Iranian Journal of Mathematical Sciences and Informatics
Vol. 10, No. 1 (2015), pp 45-58
DOI: 10.7508/ijmsi.2015.01.004

# Associated Graphs of Modules Over Commutative Rings

A. Abbasi*, H. Roshan-Shekalgourabi, D. Hassanzadeh-Lelekaami

Department of Pure Mathematics, Faculty of Mathematical Sciences,
University of Guilan, P. O. Box 41335-19141, Rasht, Iran.

E-mail: aabbasi@guilan.ac.ir
E-mail: hroshan@guilan.ac.ir
E-mail: dhmath@guilan.ac.ir

Abstract. Let $R$ be a commutative ring with identity and let $M$ be an $R$-module. In this paper we introduce a new graph associated to modules over commutative rings. We study the relationship between some algebraic properties of modules and their associated graphs. A topological characterization for the completeness of the special subgraphs is presented. Also modules whose associated graph is complete, tree or complete bipartite are studied and several characterizations are given.

**Keywords:** Associated Graph of module, Prime spectrum, Connected graph, Diameter.

**2010 Mathematics subject classification:** 13C99, 05C25, 05C10.

## 1. Introduction

The notion of a graph of zero-divisors of a commutative ring was introduced in [14], by studying the coloring of a graph constructed by all elements of a commutative ring $R$. In [6], the authors have given a counterexample for the conjecture given by Beck in [14]. After some years D. F. Anderson and P. S. Livingston investigated the interplay between the ring-theoretic properties of a commutative ring and the graph theoretic properties of the zero-divisor graph

*Corresponding Author

Received 27 April 2013; Accepted 09 September 2014

of the ring (see [8]). The zero-divisor graph of a commutative ring has been studied extensively by several authors, for example see [1, 4, 5, 7, 9, 11, 12, 18]. Recently, a lot of graphs related to the rings and modules have been defined by several authors and have been investigated by many algebraist, for example see [2, 23, 25, 27].

Throughout this paper, all rings are commutative with identity and all modules are unitary. We introduce a new graph associated to modules over commutative rings. Let $M$ be an $R$-module. We associate a graph $G(M)$ to an $R$-module $M$ whose vertices are nonzero proper submodules of $M$ in these way that two distinct vertices $N$ and $L$ are adjacent if and only if $N + L = M$. We investigate the relationship between the algebraic properties of an $R$-module $M$ and the properties of the associated graph $G(M)$. In this paper, the $R$-modules whose associated graphs are complete is completely characterized (see Theorem 2.3). Moreover, a topological characterization for completeness of $G^{\text{Spec}}(M)$, the subgraph of $G(M)$ generated (induced) by the prime spectrum of $M$, is presented (see Theorem 2.4). Modules whose associated graphs is connected (Proposition 2.12), tree (Theorem 2.13) or complete bipartite (see Theorem 2.25) are studied.

We will first define some notions which is used throughout the paper. *Coloring* of a graph $G$ is an assignment of colors (elements of some set) to the vertices of $G$, one color to each vertex, so that adjacent vertices are assigned distinct colors. If $n$ colors are used, then the coloring is referred to as an *$n$-coloring*. If there exists an $n$-coloring of a graph $G$, then $G$ is called *$n$-colorable*. The minimum $n$ for which a graph $G$ is $n$-colorable is called the *chromatic number of* $G$, and is denoted by $\chi(G)$. For a graph $G$, the *degree* of a vertex $v$ in $G$ is the number of edges of $G$ incident with $v$. Recall that a graph is said to be *connected* if for each pair of distinct vertices $v$ and $w$, there is a finite sequence $v = v_1, \ldots, v_n = w$ of distinct vertices where each pair $\{v_i, v_{i+1}\}$ is an edge. Such a sequence is said to be a path and the distance, $d(v, w)$, between connected vertices $v$ and $w$ is the length of the shortest path connecting them. A graph in which each pair of distinct vertices is joined by an edge is called a *complete* graph. The *diameter* of a connected graph is the supremum of the distances between vertices. The diameter is 0 if the graph consists of a single vertex and a connected graph with more than one vertex has diameter 1 if and only if it is complete. An *$r$-partite* graph is one which the vertex set can be partitioned into $r$ subsets so that no edge has both ends in one subset. A *complete $r$-partite* graph is an $r$-partite graph in which each vertex in a subset is joined to all vertices in another subsets. The *complete bipartite* (i.e., 2-partite) graph with part sizes $m$ and $n$ is denoted by $K_{m,n}$. In particular, $K_{1,n}$ is called a *star* graph. A *tree* is a connected graph that contains no cycle. We use $K_n$ for the complete graph with $n$ vertices. A *clique* of a graph $G$ is a maximal complete subgraph of $G$ and the number of vertices in the largest clique of graph $G$, denoted by clique $(G)$, is called the *clique number* of $G$. Obviously $\chi(G) \geq$ clique $(G)$ for a general graph $G$. A graph $G$ is said to be *totally disconnected* if no two vertices of $G$ are adjacent. The *girth* of a graph $G$, denoted by g $(G)$, is the length of a shortest cycle (if any) in $G$.

