have
\[ g(\mathcal{J}(E), F) = -g(E, \mathcal{J}(F)), \quad E, F \in \mathfrak{X}(TM) \] (3.12)

In the next Lemma, we compute the co-variant derivatives of the tensor $\mathcal{F}$.

**Lemma 3.5.** Let $(\mathcal{F}, \zeta, T, \alpha, \beta, \overline{g})$ be the structure on the tangent bundle $TM$ of an orientable real hypersurface $M$ of the Euclidean space $R^{2n}$. Then
\[
\begin{align*}
(i) \quad & (\nabla_X \mathcal{F})(Y^h) = \{(\nabla_X \varphi)(Y)\}^h - \{X(\varphi(Y) + \varphi(Y)JSX)\}^v \\
(ii) \quad & (\nabla_X \mathcal{F})(Y^v) = 0, \\
(iii) \quad & (\nabla_X \mathcal{F})(Y^h) = 0, \quad (\nabla_X \mathcal{F})(Y^v) = g(SX, \varphi Y) \circ \pi N^v + g(SX, Y) \circ \pi \xi^v.
\end{align*}
\]

**Proof.** Using the definition of $\mathcal{J}$, Lemma 2.1 and Lemma 3.3 together with equation (3.1), we get for $X, Y \in \mathfrak{X}(M)$
\[ \mathcal{J}Y^h = \mathcal{J}dF(Y^h) = \mathcal{J}\left( (df(Y))^h + g(SX, u) \circ \pi N^v \right) \]
\[ = (\varphi Y + \eta(Y)N)^h - g(SX, u) \circ \pi \xi^v \]
\[ = \mathcal{F}(Y^h) + \eta(Y) \circ \pi N^h \]

which gives
\[
\nabla_{X^h} \mathcal{J}Y^h = \nabla_{X^h}(\mathcal{J}(Y^h) + g(SX, u) \circ \pi N^v)(\nabla\varphi(Y^h) + \eta(Y) \circ \pi N^h) \]
\[ = \nabla_{X^h}(\mathcal{J}(Y^h) + g(SX, u) \circ \pi N^v) \]
\[ = \nabla_{X^h}(\mathcal{J}(Y^h) + g(SX, u) \circ \pi N^v) \]
\[ + g(SX, u) \circ \pi \xi^v + g(SX, u) \circ \pi \xi^v \]
\[ + \eta(Y) \circ \pi N^h \]
\[ + g(SX, u) \circ \pi \xi^v \]
\[ + \eta(Y) \circ \pi N^h \]
\[ = \nabla_{X^h}(\mathcal{J}(Y^h) + g(SX, u) \circ \pi N^v) \]
\[ + g(SX, u) \circ \pi \xi^v + g(SX, u) \circ \pi \xi^v \]
\[ + \eta(Y) \circ \pi N^h \]
\[ + g(SX, u) \circ \pi \xi^v \]
\[ + \eta(Y) \circ \pi N^h \]

Note that the tangent bundle $TR^{2n} = R^{4n}$ has Sasaki metric and thus using Lemma 7.2 of [10] (keeping in view that $R^{2n}$ is flat), in the above equation, we get
\[ \nabla_{X^h} \mathcal{J}Y^h = \nabla_{X^h}(\mathcal{J}(Y^h) + g(SX, u) \circ \pi N^v) \]
\[ + \eta(Y) \circ \pi N^h - \eta(Y) \circ \pi (SX)^h \] (3.12)
Similarly we have
\[
J \mathcal{D}^h X^h Y^h = J \left( \mathcal{D} \left( (df(X))^h + g(SX,u) \circ \pi N^v \right) \left( (df(Y))^h + g(SY,u) \circ \pi N^v \right) \right) \\
= J \left\{ \nabla^h X^h Y^h + h(X^h, Y^h) + X(g(SY,u) \circ \pi N^v) + g(SY,u) \circ \pi (DX)^v \right. \\
\left. \quad + g(SX,u) \circ \pi \mathcal{D}^v (df Y)^h + 0 + 0 \right\} \\
= \nabla^h (\nabla^h X^h Y^h) + \nabla^h (\nabla^h Y^h) + J h(X^h, Y^h) - X(g(SY,u) \circ \pi \xi^v) - g(SY,u) \circ \pi \mathcal{J} (SX)^v \\
- g(SY,u) \circ \pi (\varphi SX)^v - g(SY,u) \circ \pi \eta (SX^v,h(E,T) = \psi \left( \nabla^h N^v \right) \\
\right\}
\]
where we used Lemmas 2.3, 2.4 and Lemma 7.2 in [10]. Now as \( (R^3, J, \langle , \rangle) \)
is a Kaehler manifold, the equations (3.12) and (3.13) on comparing tangential components, we immediately arrive at
\[
(\nabla^h \varphi)(Y^h) = \left\{ (\nabla \varphi)(Y) \right\}^h - X(g(SY,u) + g(SY,u) JSX) \right)^v
\]
which proves (i).

