Tangent Bundle of the Hypersurfaces in a Euclidean Space

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Abstract. Let $M$ be an orientable hypersurface in the Euclidean space $\mathbb{R}^{2n}$ with induced metric $g$ and $TM$ be its tangent bundle. It is known that the tangent bundle $TM$ has induced metric $\mathcal{F}$ as submanifold of the Euclidean space $\mathbb{R}^{4n}$ which is not a natural metric in the sense that the submersion $\pi: (TM, \mathcal{F}) \rightarrow (M, g)$ is not the Riemannian submersion. In this paper, we use the fact that $\mathbb{R}^{4n}$ is the tangent bundle of the Euclidean space $\mathbb{R}^{2n}$ to define a special complex structure $\mathcal{J}$ on the tangent bundle $\mathbb{R}^{4n}$ so that $(\mathbb{R}^{4n}, \mathcal{J}, \langle \cdot, \cdot \rangle)$ is a Kaehler manifold, where $\langle \cdot, \cdot \rangle$ is the Euclidean metric which is also the Sasaki metric of the tangent bundle $\mathbb{R}^{4n}$. We study the structure induced on the tangent bundle $(TM, \mathcal{F})$ of the hypersurface $M$, which is a submanifold of the Kaehler manifold $(\mathbb{R}^{4n}, \mathcal{J}, \langle \cdot, \cdot \rangle)$. We show that the tangent bundle $TM$ is a CR-submanifold of the Kaehler manifold $(\mathbb{R}^{4n}, \mathcal{J}, \langle \cdot, \cdot \rangle)$. We find conditions under which certain special vector fields on the tangent bundle $(TM, \mathcal{F})$ are Killing vector fields. It is also shown that the tangent bundle $T\mathbb{S}^{2n-1}$ of the unit sphere $\mathbb{S}^{2n-1}$ admits a Riemannian metric $\mathcal{F}$ and that there exists a nontrivial Killing vector field on the tangent bundle $(T\mathbb{S}^{2n-1}, \mathcal{F})$.

Keywords: Tangent bundle, Hypersurface, Kaehler manifold, Almost contact structure, Killing vector field, CR-Submanifold, Second fundamental form, Wiengarten map.

1. Introduction

Recently efforts are made to study the geometry of the tangent bundle of a hypersurface $M$ in the Euclidean space $\mathbb{R}^{n+1}$ (cf. [3]), where the authors have shown that the induced metric on its tangent bundle $TM$ as submanifold of the Euclidean space $\mathbb{R}^{2n+2}$ is not a natural metric. In [4], we have extended the study initiated in [3] on the geometry of the tangent bundle $TM$ of an immersed orientable hypersurface $M$ in the Euclidean space $\mathbb{R}^{n+1}$. It is well known that Killing vector fields play an important role in shaping the geometry of a Riemannian manifold, for instance the presence of nonzero Killing vector field on a compact Riemannian manifold forces its Ricci curvature to be non-negative and this in particular implies that on a compact Riemannian manifolds of negative Ricci curvature there does not exist a nonzero Killing vector field.

The study of Killing vector fields becomes more interesting on the tangent bundle $TM$ of a Riemannian manifold $(M, g)$ as the tangent bundle $TM$ is noncompact. It is known that if the tangent bundle $TM$ of a Riemannian manifold $(M, g)$ is equipped with Sasaki metric, then the vertical lift of a parallel vector field on $M$ is a Killing vector field (cf. [15]). However if the Sasaki metric is replaced by the Cheeger-Gromoll metric, then the vertical lift of any nonzero vector field on $M$ is never Killing (cf. [14]). Note that both Sasaki metric as well as Cheeger-Gromoll metrics are natural metrics. We consider an orientable real hypersurface $M$ of the Euclidean space $\mathbb{R}^{2n}$ with the induced metric $g$. Then as the tangent bundle $TM$ of $M$ is a submanifold of codimension two in $\mathbb{R}^{4n}$, it has induced metric $\overline{g}$ and this metric $\overline{g}$ on $TM$ is not a natural metric as the submersion $\pi : (TM, \overline{g}) \to (M, g)$ is not the Riemannian submersion (cf. [3]). Let $N$ be the unit normal vector field to the hypersurface $M$ and $J$ be the natural complex structure on the Euclidean space $\mathbb{R}^{4n}$. Then we have a globally defined unit vector field $\xi$ on the hypersurface given by $\xi = -JN$ called the characteristic vector field of the real hypersurface (cf. [1, 2, 5, 6, 7, 8, 9]), and this vector field $\xi$ gives rise to two vector fields $\xi^h$ (the horizontal lift) and $\xi^v$ (the vertical lift) on the tangent bundle $(TM, \overline{g})$. In this paper, we use the fact that $R^{4n}$ is the tangent bundle of the Euclidean space $\mathbb{R}^{2n}$ and that the projection $\pi : R^{4n} \to R^{2n}$ is a Riemannian submersion, to define a special almost complex structure $\overline{J}$ on the tangent bundle $\mathbb{R}^{4n}$ which is different from the canonical complex structure of the Euclidean space $R^{4n}$ and show that $(R^{4n}, \overline{J}, \langle , \rangle)$ is a Kaehler manifold, where $\langle , \rangle$ is the Euclidean metric on $R^{4n}$. It is shown that the codimension two submanifold $(TM, \overline{g})$ of the Kaehler manifold $(R^{4n}, \overline{J}, \langle , \rangle)$ is a CR-submanifold (cf. [10]) and it naturally inherits certain special vector fields other than $\xi^h$ and $\xi^v$, and in this paper we are interested in finding conditions under which these special vector fields are Killing vector fields on $(TM, \overline{g})$. One of the interesting outcome of this study is, we have shown that the tangent bundle $TS^{2n-1}$ of the unit sphere $S^{2n-1}$ as
submanifold of $\mathbb{R}^{4n}$ admits a nontrivial Killing vector field. It is worth pointing out that on the tangent bundle $T\mathbb{S}^{2n-1}$ with Sasakian metric no vertical or horizontal lift of a vector field is Killing as this will require the corresponding vector field on $\mathbb{S}^{2n-1}$ is parallel which is impossible as $\mathbb{S}^{2n-1}$ is space of constant curvature 1. Note that on even dimensional Riemannian manifolds which are irreducible, it is difficult to find Killing vector fields, whereas on products like $\mathbb{S}^{2k-1} \times \mathbb{S}^{2l-1}, \mathbb{R}^{2k-1} \times R^{2l-1}$ one can easily find Killing vector fields. Since the tangent bundle $T\mathbb{S}^{2n-1}$ is trivial for $n = 1, 2, 4$, finding Killing vector fields is easy in these dimensions, but for $n \geq 5$, it is not trivial.