## 2. Main Results

**Definition 2.1.** We associate a graph $G(M)$ to an $R$-module $M$ whose vertices are nonzero proper submodules of $M$ and two distinct vertices $N$ and $L$ are adjacent if $N + L = M$. We use the notation $G(M)$ for the set of all vertices of $G(M)$.

We are going to give some examples of different classes of $R$-modules $M$ such that $G(M)$ is an totally disconnected graph. We initially need some definitions. A submodule $S$ of an $R$-module $M$ is called *superfluous* (or *small*) in $M$, in case for every submodule $L$ of $M$, $S + L = M$ implies that $L = M$ (see [10, p.72]). A nonzero $R$-module $M$ is called *co-uniform* in case that all of its proper submodules are superfluous in $M$ (see [10, p.294]). Recall that an $R$-module $N$ is said to be *sum-irreducible* precisely when it is nonzero and cannot be expressed as the sum of two proper submodules of itself (see [15, Definition and Exercise 7.2.8]). We recall that an $R$-module $M$ is said to be a *multiplication* module (see [13] and [16]) if every submodule $N$ of $M$ is of the form $IM$ for some ideal $I$ of $R$. An $R$-module $M$ is called *co-semisimple* if every proper submodule of $M$ is the intersection of maximal submodules (see [10, p.122]). A module $M$ is called *coatomic* if every proper submodule of $M$ is contained in a maximal submodule of $M$ (see [29]). It is well-known that co-semisimple modules, finitely generated modules and multiplication modules (see [16, Theorem 2.5]) are coatomic. An $R$-module $M$ is called a *quasi-semi-local* (resp. a *quasi-local*) module if $\mathrm{Max}(M)$, the set of all maximal submodules of $M$, is a non-empty finite (resp. a singleton) set.

**Remark 2.2.** Let $M$ be a nonzero $R$-module. It is easy to verify the following facts.

(1) $M$ is sum-irreducible if and only if the graph $G(M)$ is totally disconnected.
(2) A nonzero submodule $S$ of $M$ is superfluous if and only if $S$ is an isolated vertex in $G(M)$. In particular, if $M$ is co-uniform, then the graph $G(M)$ is totally disconnected.
(3) If $M$ is coatomic, then $M$ is quasi-local if and only if $G(M)$ is a totally disconnected graph.
(4) The graph $G(M)$ is not a star graph, for any $R$-module $M$ with $|G(M)| \geq 3$.

Let $M$ be a module over the Dedekind domain $R$ and let $K$ denotes the quotient field of $R$. Then by Remark 2.2 and [17, Lemma 2.4], the graph $G(M)$ is totally disconnected if and only if one of the following conditions hold:

(1) $M \cong R/\mathfrak{p}^n$, for some nonzero prime ideal $\mathfrak{p}$ of $R$ and an integer $n \geq 0$ (by definition, $\mathfrak{p}^0 = R$);
(2) $M \cong (K/R)_{\mathfrak{p}}$, for some nonzero prime ideal $\mathfrak{p}$ of $R$;
(3) $R$ is local and $M$ is torsion-free of rank one.

As an example, consider the $\mathbb{Z}$-module $M = \mathbb{Z}_{p^\infty}$. Then $G(\mathbb{Z}_{p^\infty})$ is a totally disconnected graph.

All semisimple $R$-modules represented as a finite direct product of simple modules are connected and $r$-partite for some $r$.

**Theorem 2.3.** *Let $M = \prod_{i=1}^n M_i$ and let $M_i$ be a simple $R$-module for all $1 \leq i \leq n$. Then $G(M)$ is a connected $n$-partite graph.*

*Proof.* If $n = 1$, the result is evident. For $n > 1$, let $V = \{(a_1, \cdots, a_n) \in \{0,1\}^n | a_i \neq 0, a_j \neq 1$ for some $i \neq i\}$. It is sufficient for us to prove the the theorem for the graph on vertex set $V$ which two vertices $(a_1, \cdots, a_n)$ and $(b_1, \cdots, b_n)$ are adjacent if and only if $(a_1, \cdots, a_n) + (b_1, \cdots, b_n) = (1, \cdots, 1)$. It is easy to see that any vertices $(a_1, \cdots, a_n)$ and $(b_1, \cdots, b_n)$ are connected. For the last part of the theorem, let $V_1 = \{(a_1, \cdots, a_n) \in V | a_1 = 0\}$ and for all $i > 1$, $V_i = \{(a_1, \cdots, a_n) | a_i = 0\} \setminus \cup_{j=1}^{i-1} V_j$. Now, $\{V_1, \cdots, V_n\}$ is a partition of $V$. In particular, $(1, \cdots, 1, 0) \in V_n$. $\square$

Obviously, for any $R$-module $M$ with $\mathrm{Max}(M) \neq \emptyset$, the subgraph $G^{\mathrm{Max}}(M)$ which is generated by $\mathrm{Max}(M)$ is complete. We are going to investigate the completeness problem of other subgraphs of $G(M)$, when $M$ is a certain $R$-module. Before that we need some definitions.