Now, using \( h(X^v, Y^v) = 0 \) and \( \nabla_{\varphi Y}^v X^v = 0 \) together with \( \mathcal{D}^v \mathcal{J}^v = \mathcal{J} \mathcal{D}^v \mathcal{J}^v \), and comparing tangential components, we immediately arrive at
\[
(\nabla^h \varphi)(Y^v) = 0
\]
Next, we have \( \nabla^h \varphi(Y^h) = \nabla^h \left( (\varphi Y)^h - g(SX,u) \circ \pi \xi^v \right) = \nabla^h (\varphi Y)^h = g(SX,\varphi Y) \circ \pi N^v \) and \( \nabla \mathcal{J} (\varphi Y^h) = g(SX,\varphi Y) \circ \pi \mathcal{J}(N^v) = -g(SX,Y) \circ \pi \xi^v \). Thus, we get
\[
(\nabla^h \varphi)(Y^h) = g(SX,\varphi Y) \circ \pi N^v + g(SX,Y) \circ \pi \xi^v
\]
Finally, using \( h(X^h, Y^v) = 0 \) and \( \nabla_{\varphi Y}^v X^h = 0 \) together with \( \mathcal{D}^h \mathcal{J}^v = \mathcal{J} \mathcal{D}^h \mathcal{J}^v \), and comparing tangential components, we immediately arrive at
\[
(\nabla^h \varphi)(Y^v) = 0
\]
\[ \square \]

**Lemma 3.6.** Let \( (\varphi, \zeta, T, \alpha, \beta, g) \) be the structure on the tangent bundle \( TM \) of an orientable real hypersurface \( M \) of the Euclidean space \( R^{2n} \). Then for \( E \in \mathcal{X}(TM) \),
\[
\nabla_E \xi = \nabla \left( \nabla_{\xi} N^v \right) - J \left( \nabla_{E} N^v \right), \quad h(E, \xi) = \nabla \left( \nabla_{E} N^v \right), \quad h(E, T) = \nabla \left( \nabla_{E} N^v \right)
\]

Proof. Using equation (2.2), we have
\[ \nabla E \zeta = D E \zeta - h(E, \zeta) \]
\[ = - \mathcal{J} D E N^* - h(E, \zeta) \]
\[ = \mathcal{J} (S N^*(E)) - \mathcal{J} (\nabla E N^*) - h(E, \zeta) \]
\[ = \phi (S N^*(E)) + \psi (S N^*(E)) - \mathcal{J} (\nabla E N^*) - h(E, \zeta) \]
Since \( \mathcal{J}(N) \in \mathfrak{X}(TM) \) for each normal \( N \in \Gamma(T^\perp TM) \), equation tangential and normal components in above equation, we get the first part. The second part follows similarly using \( T = -\mathcal{J} N^h \). □

Now, we prove the following:

**Theorem 3.7.** The tangent bundle \( TM \) of an orientable real hypersurface \( M \) of the Euclidean space \( \mathbb{R}^{2n} \) is a CR-submanifold of the Kaehler manifold \( (\mathbb{R}^{4n}, \mathcal{J}, \langle \cdot, \cdot \rangle) \).