2. Preliminaries

Let $(M,g)$ be a Riemannian manifold and $TM$ be its tangent bundle with projection map $\pi : TM \rightarrow M$. Then for each $(p, u) \in TM$, the tangent space $T_{(p, u)}TM = H_{(p, u)} \oplus \mathfrak{H}_{(p, u)}$, where $\mathfrak{H}_{(p, u)}$ is the kernel of $d\pi_{(p, u)} : T_{(p, u)}(TM) \rightarrow T_{p}M$ and $H_{(p, u)}$ is the kernel of the connection map $K_{(p, u)} : T_{(p, u)}(TM) \rightarrow T_{p}M$ with respect to the Riemannian connection on $(M,g)$. The subspaces $H_{(p, u)}, \mathfrak{H}_{(p, u)}$ are called the horizontal and vertical subspaces respectively. Consequently, the Lie algebra of smooth vector fields $\mathfrak{X}(TM)$ on the tangent bundle $TM$ admits the decomposition $\mathfrak{X}(TM) = H \oplus \mathfrak{H}$ where $H$ is called the horizontal distribution and $\mathfrak{H}$ is called the vertical distribution on the tangent bundle $TM$. For each $X_{p} \in T_{p}M$, the horizontal lift of $X_{p}$ to a point $z = (p, u) \in TM$ is the unique vector $X_{p}^{h} \in H_{z}$ such that $d\pi(X_{p}^{h}) = X_{p} \circ \pi$ and the vertical lift of $X_{p}$ to a point $z = (p, u) \in TM$ is the unique vector $X_{p}^{v} \in \mathfrak{H}_{z}$ such that $X_{z}^{v}(df) = X_{p}(f)$ for all functions $f \in C^{\infty}(M)$, where $df$ is the function defined by $(df)(p, u) = u(f)$. Also for a vector field $X \in \mathfrak{X}(M)$, the horizontal lift of $X$ is a vector field $X^{h} \in \mathfrak{X}(TM)$ whose value at a point $(p, u)$ is the horizontal lift of $X(p)$ to $(p, u)$, the vertical lift $X^{v}$ of $X$ is defined similarly. For $X \in \mathfrak{X}(M)$ the horizontal and vertical lifts $X^{h}, X^{v}$ of $X$ are uniquely determined vector fields on $TM$ satisfying

$$d\pi(X_{p}^{h}) = X_{\pi(z)}, K(X_{p}^{h}) = 0, d\pi(X_{p}^{v}) = 0, K(X_{p}^{v}) = X_{\pi(z)}$$

Also, we have for a smooth function $f \in C^{\infty}(M)$ and vector fields $X, Y \in \mathfrak{X}(M)$, that $(fX)^{h} = (f \circ \pi)X^{h}$, $(fX)^{v} = (f \circ \pi)X^{v}$, $(X + Y)^{h} = X^{h} + Y^{h}$ and $(X + Y)^{v} = X^{v} + Y^{v}$. If $\dim M = m$ and $(U, \varphi)$ is a chart on $M$ with local coordinates $x^{1}, x^{2}, \ldots, x^{m}$, then $(\pi^{-1}(U), \varphi)$ is a chart on $TM$ with local coordinates $x^{1}, x^{2}, \ldots, x^{m}, y^{1}, y^{2}, \ldots, y^{m}$, where $x^{i} = x^{i} \circ \pi$ and $y^{i} = dx^{i}$, $i = 1, 2, \ldots, m$.

A Riemannian metric $\overline{g}$ on the tangent bundle $TM$ is said to be natural metric with respect to $g$ on $M$ if $\overline{g}_{(p, u)}(X^{h}, Y^{h}) = g_{p}(X, Y)$ and $\overline{g}_{(p, u)}(X^{h}, Y^{v}) = 0$, for all vector fields $X, Y \in \mathfrak{X}(M)$ and $(p, u) \in TM$, that is the projection map $\pi : TM \rightarrow M$ is a Riemannian submersion.
Let $M$ be an orientable hypersurface of the Euclidean space $\mathbb{R}^{2n}$ with immersion $f : M \rightarrow \mathbb{R}^{2n}$ and $TM$ be its tangent bundle. Then as $F = df : TM \rightarrow T\mathbb{R}^{4n} = TR^{2n}$ is also an immersion, $TM$ is an immersed submanifold of the Euclidean space $R^{4n}$. We denote the induced metrics on $M$, $TM$ by $g, \bar{g}$ respectively and the Euclidean metric on $R^{2n}$ as well as on $R^{4n}$ by $\langle, \rangle$. Also, we denote by $\nabla, \bar{\nabla}, D$ and $\bar{D}$ the Riemannian connections on $M$, $TM$, $R^{2n}$, and $R^{4n}$ respectively. Let $N$ and $S$ be the unit normal vector field and the shape operator of the hypersurface $M$. For the hypersurface $M$ of the Euclidean space $R^{2n}$ we have the following Gauss and Weingarten formulae