A submodule $N$ of an $R$-module $M$ is said to be *prime* if $N \neq M$ and whenever $rm \in N$ (where $r \in R$ and $m \in M$) then $r \in (N : M)$ or $m \in N$. If $N$ is prime, then ideal $\mathfrak{p} = (N : M)$ is a prime ideal of $R$. In this circumstance, $N$ is said to be $\mathfrak{p}$-*prime* (see [19]). The set of all prime submodules of an $R$-module $M$ is called the *prime spectrum of* $M$ and denoted by $\mathrm{Spec}(M)$. We remark that $\mathrm{Spec}(\mathbf{0}) = \emptyset$ and that $\mathrm{Spec}(M)$ may be empty for some nonzero $R$-module $M$. For example, $\mathbb{Z}_{p^\infty}$ as a $\mathbb{Z}$-module has no prime submodule for any prime integer $p$ (see [20]). Such a module is said to be *primeless*. For any submodule $N$ of $M$ the set of all prime submodules of $M$ containing $N$ is denoted by $V^*(N)$ (see [24]). Suppose that $M$ is an $R$-module. Set $\mathsf{Z}^*(M) = \{V^*(N) \,|\, N$ is a submodule of $M\}$. There is a topology, $\tau^*$ say, on $\mathrm{Spec}(M)$ due to $\mathsf{Z}^*(M)$ as the collection of all closed sets if and only if $\mathsf{Z}^*(M)$ is closed under finite union. When this is the case, we call the topology $\tau^*$ the *quasi-Zariski topology on* $\mathrm{Spec}(M)$ and $M$ is called a *top* module (see [24]). There are plentiful examples of top modules in [3] and [24].

For a non-primeless $R$-module $M$, let $G^{\mathrm{Spec}}(M)$ denote the subgraph of $G(M)$ which is generated by elements of $\mathrm{Spec}(M)$. In the sequel, we introduce a topological characterization for completeness of $G^{\mathrm{Spec}}(M)$. We recall that, if $M$ is a top $R$-module, the closure of $\{P\}$ is $V^*(P)$, for every $P \in \mathrm{Spec}(M)$. Recall that a topological space is a $T_1$-*space* if and only if every singleton subset is closed.

**Theorem 2.4.** *Let $M$ be a non-primeless top $R$-module. Then $G^{\mathrm{Spec}}(M)$ is a complete graph if and only if $\mathrm{Spec}(M)$ is a topological $T_1$-space.*

*Proof.* Suppose that $G^{\mathrm{Spec}}(M)$ is a complete graph and let $Q$ and $P$ be two prime submodules of $M$ such that $Q \in V^*(P)$. If $P \neq Q$, then since $P + Q = M$ we infer that $Q = M$ which is impossible. Therefore, every singleton subset is closed, and so, $\mathrm{Spec}(M)$ is a topological $T_1$-space. Conversely, suppose that $\mathrm{Spec}(M)$ is a topological $T_1$-space. Therefore every prime submodule is a maximal element in the set of all prime submodules of $M$. Hence, $G^{\mathrm{Spec}}(M)$ is a complete graph. □

**Corollary 2.5.** *Let $M$ be a non-primeless top $R$-module. If $\mathrm{Spec}(R)$ is a $T_1$-space, then $G^{\mathrm{Spec}}(M)$ is a complete graph. However, the converse is not true in general.*

*Proof.* Suppose $Q$ is a prime submodule of $M$. If $P$ is a prime submodule of $M$ belongs to the closure of $\{Q\}$, then $(Q : M) = (P : M)$. Hence, $Q = P$ by [24, Theorem 3.5]. This implies that $\mathrm{Spec}(M)$ is a $T_1$-space and the result follows from Theorem 2.4.

For the last statement, consider $M = \bigoplus_p \mathbb{Z}/p\mathbb{Z}$ as a $\mathbb{Z}$-module, where $p$ runs through the set of all prime integers. Since $\mathrm{Max}(M) = \mathrm{Spec}(M)$, $G^{\mathrm{Spec}}(M)$ is a complete graph but $\mathrm{Spec}(\mathbb{Z})$ is not a $T_1$-space. □

It is well known that any ring $R$ is a top $R$-module. So, the next corollaries are direct consequences of Theorem 2.4.

**Corollary 2.6.** *Let $R$ be a ring. Then the following are equivalent:*

(1) *$G^{\mathrm{Spec}}(R)$ is a complete graph;*
(2) *$\mathrm{Spec}(R)$ is a $T_1$-space;*
(3) *$\mathrm{Spec}(R)$ is a Hausdorff space.*

**Corollary 2.7.** *Let $R$ be a Noetherian ring. Then the following are equivalent:*

(1) *$G^{\mathrm{Spec}}(R)$ is a complete graph;*
(2) *$\mathrm{Spec}(R)$ is discrete and finite;*
(3) *$\mathrm{Spec}(R)$ is a discrete space.*

**Proposition 2.8.** *Let $M$ be a coatomic $R$-module. Then the following hold:*

(1) *The clique number and the chromatic number of $G(M)$ are equal to the cardinal number of the set of maximal submodules of $M$.*