Proof. Use the structure \( (\phi, \zeta, T, \alpha, \beta, g) \) on the submanifold \( TM \) of \( \mathbb{R}^{4n} \) to define the distribution \( D \) by
\[ D = \{ E \in \mathfrak{X}(TM) : \alpha(E) = \beta(E) = 0 \} \]
and \( D^\perp \) be the distribution spanned by the orthogonal vector fields \( \zeta \) and \( T \). Note that \( \zeta \) is unit vector field on \( TM \) and the length of the vector field \( T \) satisfies
\[ ||T||^2 = 1 + 2g(S(\xi), v)^2 \geq 1 \]
which shows that \( D^\perp \) is 2-dimensional distribution on \( TM \) and that \( \mathcal{J} D^\perp = \Gamma(T^\perp TM) \). It is easy to see that \( D \) and \( D^\perp \) are orthogonal complementary distributions and that \( \dim D = 4(n - 1) \). Note that for \( E \in \mathfrak{X}(TM) \), we have
\[ \phi(E) = \langle \phi(E), N^* \rangle N^* + \langle \phi(E), N^h \rangle N^h = \alpha(E) N^* + \beta(E) N^h \]
and consequently if \( E \in D \), then above equation gives \( \mathcal{J} E = \phi E \) which is orthogonal to both \( \zeta \) and \( T \) and that \( \mathcal{J} E \in D \), which implies \( \mathcal{J} D = D \). This proves that \( TM \) is a CR-submanifold of the Kaehler manifold \( (\mathbb{R}^{4n}, \mathcal{J}, \langle \cdot, \cdot \rangle) \) (cf. [8]). □

4. **Killing Vector Fields on \( TM \)**

Let \( TM \) be the tangent bundle of an orientable real hypersurface \( M \) of the Euclidean space \( \mathbb{R}^{2n} \). Recall that a vector field \( \zeta \in \mathfrak{X}(TM) \) on the Riemannian manifold \( (TM, \bar{g}) \) is said to be Killing if
\[ (\mathcal{L}_\zeta \bar{g})(E, F) = 0, \quad E, F \in \mathfrak{X}(TM) \]
where \( \mathcal{L}_\zeta \) is the Lie derivative with respect to the vector field \( \zeta \). We have seen in previous section that the tangent bundle \( (TM, \bar{g}) \) admits a structure \( (\phi, \zeta, T, \alpha, \beta, \bar{g}) \), that is similar to the almost contact structure. In this section
we are interested in finding conditions under which the special vector fields $\zeta$ and $T$ are Killing vector fields and as a particular case we get that the tangent bundle $(TS^{2n-1}, \bar{g})$ of the unit sphere $S^{2n-1}$ in the Euclidean space $R^{2n}$ admits a nontrivial Killing vector field.

**Theorem 4.1.** Let $(\varphi, \zeta, T, \alpha, \beta, \bar{g})$ be the structure on the tangent bundle $TM$ of an orientable real hypersurface $M$ of the Euclidean space $R^{2n}$. Then the vector field $\zeta$ is Killing.

**Proof.** First note that on taking inner product with $N^*$ in each part of Lemma 2.5, we conclude that $\overline{S}_{N^*}(X^h) = \overline{S}_{N^*}(X^v) = 0$, $X \in \mathfrak{X}(M)$ and consequently,

$$\overline{S}_{N^*}(E) = 0, \ E \in \mathfrak{X}(TM) \quad (4.1)$$

Also using second part of equation (2.2) in (ii) and (iv) of Lemma 2.4, we conclude that $\nabla_E^\perp N^h = 0$, $E \in \mathfrak{X}(TM)$, that is $N^h$ is parallel on the normal bundle of $TM$. Moreover, we have

$$\nabla_E^\perp N^* = \overline{\langle \nabla_E N^*, N^h \rangle} N^h = - \overline{\langle N^*, \nabla_E^\perp N^h \rangle} N^h = 0$$

that is $N^*$ is parallel in the normal bundle of $TM$. Thus using equation (4.1) in Lemma 3.5, it follows that $\zeta$ is a parallel vector field and consequently, it is a Killing vector field. \(\square\)