$$D_X Y = \nabla_X Y + \langle S(X), Y \rangle N, \quad D_X N = -S(X), \quad X, Y \in \mathfrak{X}(M) \quad (2.1)$$

where $S$ is the shape operator (Weingarten map). Similarly for the submanifold $TM$ of the Euclidean space $R^{4n}$ we have the Gauss and Weingarten formulae

$$\bar{D}_E F = \bar{\nabla}_E F + h(E, F), \quad \bar{D}_E \bar{N} = -\bar{S}(E) + \bar{\nabla}_{E}^\bot \bar{N} \quad (2.2)$$

where $E, F \in \mathfrak{X}(TM), \bar{\nabla}^\bot$ is the connection in the normal bundle of $TM$ and $\bar{S}$ denotes the Weingarten map in the direction of the normal $\bar{N}$ and is related to the second fundamental form $h$ by

$$\langle h(X, Y), N \rangle = \bar{g}(\bar{S}(X), Y) \quad (2.3)$$

Also we observe that for $X \in \mathfrak{X}(M)$ the vertical lift $X_v$ of $X$ to $TM$, as $X_v \in \ker d\pi$, where $\pi : TM \rightarrow M$ is the natural submersion, we have $d\pi(X_v) = 0$ that is $df(d\pi(X_v)) = 0$ or equivalently we get $d(f \circ \pi)(X_v) = 0$, that is $d(\tilde{\pi} \circ F)(X_v) = 0$ ($\pi : TR^{2n} \rightarrow R^{2n}$), which gives $dF(X_v) \in \ker d\tilde{\pi} = \mathcal{G}$.

Now we state the following results which are needed in our work.

**Lemma 2.1.** [3] Let $N$ be the unit normal vector field to the hypersurface $M$ of $R^{2n}$ and $P = (p, X_p) \in TM$. Then the horizontal and vertical lifts $Y^h_p, Y^v_p$ of $Y_p \in T_pM$ satisfy

$$dF_P(Y^h_p) = (d\bar{F}_p(Y_p))^h + V_p, \quad dF_P(Y^v_p) = (d\bar{F}_p(Y_p))^v$$

where $V_P \in \mathcal{G}_P$ is given by $V_P = (S_p(X_p), Y_p) N^v_p$, $N^v_p$ being the vertical lift of the unit normal $N$ to with respect to the tangent bundle $\pi : R^{4n} \rightarrow R^{2n}$.

**Lemma 2.2.** [3] If $(M, g)$ is an orientable hypersurface of $R^{2n}$, and $(TM, \bar{g})$ is its tangent bundle as submanifold of $R^{4n}$, then the metric $\bar{g}$ on $TM$ for $P = (p, u) \in TM$, satisfies:

1. $\bar{g}_P(X^h_p, Y^h_p) = g_p(X_p, Y_p) + g_p(S_p(X_p), u) g_p(S_p(Y_p), u)$.
2. $\bar{g}_P(X^h_p, Y^v_p) = 0$.
3. $\bar{g}(X^v, Y^v) = g_p(X_p, Y_p)$.

**Remark 2.3.** It is well known that a metric $\bar{g}$ defined on $TM$ using the Riemannian metric $g$ of $M$ (such as Sasaki metric, Cheeger-Gromoll metric) are
natural metrics in the sense that the submersion $\pi : (TM, \mathcal{G}) \to (M, g)$ becomes a Riemannian submersion with respect to these metrics. However, as seen from above Lemmas, the induced metric on the tangent bundle $TM$ of a hypersurface $M$ of the Euclidean space $R^{2n}$, as a submanifold of $R^{4n}$, is not a natural metric because of the present of the term $g_{p}(S_{p}(X_{p}), u)g_{p}(S_{p}(Y_{p}), u)$ in the inner product of horizontal vectors on $TM$. Moreover, note that for an orientable hypersurface $M$ of the Euclidean space $R^{2n}$, the vertical lift $N^{v}$ of the unit normal is tangential to the submanifold $TM$ of $R^{4n}$ as seen in 2.1.

In what follows, we drop the suffixes like in $g_{p}(S_{p}(X_{p}), u)$ and it will be understood from the context of the entities appearing in the equations.

**Theorem 2.4.** [3] Let $(M, g)$ be an orientable hypersurface of $R^{2n}$, and $(TM, \mathcal{G})$ be its tangent bundle as submanifold of $R^{4n}$. If $\nabla$ and $\nabla$ denote the Riemannian connections on $(M, g)$ and $(TM, \mathcal{G})$, respectively, then

(i) $\nabla_{X^{h}}Y^{h} = (\nabla X Y)^{h} - \frac{1}{2}(R(X, Y)u)^{v}$,

(ii) $\nabla_{X^{h}}Y^{h} = g(S(X), Y) \circ \pi N^{v}$

(iii) $\nabla_{X^{h}}Y^{v} = 0$, (iv) $\nabla_{X^{h}}Y^{v} = (\nabla X Y)^{v} + g(S(X), Y) \circ \pi N^{v}$.