(2) *The girth of* $G(M)$ *is always* 3 *except when* $M$ *has at most two maximal submodules.*

*Proof.* (1) Let $S$ be a complete subgraph of $G(M)$. For any vertex $N$ of $S$, choose a maximal submodule $P_N$ of $M$ with $N \subseteq P_N$. For any distinct vertices $N$ and $L$ of $S$, since $N + L = M$, we have $P_N + P_L = M$, and so $P_N \neq P_L$. Thus the subgraph of $G(M)$ induced by $\{P_N | N \text{ is a vertex of } S\}$ is a complete graph which its cardinality is at least the cardinality of $S$. On the other hand as we mentioned the subgraph $G^{\text{Max}}(M)$ is complete. Therefore, the clique number of $G(M)$ is the cardinal number of the set of maximal submodules of $M$.

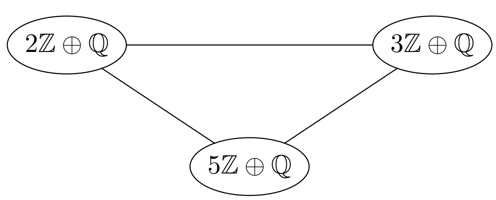
To find the chromatic number of $G(M)$, let $\{P_\lambda | \lambda \in \Lambda\}$ be the set of all maximal submodules of $M$ and suppose that $<$ is a well ordering on $\Lambda$. For any $\lambda \in \Lambda$, let $G_\lambda(M) = \{N \subseteq M | 0 \neq N \subseteq P_\lambda \text{ and } N \notin \bigcup_{\lambda' < \lambda} G_{\lambda'}(M)\}$. Then for each $\lambda \in \Lambda$, $P_\lambda \in G_\lambda(M)$ and so $G_\lambda(M) \neq \emptyset$. Also $\{G_\lambda(M) | \lambda \in \Lambda\}$ forms a partition for the set of all vertices of $G(M)$. Since for every $\lambda \in \Lambda$, any two vertices in $G_\lambda(M)$ are not adjacent, all vertices in $G_\lambda(M)$ can have the same color. However, the $P_\lambda$'s must have different colors. Therefore the chromatic number of $G(M)$ is equal to $|\Lambda|$.

(2) If $P_1, P_2, P_3$ are three distinct maximal submodules of $M$, then they are the vertices of a triangle in $G(M)$.

□

In the next example we show that the coatomicness of the $R$-module $M$ in the part (2) of Proposition 2.8 is *not* a necessary condition.

**Example 2.9.** Let $R = \mathbb{Z}$ and consider the $R$-module $M = \mathbb{Z} \oplus \mathbb{Q}$. Note that $\text{Max}(M) = \{pM \,|\, p \text{ is a prime integer}\}$. It is easy to see that $M$ has no maximal submodule $P$ such that $\mathbb{Z} \oplus (0) \subseteq P$. Hence, $M$ is not coatomic. But we have the below triangle in $G(M)$;




Therefore, $\text{g}\,(G(M)) = 3$.

Two vertices $N$ and $L$ are *orthogonal* in $G(M)$ if we have $N + L = M$, and for any vertex $K \in G(M)$ either $N + K \neq M$ or $L + K \neq M$. If $M$ has at least three maximal submodules, then they cannot be orthogonal to each other. We will use the notion of orthogonality to find girth of certain modules (see Example 2.11).

**Theorem 2.10.** *Let $M$ be a coatomic $R$-module. Then the following are equivalent:*

(1) *$G(M)$ has no triangle.*
(2) *Every two adjacent submodules are orthogonal.*
(3) *$M$ has at most two maximal submodules.*

*Proof.* (1) $\Rightarrow$ (2) Let $N$ and $L$ be two adjacent vertices which are not orthogonal. Then by the definition of orthogonality, there exists another vertex $K$ which is adjacent to both $N$ and $L$. This means that there is a triangle in $G(M)$, which is a contradiction.

(2) $\Rightarrow$ (3) If there exist at least three maximal submodules, then they cannot be orthogonal to each other.

(3) $\Rightarrow$ (1) Let $P_1$ and $P_2$ be the only two maximal submodules of $M$. Then for any three vertices $N_1, N_2$ and $N_3$ of $G(M)$, at least two of them are contained in one of $P_1$ or $P_2$ (since $M$ is coatomic) and hence they are not adjacent. Therefore there is no triangle in $G(M)$. □

**Example 2.11.** Consider the finitely generated $\mathbb{Z}$-module $M = \frac{\mathbb{Z}}{4\mathbb{Z}} \oplus \frac{\mathbb{Z}}{9\mathbb{Z}}$. This module has exactly two maximal submodules and $G(M)$ has a cycle of length four; $2M - 9M - 4M - 3M - 2M$. By Theorem 2.10, $G(M)$ has no triangle. Therefore $\mathrm{g}\,(G(M)) = 4$.