**Theorem 4.2.** Let $(\varphi, \zeta, T, \alpha, \beta, \bar{g})$ be the structure on the tangent bundle $TM$ of an orientable real hypersurface $M$ of the Euclidean space $R^{2n}$. Then the vector field $T$ is Killing if and only if the following condition holds

$$g((\varphi \circ \overline{S}_{N^h} - \overline{S}_{N^h} \circ \varphi)(X^h), Y^h) = 0, \ X, Y \in \mathfrak{X}(M)$$

**Proof.** Since $N^h$ is parallel in the normal bundle of $TM$, by Lemma 3.5, we have

$$\nabla_E T = \varphi(\overline{S}_{N^h}(E)), \ E \in \mathfrak{X}(TM) \quad (4.2)$$

Also using Lemma 2.4, we conclude that

$$\overline{S}_{N^h}(X^v) = 0, \overline{S}_{N^h}(X^h) = (S(X))^h, \ X \in \mathfrak{X}(M) \quad (4.3)$$

Then using skew-symmetry of the tensor $\varphi$, and equations (4.2) and (4.3) together with Lemma 3.3, we immediately arrive at

$$(\mathcal{L}_T \bar{g})(X^v, Y^v) = 0 \quad (4.4)$$

$$\begin{align*}
(\mathcal{L}_T \bar{g}) (X^h, Y^v) &= \bar{g}(\varphi \circ \overline{S}_{N^h} (X^h), Y^v) = -\bar{g}(\overline{S}_{N^h} (X^h), \varphi(Y^v)) \\
&= -\bar{g}(\overline{S}_{N^h} (X^h), (\varphi(Y))^v + \eta(X) \circ \pi N^v) = 0 \quad (4.5)
\end{align*}$$

and the equations (4.4)-(4.6) prove the Theorem. \(\square\)
Consider the unit sphere $S^{2n-1}$ in the Euclidean space $\mathbb{R}^{2n}$, whose shape operator is given by $\S = -I$. Using Lemma 2.4, we get on the tangent bundle $TS^{2n-1}$ that
\[
\overline{S}_{N^h}(X^h) = (\S(X))^h = -X^h, \quad \overline{S}_{N^h}(X^v) = 0
\]
Then the Lemma 3.3 together with above equation, gives
\[
(\varphi \circ \overline{S}_{N^h} - \overline{S}_{N^h} \circ \varphi)(X^h) = -\varphi(X^h) - \overline{S}_{N^h}((\varphi(X))^h - g(S(X), u) \circ \pi^v) = -g(X, u) \circ \pi^v, \quad X \in \mathcal{X}(S^{2n-1})
\]
and consequently,
\[
\overline{g}((\varphi \circ \overline{S}_{N^h} - \overline{S}_{N^h} \circ \varphi)(X^h), Y^h) = 0, \quad X, Y \in \mathcal{X}(S^{2n-1})
\]
Thus as a particular case of the Theorem 4.2, we have

**Corollary 4.3.** Let $(\varphi, \zeta, T, \alpha, \beta, \overline{g})$ be the structure on the tangent bundle $TS^{2n-1}$ of the unit sphere $S^{2n-1}$ in the Euclidean space $\mathbb{R}^{2n}$, $n > 1$. Then the vector field $T$ is a nontrivial Killing vector field.

**Proof.** It remains to be shown that $T$ is nontrivial. Since, $N^h$ is parallel in the normal bundle of $TS^{2n-1}$, by Lemmas 2.4 and 3.5, we have
\[
\nabla_{X^h} T = -\varphi(X^h), \quad X \in \mathcal{X}(S^{2n-1}) \tag{4.7}
\]
where we used the fact that the shape operator $S$ of the unit sphere $S^{2n-1}$ is given by $S = -I$. The Lemma 3.4 gives the rank of $\varphi$ is $4(n - 1)$ and consequently, equation (4.7) gives that the Killing vector field $T$ is not parallel, that is $T$ is a nontrivial Killing vector field. \qed

**Acknowledgments**

This Work is supported by King Saud University, Deanship of Scientific Research, College of Science Research Center.

**References**