**Lemma 2.5.** [4] Let $TM$ be the tangent bundle of an orientable hypersurface $M$ of $R^{2n}$. Then for $X,Y \in \mathfrak{X}(M)$,

(i) $h(X^{v}, Y^{v}) = 0$,

(ii) $h(X^{v}, Y^{h}) = 0$,

(iii) $h(X^{h}, Y^{h}) = g(S(X), Y) \circ \pi N^{h}$.

**Lemma 2.6.** [4] For the tangent bundle $TM$ of an orientable hypersurface $M$ of $R^{2n}$ and $X \in \mathfrak{X}(M)$, we have

(i) $\mathcal{D}_{X^{v}}N^{v} = 0$,

(ii) $\mathcal{D}_{X^{v}}N^{h} = 0$,

(iii) $\mathcal{D}_{X^{h}}N^{v} = -(S(X))^{v}$, (iv) $\mathcal{D}_{X^{h}}N^{h} = -(S(X))^{h}$.

Let $J$ be the natural complex structure on the Euclidean space $R^{2n}$, which makes $(R^{2n}, J, \langle \cdot , \cdot \rangle)$ a K"{a}hler manifold. Then on an orientable real hypersurface $M$ of $R^{2n}$ with unit normal $N$, we define a unit vector field $\xi \in \mathfrak{X}(M)$ by $\xi = -JN$, with its dual 1-form $\eta(X) = g(X, \xi)$, where $g$ is the induced metric on $M$. For $X \in \mathfrak{X}(M)$, we express $JX = \varphi(X) + \eta(X)N$, where $\varphi(X)$ is the tangential component of $JX$, and it follows that $\varphi$ is a $(1,1)$ tensor field on $M$, and that $(\varphi, \xi, \eta, g)$ defines an almost contact metric structure on $M$ (cf. [5], [8], [9]), that is

$\varphi^{2}X = -X + \eta(X)\xi, \eta(\xi) = 1, \eta \circ \varphi = 0, \varphi(\xi) = 0$

and

$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad X,Y \in \mathfrak{X}(M)$

Moreover, we have the following.
Lemma 2.7. [8] Let $M$ be an orientable real hypersurface of $R^{2n}$. Then the structure $(\phi, \xi, \eta, g)$ on $M$ satisfies

(i) $(\nabla_X \phi)(Y) = \eta(Y)SX - g(SX, Y)\xi$,
(ii) $\nabla_X \xi = \phi SX$, $X, Y \in \mathfrak{X}(M)$.

3. A Structure on $(TM, g)$

We know that the Euclidean space $R^{4n}$ has many complex structures, however in this section we treat $R^{4n}$ as the tangent bundle of $R^{2n}$ and consider a specific complex structure on the Euclidean space $R^{4n}$. Let $\pi : R^{4n} = TR^{2n} \rightarrow R^{2n}$ be the submersion of the tangent bundle of $R^{2n}$. Then it is easy to show that the Euclidean metric $\langle \cdot, \cdot \rangle$ on the tangent bundle $R^{4n}$ is Sasaki metric and using the canonical almost complex structure $J$ of $R^{2n}$, we define $J : \mathfrak{X}(R^{4n}) \rightarrow \mathfrak{X}(R^{4n})$ by

$$J(E^h) = (JE)^h, \quad J(E^v) = (JE)^v, \quad E \in \mathfrak{X}(R^{2n})$$

and it is easily follows that $J$ is an almost complex structure, satisfying $\langle JE, JF \rangle = \langle E, F \rangle$ with respect to the Euclidean metric $\langle \cdot, \cdot \rangle$ on $R^{4n}$ and that $(R^{4n}, J, \langle \cdot, \cdot \rangle)$ is a Kaehler manifold. Regarding the complex structure $J$ defined above, we have the following

Lemma 3.1. Let $\pi : R^{4n} \rightarrow R^{2n}$ be the submersion of the tangent bundle $R^{4n} = TR^{2n}$. Then complex structure $J$ on $R^{4n}$ satisfies

$$J \circ d\pi = d\pi \circ J$$

Proof. Take $X \in \mathfrak{X}(R^{2n})$, then for the horizontal lift $X^h$, we have:

$$J \circ d\pi(X^h) = J(d\pi(X^h)) = JX \circ \pi$$

and

$$d\pi \circ J(X^h) = d\pi(JX)^h = JX \circ \pi$$

which proves

$$J \circ d\pi(X^h) = d\pi \circ J(X^h)$$

Similarly for the vertical lift $X^v$we have

$$J \circ d\pi(X^v) = J(d\pi(X^v)) = 0$$

and

$$d\pi \circ J(X^v) = d\pi(JX)^v = 0$$

This proves the Lemma.  

Remark 3.2. If $M$ is an orientable real hypersurface of the Euclidean space $R^{2n}$ with immersion $f$, then $F = df$ is the immersion of the tangent bundle $TM$ into the Euclidean space $R^{4n}$ and as immersions are local embeddings, in general, we identify the local quantities on submanifold with those of the ambient space for instance we identify $df(X)$ with $X$ for $X \in \mathfrak{X}(M)$. However,
while dealing with the immersion $F$ of $TM$ in $R^{4n}$ one need to be cautious specially while dealing with the horizontal lifts (cf. 2.1). Therefore in what follows, we shall bring $dF$ in to play whenever it is needed specially in the case of horizontal lifts.