The *(Jacobson) radical* of an $R$-module $M$ is the intersection of all maximal submodules of $M$ and is denoted by $\mathrm{Rad}(M)$. If $M$ has no maximal submodule, then we define $\mathrm{Rad}(M) = M$. The radical of $M$ is the smallest submodule of $M$ that contains all superfluous submodules of $M$. However, the radical of $M$ need not be superfluous. If $M$ is coatomic, then $\mathrm{Rad}(M)$ is superfluous in $M$ (see [28, 21.6]).

In the following we investigate the connectedness of $G(M)$ for an $R$-module $M$.

**Proposition 2.12.** *Let $M$ be an $R$-module. Then $G(M)$ is connected if and only if* $\mathrm{Rad}(M) = 0$.

*Proof.* Suppose that $\mathrm{Rad}(M) = 0$. Then $\mathrm{Max}(M) \neq \emptyset$. Let $N$ and $L$ be two distinct elements of $G(M)$. There are maximal submodules $P_1$ and $P_2$ of $M$ such that $N \nsubseteq P_1$ and $L \nsubseteq P_2$. So, there is a path from $N$ to $L$ in $G(M)$.

Conversely suppose that $G(M)$ is connected and $\mathrm{Rad}(M) \neq 0$. Let $\mathrm{Rad}(M) = M$. If $L$ is a vertex of $G(M)$, then for each nonzero element $m \in L$, the proper submodule $Rm$ of $M$ is superfluous (see [28, p.177]). So, by Remark 2.2, $Rm$ is an isolated vertex of $G(M)$, which is impossible, since $G(M)$ is connected. Now, suppose that $\mathrm{Rad}(M) \neq M$. Then for each nonzero element $x \in \mathrm{Rad}(M)$, the proper submodule $Rx$ of $M$ is superfluous, i.e., $Rx$ is an isolated vertex of $G(M)$, that is a contradiction. □

**Theorem 2.13.** *Let $M$ be an $R$-module and $G(M)$ be a tree graph. Then* $|\mathrm{Max}(M)| = 2$.

*Proof.* Since $G(M)$ has no cycle, we must have $|\mathrm{Max}(M)| < 3$. By Proposition 2.12, $\mathrm{Rad}(M) = 0$. Hence $|\mathrm{Max}(M)| > 1$. Therefore $|\mathrm{Max}(M)| = 2$. □

From the proof of Proposition 2.12, we infer:

**Corollary 2.14.** *Let $M$ be an $R$-module with* $\mathrm{Rad}(M) = 0$. *Then* $\operatorname{diam} G(M) \leq 3$.

In next example we introduce an $R$-module $M$ such that $\operatorname{diam} G(M) = 2$.

**Example 2.15.** Let $R$ be an infinite $PID$ and consider the $R$-module $M := R$. Then for any two distinct vertices $Ra$ and $Rb$ of $G(M)$, there exists a prime element $p$ such that $p$ does not divide $a$ and $b$. Therefore $Ra+Rp = Rb+Rp = R$. So $d(Ra, Rb) \leq 2$. Since $Rab$ and $Ra$ are not adjacent we deduce that $\operatorname{diam} G(M) = 2$.

**Definition 2.16.** Let $M$ be an $R$-module. Put

$$\Theta = \{N \in G(M) \,|\, N = IM \text{ for some ideal } I \text{ of } R\}.$$

Then the subgraph of $G(M)$ generated by the set $\Theta$ is denoted by $G'(M)$.

Note that for a multiplication $R$-module $M$ we have $G'(M) = G(M)$.

**Proposition 2.17.** *Let $M$ be an $R$-module such that $\mathfrak{m}M \neq M$ for each maximal ideal $\mathfrak{m}$ of $R$. If* $\mathrm{Jac}(R) = 0$*, then $G'(M)$ is connected.*

*Proof.* Suppose $IM$ and $JM$ are two vertices of $G'(M)$, where $I$ and $J$ are two ideals of $R$. By assumption there are maximal ideals $\mathfrak{m}$ and $\mathfrak{m}'$ of $R$ such that $I \not\subseteq \mathfrak{m}$ and $J \not\subseteq \mathfrak{m}'$. Hence $I + \mathfrak{m} = R$ and $J + \mathfrak{m}' = R$. Therefore $IM + \mathfrak{m}M = M$ and $JM + \mathfrak{m}'M = M$. By assumption $\mathfrak{m}M$ and $\mathfrak{m}'M$ are two vertices of $G'(M)$. Either $\mathfrak{m}M = \mathfrak{m}'M$ or $\mathfrak{m}M \neq \mathfrak{m}'M$. In either case we have a path from $IM$ to $JM$ in $G'(M)$, which shows that $IM$ and $JM$ are connected. □

According to the Proposition 2.17, if $M$ is a faithfully flat $R$-module and $\mathrm{Jac}(R) = 0$, then $G'(M)$ is connected. In the next corollary we extend this result. An $R$-module $M$ is called *primeful* if either $M = (\mathbf{0})$ or $M \neq (\mathbf{0})$ and the map $\psi : \mathrm{Spec}(M) \to \mathrm{Spec}(R/\mathrm{Ann}(M))$ defined by $\psi(P) = (P : M)/\mathrm{Ann}(M)$ for every $P \in \mathrm{Spec}(M)$, is surjective. The class of primeful modules contains all modules which are Finitely generated or faithfully flat (see [21]).