Observe that if $M$ is an orientable real hypersurface of the Euclidean space $R^{2n}$ with unit normal vector field $N$, then we know that horizontal lift $N^h$ is a unit normal vector field to the submanifold $TM$ of $R^{4n}$ and that the vertical lift $N^v \in \mathfrak{x}(TM)$ (cf.[1]). We have

$$\overline{J}N^h = (JN)^h = -(df(\xi))^h = -dF(\xi^h) + g(S(\xi), u)N^v \in \mathfrak{x}(TM) \quad (3.1)$$

and

$$\overline{J}N^v = (JN)^v = -\xi^v \in \mathfrak{x}(TM) \quad (3.2)$$

Let $M$ be an orientable real hypersurface of the Kaehler manifold $(R^{2n}, J, \langle \cdot, \cdot \rangle)$. Then as $TM$ is submanifold of the Kaehler manifold $(R^{4n}, \overline{J}, \langle \cdot, \cdot \rangle)$, we denote by $\Gamma(T^\perp TM)$ the space of smooth normal vector fields to $TM$. The restriction of the complex structure $\overline{J}$ on $R^{2n}$ to $\mathfrak{x}(TM)$ and $\Gamma(T^\perp TM)$ can be expressed as

$$\overline{J}(E) = \overline{\varphi}(E) + \overline{\psi}(E), \overline{J}(\overline{N}) = \overline{G}(\overline{N}) + \overline{\chi}(\overline{N}), \quad E \in \mathfrak{x}(TM), \overline{N} \in \Gamma(T^\perp TM)$$

where $\varphi(E)$, $\overline{G}(\overline{N})$ are the tangential and $\overline{\psi}(E)$, $\overline{\chi}(\overline{N})$ are the normal components of $\overline{J}E$, and $\overline{J}(\overline{N})$ respectively. Note that the horizontal lift $N^h$ of the unit normal $N$ to the hypersurface $M$ is normal to $TM$ that is $N^h \in \Gamma(T^\perp TM)$, where as the vertical lift $N^v \in \mathfrak{x}(TM)$.

**Lemma 3.3.** Let $TM$ be the tangent bundle of an orientable real hypersurface of $R^{2n}$. Then for $X \in \mathfrak{x}(M)$,

$$\overline{\varphi}(X^h) = (\varphi(X))^h - g(S(X), u)\xi^v, \quad \overline{\varphi}(X^v) = (\varphi(X))^v + \eta(X) \circ \pi N^v$$

$$\overline{\psi}(X^h) = \eta(X) \circ \pi N^h, \quad \overline{\psi}(X^v) = 0$$

**Proof.** Note that for the horizontal lift $X^h$ we have

$$\overline{J}X^h = \overline{J}dF(X^h) = \overline{J}((df(\xi))^h + g(SX, u) \circ \pi N^v)$$

$$= (JdF(\xi))^h + g(SX, u) \circ (JN)^v$$

$$= (\varphi X + \eta(X)N)^h - g(SX, u) \circ \pi \xi^v$$

$$= (\varphi(X))^h - g(SX, u) \circ \pi \xi^v + \eta(X) \circ \pi N^h$$

which together with the definition $\overline{J}X^h = \overline{\varphi}(X^h) + \overline{\psi}(X^h)$, on equating tangential and normal components give

$$\overline{\varphi}(X^h) = (\varphi(X))^h - g(S(X), u)\xi^v$$

and

$$\overline{\psi}(X^h) = \eta(X) \circ \pi N^h$$

Similarly for the vertical lift $X^v$, we have

$$\overline{J}X^v = \overline{\varphi}(X^v) + \overline{\psi}(X^v) = (JX)^v = (\varphi X + \eta(X)N)^v$$
which gives
\[(\overline{\varphi}(X^v)) + \overline{\psi}(X^v) = (\varphi X)^v + \eta(X) \circ \pi N^v\]
Comparing the tangential and normal components we conclude
\[\overline{\varphi}(X^v) = (\varphi(X))^v + \eta(X) \circ \pi N^v, \text{ and } \overline{\psi}(X^v) = 0.\]

\[\square\]

We choose a unit normal vector field \(N^* \in \Gamma(T^\perp TM)\) such that \(\{N^*, N^h\}\) is a local orthonormal frame of normals for the submanifold \(TM\). It is known that \(N^*\) is vertical vector field on the tangent bundle \(\mathbb{R}^{4n}\) (cf. [1]). Since, \(\langle JN^*, N^* \rangle = 0, \langle JN^*, N^h \rangle = \langle N^*, \xi^h \rangle = 0\), it follows that \(JN^* \in \mathfrak{X}(TM)\) and we define unit vector field \(\zeta \in \mathfrak{X}(TM)\) by
\[\zeta = -JN^* \tag{3.3}\]

Now, for any normal vector field \(N \in \Gamma(T^\perp TM)\), we have
\[N = \langle N, N^* \rangle N^* + \langle N, N^h \rangle N^h\]
which together with equations (3.1), (3.2) and (3.3) gives \(\chi(N) = 0\) and that \(J(N) \in \mathfrak{X}(TM)\), is given by
\[J(N) = \langle J(N), \zeta \rangle \zeta + \langle J(N), T \rangle T \tag{3.4}\]
where \(T \in \mathfrak{X}(TM)\), is given by
\[T = \xi^h - g(S(\xi), u)N^v = -JN^h \tag{3.5}\]
Also, using equation (3.2), we have
\[-\xi^v = JN^v = \overline{\varphi}(N^v) + \overline{\psi}(N^v)\]
which gives
\[\overline{\varphi}(N^v) = -\xi^v \text{ and } \overline{\psi}(N^v) = 0 \tag{3.6}\]
Moreover, we have
\[\overline{\varphi}(\zeta) = 0 \text{ and } \overline{\psi}(\zeta) = N^*, \overline{\psi}(\xi^h) = N^h \tag{3.7}\]
If we denote by \(\alpha, \beta\) the smooth 1-forms on \(TM\) dual to the vector field \(\zeta\) and \(T\) respectively, then for \(E \in \mathfrak{X}(TM)\), it follows that
\[J(\overline{\psi}(E)) = -\alpha(E)\zeta - \beta(E)T\]
and consequently, operating \(J\) on \(J(E) = \overline{\varphi}(E) + \overline{\psi}(E), E \in \mathfrak{X}(TM)\), we get
\[\overline{\varphi}^2 = -I + \alpha \otimes \zeta + \beta \otimes T \text{ and } \overline{\psi} \circ \overline{\varphi} = 0 \tag{3.8}\]
Using Lemma 2.1 and equations (3.3), (3.5), (3.6), (3.8), we see that the vector fields \(\zeta, T\) and 1-forms \(\alpha, \beta\) satisfy
\[\overline{\varphi}(\zeta) = 0, \overline{\varphi}(T) = 0, \overline{\psi}(\zeta, T) = 0, \alpha \circ \overline{\varphi} = 0, \beta \circ \overline{\varphi} = 0 \tag{3.9}\]
Also, as $\overline{g}$ is the induced metric on the submanifold $TM$ and $\mathcal{J}$ is skew symmetric with respect to the Hermitian metric $\langle \cdot, \cdot \rangle$, we have
\[
\overline{g}(\varphi(E), F) = -\overline{g}(E, \varphi(F)), \quad E, F \in \mathfrak{X}(TM)
\] (3.10)
Then using equations (3.8), (3.9) and (3.10), we have
\[
\overline{g}(\varphi(E), \varphi(F)) = \overline{g}(E, F) - \alpha(E)\alpha(F) - \beta(E)\beta(F), \quad E, F \in \mathfrak{X}(TM)
\] (3.11)
Thus we have proved the following

**Lemma 3.4.** Let $TM$ be the tangent bundle of an orientable real hypersurface of $R^{2n}$. Then there is a structure $(\overline{\varphi}, \zeta, T, \alpha, \beta, \overline{g})$ similar to contact metric structure on $TM$, where $\overline{\varphi}$ is a tensor field of type $(1, 1)$, $\zeta, T$ are smooth vector fields and $\alpha, \beta$ are smooth 1-forms dual to $\zeta, T$ with respect to the Riemannian metric $\overline{g}$ satisfying
\[
\overline{\varphi}^2 = -I + \alpha \otimes \zeta + \beta \otimes T, \quad \overline{\varphi}(\zeta) = 0, \quad \overline{\varphi}(T) = 0, \quad \alpha \circ \overline{\varphi} = 0, \quad \beta \circ \overline{\varphi} = 0, \quad \overline{g}(\zeta, T) = 0
\]

In the next Lemma, we compute the co-variant derivatives of the tensor $\overline{\varphi}$.

**Lemma 3.5.** Let $(\overline{\varphi}, \zeta, T, \alpha, \beta, \overline{g})$ be the structure on the tangent bundle $TM$ of an orientable real hypersurface $M$ of the Euclidean space $R^{2n}$. Then

(i) \((\nabla_{X^h} \overline{\varphi})(Y^h) = \{(\nabla_X \overline{\varphi})(Y)^h - \{X(g(SY, u) + g(SY, u)JSX)\} + v^v\}

(ii) \((\nabla_{X^h} \overline{\varphi})(\eta^v) = 0.

(iii) \((\nabla_{X^h} \overline{\varphi})(\eta^v) = 0, \quad (\nabla_{X^h} \overline{\varphi})(\eta^v) = g(SX, \varphi Y) \circ \pi N^v + g(SX, Y) \circ \pi \xi^v.

**Proof.** Using the definition of $\mathcal{J}$, Lemma 2.1 and Lemma 3.3 together with equation (3.1), we get for $X, Y \in \mathfrak{X}(M)$
\[
\mathcal{J}Y^h = \mathcal{J}dF(Y^h) = \mathcal{J}((df(Y))^h + g(SY, u) \circ \pi N^v)
\]
\[
= (\varphi Y + \eta(Y))^h - g(SY, u) \circ \pi \xi^v
\]
\[
= \overline{\varphi}(Y^h) + \eta(Y) \circ \pi N^h
\]
which gives
\[
\overline{D}_{X^h} \mathcal{J}Y^h = \overline{D}_{(df(X))^h + g(SX, u) \circ \pi N^v} (\overline{\varphi}(Y^h) + \eta(Y) \circ \pi N^h)
\]
\[
= \overline{D}_{(df(X))^h \varphi}(Y^h) + X(\eta(Y)) \circ \pi N^h + \eta(Y) \circ \pi \overline{D}_{(df(X))^h} N^h
\]
\[
+ g(SX, u) \circ \pi \eta(Y) \circ \pi \overline{D}_{N^v} N^h
\]
Note that the tangent bundle $TR^{2n} = R^{4n}$ has Sasaki metric and thus using Lemma 7.2 of [10] (keeping in view that $R^{2n}$ is flat), in the above equation, we get
\[
\overline{D}_{X^h} \mathcal{J}Y^h = \nabla_{X^h} \overline{\varphi}(Y^h) + h(X^h, \overline{\varphi}(Y^h) + X(\eta(Y)) \circ \pi N^h - \eta(Y) \circ \pi (SX)^h
\]
Similarly we have
\[
\mathcal{D} X^h Y^h = \mathcal{D} \left( (df(X))^h + g(SX,u) \circ \pi N^v \right) \left( (df(Y))^h + g(SY,u) \circ \pi N^v \right)
\]
\[
= \mathcal{D} \left\{ \nabla X^h Y^h + h(X^h, Y^h) + X(g(SY,u) \circ \pi N^v) + g(SY,u) \circ \pi (D_X N)^v 
+ g(SX,u) \circ \pi D_N^v (dfY)^h + 0 + 0 \right\}
\]
\[
= \nabla (X^h Y^h) + \nabla h(X^h, Y^h) - X(g(SY,u) \circ \pi \xi^v)
- g(SY,u) \circ \pi \mathcal{J}(SX)^v
\]
\[
= \nabla (X^h Y^h) + \nabla h(X^h, Y^h) - g(SX,Y) \circ \pi \xi^h - X(g(SY,u) \circ \pi \xi^v)
- g(SY,u) \circ \pi (\varphi SX)^v - g(SY,u) \circ \pi \eta(SX)N^v \tag{3.13}
\]