**Corollary 2.18.** *Let $M$ be a nonzero faithful and primeful $R$-module. If* $\mathrm{Jac}(R) = 0$*, then $G'(M)$ is connected.*

*Proof.* Use Proposition 2.17 and [21, Result 2]. □

**Example 2.19.** Let $\Omega$ be the set of all prime integers $p$, and let $M = \prod_p \mathbb{Z}/p\mathbb{Z}$, where $p$ runs through $\Omega$. By [21, Example 1], $M$ is a faithful and primeful $\mathbb{Z}$-module. Hence by Corollary 2.18, $G'(M)$ is connected.

**Corollary 2.20.** *Let $M$ be a nonzero faithful and finitely generated multiplication $R$-module. If* $\mathrm{Jac}(R) = 0$*, then $G(M)$ is connected.*

*Proof.* Use Proposition 2.17 and [16, Theorem 3.1]. □

**Definition 2.21.** Let $M$ be an $R$-module such that $\mathrm{Max}(M) \neq \emptyset$. Put

$$\Lambda = \{N \in G(M) \,|\, N \nsubseteq \mathrm{Rad}(M)\}.$$

Then the subgraph generated by the set $\Lambda$ is denoted by $G^*(M)$. Note that if $\mathrm{Rad}(M) = 0$, then $G^*(M) = G(M)$.

It was mentioned in Remark 2.2 that each superfluous submodule of $M$ is an isolated vertex of $G(M)$. As we mentioned, the radical of $M$ is the smallest submodule of $M$ containing all superfluous submodules of $M$. As the second part of our work, we would like to study the subgraph $G^*(M)$ of $G(M)$.

**Proposition 2.22.** *Let $M$ be an $R$-module. Then the graph $G^*(M)$ is connected and* $\mathrm{diam}\, G^*(M) \leq 3$*.*

*Proof.* Let $N$ and $L$ be two distinct elements of $G^*(M)$. There are maximal submodules $P_1$ and $P_2$ of $M$ such that $N \nsubseteq P_1$ and $L \nsubseteq P_2$. Either $P_1 = P_2$ or $P_1 \neq P_2$. In either case we have a path from $N$ to $L$, in $G^*(M)$. Also, we infer that $\mathrm{diam}\, G^*(M) \leq 3$. □

For any subset $X$ of an $R$-module $M$, we define $\mathcal{V}(X) := \{P \in \mathrm{Max}(M) \,|\, X \subseteq P\}$.

**Proposition 2.23.** *Let $M$ be an $R$-module such that* $\mathrm{Max}(M)$ *is an infinite set. Then there exists an element $N \in G^*(M)$ such that* $\mathrm{Max}(M) \setminus \mathcal{V}(N)$ *is infinite.*

*Proof.* Assume to the contrary that for any element $N \in G^*(M)$, $\mathrm{Max}(M) \setminus \mathcal{V}(N)$ is a finite set. Let $L$ and $K$ be two distinct elements of $G^*(M)$. Since $\mathrm{Max}(M)$ is an infinite set, there exists a maximal submodule $Q$ of $M$ such that $L \subseteq Q$ and $K \subseteq Q$. So, $L + K \subseteq Q \neq M$ and we deduce that $G^*(M)$ is totally disconnected, which contradicts Proposition 2.22. □

In next example we compute the diam of $G^*(\mathbb{Z} \otimes \mathbb{Z}_{p^\infty})$.

**Example 2.24.** Consider the $\mathbb{Z}$-module $M = \mathbb{Z} \oplus \mathbb{Z}_{p^\infty}$, where $p$ is a prime integer. We claim that $\mathrm{diam}\, G^*(M) = 2$. Note that $\mathrm{Rad}(M) = (0) \oplus \mathbb{Z}_{p^\infty}$. Let $N$ and $L$ be two distinct vertices of $G^*(M)$ which are not adjacent. Then either $N \cap \mathbb{Z} \neq \mathbb{Z}$ or $N \cap \mathbb{Z}_{p^\infty} \neq \mathbb{Z}_{p^\infty}$. Similarly, $L \cap \mathbb{Z} \neq \mathbb{Z}$ or $L \cap \mathbb{Z}_{p^\infty} \neq \mathbb{Z}_{p^\infty}$. Consider the following four cases:

**Case 1:** Suppose that $N \cap \mathbb{Z} \neq \mathbb{Z}$ and $L \cap \mathbb{Z} \neq \mathbb{Z}$. There are integers $s$ and $r$ such that $N \cap \mathbb{Z} = r\mathbb{Z}$ and $L \cap \mathbb{Z} = s\mathbb{Z}$. We note that $r \neq 0$ and $s \neq 0$, since $N \nsubseteq \mathrm{Rad}(M)$ and $L \nsubseteq \mathrm{Rad}(M)$. There exists a prime integer $p$ such that $p$ does not divide $r$ and $s$. Now, $pM$ is a maximal submodule of $M$ such that $N + pM = L + pM = M$. Thus we have $N - pM - L$, a path from $N$ to $L$.