where we used Lemmas 2.3, 2.4 and Lemma 7.2 in [10]. Now as \((\mathbb{R}^n, \mathcal{J}, \langle \cdot , \cdot \rangle )\) is a Kaehler manifold, the equations (3.12) and (3.13) on comparing tangential we get
\[
(\nabla X^h \varphi)(Y^h) = ((\nabla X \varphi)(Y))^h - \{ X (g(SY,u) + g(SY,u)JSX) \}^v
\]
which proves (i).

Now, using \(h(X^v, Y^v) = 0\) and \(\tilde{S}_{\varphi Y^v} X^v = 0\) together with \(\mathcal{D} X^v \mathcal{J} Y^v = \mathcal{D} X^v Y^v\), and comparing tangential components, we immediately arrive at
\[
(\nabla X^v \varphi)(Y^v) = 0
\]
Next, we have \(\nabla X^v \varphi(Y^h) = \nabla X^v \left( (\varphi Y)^h - g(SX,u) \circ \pi \xi^v \right) = \nabla X^v (\varphi Y)^h = g(SX,\varphi Y) \circ \pi N^v\) and \(\nabla \varphi (\nabla X^v Y^h) = g(SX,\varphi Y) \circ \pi \varphi (N^v) = -g(SX,Y) \circ \pi \xi^v\).
Thus, we get
\[
(\nabla X^v \varphi)(Y^h) = g(SX,\varphi Y) \circ \pi N^v + g(SX,Y) \circ \pi \xi^v
\]
Finally, using \(h(X^h, Y^v) = 0\) and \(\tilde{S}_{\varphi(Y^v)} X^h = 0\) together with \(\mathcal{D} X^h \mathcal{J} Y^v = \mathcal{D} X^h Y^v\), and comparing tangential components, we immediately arrive at
\[
(\nabla X^h \varphi)(Y^v) = 0
\]

\[\square\]

**Lemma 3.6.** Let \((\varphi, \zeta, T, \alpha, \beta, \varphi)\) be the structure on the tangent bundle \(TM\) of an orientable real hypersurface \(M\) of the Euclidean space \(\mathbb{R}^{2n}\). Then for \(E \in \mathfrak{X}(TM)\),
\[
\nabla_E \zeta = \varphi (\mathcal{S}^{N^*} (E)) - \mathcal{J} \left( \nabla^h_E N^* \right), \quad h(E, \zeta) = \varphi (\mathcal{S}^{N^*} (E))
\]
\[
\nabla_E T = \varphi (\mathcal{S}^{N^h} (E)) - \mathcal{J} \left( \nabla^h_E N^h \right), \quad h(E, T) = \varphi (\mathcal{S}^{N^h} (E))
\]
Proof. Using equation (2.2), we have
\[ \nabla_E \zeta = D_E \zeta - h(E, \zeta) = -J D_E N^* - h(E, \zeta) \]
\[ = J(S_N^*(E)) - J(\nabla^\perp_E N^*) - h(E, \zeta) \]
\[ = \phi(S_N^*(E)) + \psi(S_N^*(E)) - J(\nabla^\perp_E N^*) - h(E, \zeta) \]
Since \( J(N) \in \mathfrak{X}(TM) \) for each normal \( N \in \Gamma(T^\perp TM) \), equation tangential and normal components in above equation, we get the first part. The second part follows similarly using \( T = -JN^h \). \( \square \)

Now, we prove the following:

**Theorem 3.7.** The tangent bundle \( TM \) of an orientable real hypersurface \( M \) of the Euclidean space \( \mathbb{R}^{2n} \) is a CR-submanifold of the Kaehler manifold \((\mathbb{R}^{4n}, J, \langle \cdot, \cdot \rangle)\).

Proof. Use the structure \( (\varphi, \zeta, T, \alpha, \beta, g) \) on the submanifold \( TM \) of \( \mathbb{R}^{4n} \) to define the distribution \( D \) by
\[ D = \{ E \in \mathfrak{X}(TM) : \alpha(E) = \beta(E) = 0 \} \]
and \( D^\perp \) be the distribution spanned by the orthogonal vector fields \( \zeta \) and \( T \). Note that \( \zeta \) is unit vector field on \( TM \) and the length of the vector field \( T \) satisfies
\[ ||T||^2 = 1 + 2g(S(\xi), u)^2 \geq 1 \]
which shows that \( D^\perp \) is 2-dimensional distribution on \( TM \) and that \( JD^\perp = \Gamma(T^\perp TM) \). It is easy to see that \( D \) and \( D^\perp \) are orthogonal complementary distributions and that \( \text{dim } D = 4(n - 1) \). Note that for \( E \in \mathfrak{X}(TM) \), we have
\[ \psi(E) = \langle \psi(E), N^* \rangle N^* + \langle \psi(E), N^{h} \rangle N^{h} = \alpha(E)N^* + \beta(E)N^h \]
and consequently if \( E \in D \), then above equation gives \( J E = \varphi E \) which is orthogonal to both \( \zeta \) and \( T \) and that \( J E \in D \), which implies \( JD = D \). This proves that \( TM \) is a CR-submanifold of the Kaehler manifold \((\mathbb{R}^{4n}, J, \langle \cdot, \cdot \rangle)\) (cf. [8]). \( \square \)