**Case 2:** Suppose that $N \cap \mathbb{Z} = \mathbb{Z}$ and $N \cap \mathbb{Z}_{p^\infty} \neq \mathbb{Z}_{p^\infty}$ and $L \cap \mathbb{Z} \neq \mathbb{Z}$ and $L \cap \mathbb{Z}_{p^\infty} \neq \mathbb{Z}_{p^\infty}$. There exists a nonzero integer $r$ such that $L \cap \mathbb{Z} = r\mathbb{Z}$. Suppose that $H := p\mathbb{Z} \oplus \mathbb{Z}_{p^\infty}$, where $p$ is a prime integer such that $p$ does not divide $r$. Then $H$ is a nonzero proper submodule of $M$ and $H \nsubseteq \mathrm{Rad}(M)$. Hence, $N + H = M$ and $L + H = M$. So, in this case we have $N - pM - L$.

**Case 3:** Suppose that $N \cap \mathbb{Z} = \mathbb{Z}$ and $N \cap \mathbb{Z}_{p^\infty} \neq \mathbb{Z}_{p^\infty}$ and $L \cap \mathbb{Z} \neq \mathbb{Z}$ and $L \cap \mathbb{Z}_{p^\infty} = \mathbb{Z}_{p^\infty}$. Note that $\mathbb{Z}_{p^\infty} \subseteq L$ and $\mathbb{Z} \subseteq N$. There exists a prime integer $p$ such that $N + pM = M$ and $L + pM = M$.

**Case 4:** Suppose that $N \cap \mathbb{Z} = \mathbb{Z}$ and $N \cap \mathbb{Z}_{p^\infty} \neq \mathbb{Z}_{p^\infty}$ and $L \cap \mathbb{Z} = \mathbb{Z}$ and $L \cap \mathbb{Z}_{p^\infty} \neq \mathbb{Z}_{p^\infty}$. However, $N$ and $L$ are not adjacent (since the set of all submodules of $\mathbb{Z}_{p^\infty}$ is totally ordered), for any arbitrary prime integer $p$, we have $N - pM - L$.

Consequently, $\operatorname{diam} G^*(M) = 2$.

**Theorem 2.25.** *Let $M$ be a coatomic $R$-module. Then the following are equivalent:*

(1) *$G^*(M)$ is a complete bipartite graph.*
(2) *The cardinal number of the set* $\mathrm{Max}(M)$ *is equal 2.*

*Proof.* (1) $\Rightarrow$ (2). Suppose that $G^*(M)$ is a complete bipartite graph with two parts $V_1$ and $V_2$. By assumption and by Remark 2.2, $|\mathrm{Max}(M)| \geq 2$. Suppose that $|\mathrm{Max}(M)| > 2$. Then by the Pigeon Hole Principal, two of the maximal submodules should belong to one of $V_i$'s, that is a contradiction.

(2) $\Rightarrow$ (1). Suppose that $\mathrm{Max}(M) = \{P_1, P_2\}$. Since $M$ is coatomic, every submodule of $M$ is contained in $P_1$ or $P_2$. Set $V_1 = \{N \in G^*(M) | N \subseteq P_1\}$ and $V_2 = \{N \in G^*(M) \,|\, N \subseteq P_2\}$. Clearly, $V_1 \cap V_2 = \emptyset$, $G^*(M) = V_1 \cup V_2$ and the elements of $V_i$ are not adjacent. Now suppose that $L \in V_1$ and $N \in V_2$. Since $N + L \nsubseteq P_1$ and $N + L \nsubseteq P_2$ and $M$ is coatomic, we must have $N + L = M$. This implies that $G^*(M)$ is a complete bipartite graph. □

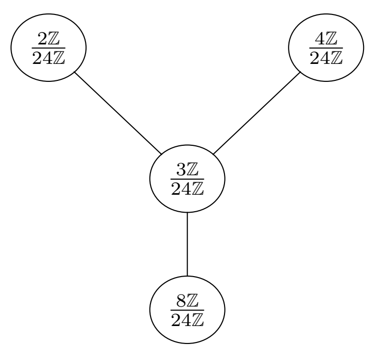
**Proposition 2.26.** *Let $M$ be a coatomic $R$-module and $n > 1$. If $|\mathrm{Max}(M)| = n < \infty$, then $G^*(M)$ is $n$-partite.*

*Proof.* Let $\mathrm{Max}(M) = \{P_1, \ldots, P_n\}$ and set $A_i = \{N \in G^*(M) \,|\, N \subseteq P_i\}$. Suppose that $V_1$ is the vertex set of subgraph generated by $A_1$ and $V_i$ is the vertex set of subgraph generated by $A_i \setminus \bigcup_{j=1}^{i-1} A_j$ for each $i \geq 2$. Clearly, for each $i$, $P_i \in V_i$ and so $V_i \neq \emptyset$. Also $G^*(M) = V_1 \cup \ldots \cup V_n$. Now, let $L, N \in V_i$ for some $i$. If $L$ and $N$ are adjacent, then $M = L + N \subseteq P_i$, that is a contradiction. □