4. Killing Vector Fields on \( TM \)

Let \( TM \) be the tangent bundle of an orientable real hypersurface \( M \) of the Euclidean space \( \mathbb{R}^{2n} \). Recall that a vector field \( \zeta \in \mathfrak{X}(TM) \) on the Riemannian manifold \((TM, \bar{g})\) is said to be Killing if
\[ (\mathcal{L}_\zeta \bar{g})(E, F) = 0, \quad E, F \in \mathfrak{X}(TM) \]
where \( \mathcal{L}_\zeta \) is the Lie derivative with respect to the vector field \( \zeta \). We have seen in previous section that the tangent bundle \((TM, \bar{g})\) admits a structure \( (\mathfrak{V}, \zeta, T, \alpha, \beta, \bar{g}) \), that is similar to the almost contact structure. In this section
we are interested in finding conditions under which the special vector fields $\zeta$ and $T$ are Killing vector fields and as a particular case we get that the tangent bundle $(TS^{2n-1}, \mathcal{g})$ of the unit sphere $S^{2n-1}$ in the Euclidean space $\mathbb{R}^{2n}$ admits a nontrivial Killing vector field.

**Theorem 4.1.** Let $(\varphi, \zeta, T, \alpha, \beta, \mathcal{g})$ be the structure on the tangent bundle $TM$ of an orientable real hypersurface $M$ of the Euclidean space $\mathbb{R}^{2n}$. Then the vector field $\zeta$ is Killing.

**Proof.** First note that on taking inner product with $N^*$ in each part of Lemma 2.5, we conclude that $\mathcal{g}(\nabla_E N^*, N^h) = 0$, $E \in \mathfrak{X}(TM)$ and consequently,

$$\mathcal{g}(\nabla_E N^*, N^h) = 0, \quad E \in \mathfrak{X}(TM) \quad (4.1)$$

Also using second part of equation (2.2) in (ii) and (iv) of Lemma 2.4, we conclude that $\nabla_E N^h = 0$, $E \in \mathfrak{X}(TM)$, that is $N^h$ is parallel on the normal bundle of $TM$. Moreover, we have

$$\nabla_E N^* = \left\langle \nabla_E N^*, N^h \right\rangle N^h = - \left\langle N^*, \nabla_E N^h \right\rangle N^h = 0$$

that is $N^*$ is parallel in the normal bundle of $TM$. Thus using equation (4.1) in Lemma 3.5, it follows that $\zeta$ is a parallel vector field and consequently, it is a Killing vector field. $\square$

**Theorem 4.2.** Let $(\varphi, \zeta, T, \alpha, \beta, \mathcal{g})$ be the structure on the tangent bundle $TM$ of an orientable real hypersurface $M$ of the Euclidean space $\mathbb{R}^{2n}$. Then the vector field $T$ is Killing if and only if the following condition holds

$$\mathcal{g}((\varphi \circ S_{Nh} - S_{Nh} \circ \varphi)(X^h), Y^h) = 0, \quad X, Y \in \mathfrak{X}(M) \quad (4.6)$$

and the equations (4.4)-(4.6) prove the Theorem. $\square$
Consider the unit sphere $S^{2n-1}$ in the Euclidean space $R^{2n}$, whose shape operator is given by $S = -I$. Using Lemma 2.4, we get on the tangent bundle $TS^{2n-1}$ that

$$
\overline{S}^*_N (X^h) = (S(X))^h = -X^h, \quad \overline{S}^*_N (X^v) = 0
$$

Then the Lemma 3.3 together with above equation, gives

$$
(\varphi \circ \overline{S}^*_N - \overline{S}^*_N \circ \varphi) (X^h) = -\varphi (X^h) - \overline{S}^*_N ( ((\varphi(X))^h - g(S(X), u) \circ \pi^v) )
$$

and consequently,

$$
\overline{g} ( (\varphi \circ \overline{S}^*_N - \overline{S}^*_N \circ \varphi)(X^h), Y^h ) = 0, \quad X, Y \in \mathcal{X}(S^{2n-1})
$$

Thus as a particular case of the Theorem 4.2, we have

**Corollary 4.3.** Let $(\varphi, \zeta, T, \alpha, \beta, \overline{g})$ be the structure on the tangent bundle $TS^{2n-1}$ of the unit sphere $S^{2n-1}$ in the Euclidean space $R^{2n}$, $n > 1$. Then the vector field $T$ is a nontrivial Killing vector field.

**Proof.** It remains to be shown that $T$ is nontrivial. Since, $N^h$ is parallel in the normal bundle of $TS^{2n-1}$, by Lemmas 2.4 and 3.5, we have

$$
\nabla^X T = -\varphi (X^h), \quad X \in \mathcal{X}(S^{2n-1}) \tag{4.7}
$$

where we used the fact that the shape operator $S$ of the unit sphere $S^{2n-1}$ is given by $S = -I$. The Lemma 3.4 gives the rank of $\varphi$ is $4(n-1)$ and consequently, equation (4.7) gives that the Killing vector field $T$ is not parallel, that is $T$ is a nontrivial Killing vector field. \hfill \Box

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