**Theorem 2.27.** *Let $M$ be an $R$-module and $G^*(M)$ be a star graph. Then $|\mathrm{Max}(M)| = 2$ and $M$ is coatomic.*

*Proof.* Since $G^*(M)$ is a star graph, $|\mathrm{Max}(M)| < 3$ and there exists a vertex $P \in G^*(M)$ such that $P$ is adjacent to any other vertices. We claim that $P$ is a maximal submodule of $M$. Let $N$ be a proper submodule of $M$ such that $P \subseteq N$, then $P + N \neq M$. So, $P \subseteq N \subseteq \mathrm{Rad}(M)$, that is a contradiction. This implies that $P$ is a maximal submodule of $M$. It is trivial that $|\mathrm{Max}(M)| \neq 1$, otherwise $\mathrm{Rad}(M) = P$ and $P \notin G^*(M)$, that is a contradiction. Therefore $|\mathrm{Max}(M)| = 2$. Suppose that $Q \neq P$ is the second maximal submodule of $M$. Let $N$ be a proper submodule of $M$. If $N \subseteq \mathrm{Rad}(M)$, we are done. Otherwise, $N + P = M$. So, $N \nsubseteq P$. If $N \nsubseteq Q$, then $N + Q = M$. Hence $N - Q - P - N$ is a cycle in $G^*(M)$. That is a contradiction. Therefore $N \subseteq Q$ and $M$ is coatomic. □

**Example 2.28.** Let $R = \mathbb{Z}/48\mathbb{Z}$ and consider the $R$-module $M = \mathbb{Z}/24\mathbb{Z}$. Then, the graph $G^*(M)$ is a star graph. Therefore, $|\mathrm{Max}(M)| = 2$. Indeed, $\mathrm{Rad}(M) = 6\mathbb{Z}/24\mathbb{Z}$ and $G^*(M) = \{2\mathbb{Z}/24\mathbb{Z}, 3\mathbb{Z}/24\mathbb{Z}, 4\mathbb{Z}/24\mathbb{Z}, 8\mathbb{Z}/24\mathbb{Z}\}$.




The graph of $G^*(M)$.

**Proposition 2.29.** *Let $M$ be a non-quasi-local coatomic $R$-module. Then $G^*(M)$ is a star graph or $\mathrm{g}\,(G(M)) \leq 4$.*

*Proof.* Since $M$ is non-quasi-local coatomic, $|\mathrm{Max}(M)| > 1$. If $|\mathrm{Max}(M)| = 2$, $G^*(M)$ is a complete bipartite graph by Theorem 2.25. Thus $G^*(M)$ is a star graph or the girth of $G(M)$ is 4. If $|\mathrm{Max}(M)| \geq 3$, then the girth of $G(M)$ is 3, as desired. □

**Corollary 2.30.** *Let $M$ be a coatomic $R$-module. Then the following statements are equivalent:*

(1) *$G^*(M)$ is a tree graph.*
(2) *$G^*(M)$ is a star graph.*

The *eccentricity* of the vertex $x$ of a graph $\Gamma$ is the distance between $x$ and the vertex which is at the greatest distance from $x$, $e(x) = \max\{d(x, y) \mid y \in \Gamma\}$.

The *radius* of the graph $\Gamma$, $r(\Gamma)$, is defined by $r(\Gamma) = \min\{e(x) \mid x \in \Gamma\}$, and the *center* of the graph is the set of all of its vertices whose eccentricity is minimal.

Now let us to discuss the radius of the graph $G(M)$.

**Proposition 2.31.** *Let $M$ be a coatomic $R$-module. If an element $N \in G^*(M)$ belongs to the center of $G(M)$, then*

$$A_N := \{L \in G^*(M) \mid L \subseteq \bigcup_{P \in \mathcal{V}(N)} P \setminus \bigcup_{P' \in \mathrm{Max}(M) \setminus \mathcal{V}(N)} P'\}.$$

*is a subset of the center.*

*Proof.* Suppose that $N$ belongs to the center. It is easy to see that for any element $L \in A_N$, we have $\mathcal{V}(L) \subseteq \mathcal{V}(N)$. Suppose that $N$ is adjacent to an element $K$. Then $N + K = M$ and $\mathcal{V}(N) \cap \mathcal{V}(K) = \emptyset$. Since $M$ is coatomic and $\mathcal{V}(L) \cap \mathcal{V}(K) = \emptyset$, $K + L = M$. This implies that $L$ belongs to the center. □

**Theorem 2.32.** *Let $M$ be a multiplication $R$-module and let $S \subset \mathrm{Max}(M)$ be such that*

$$\bigcap_{P \in S} P \setminus \bigcup_{P' \in \mathrm{Max}(M) \setminus S} P' \neq \emptyset$$

*while, for every $T \subset S$,*

$$\bigcap_{P \in T} P \setminus \bigcup_{P' \in \mathrm{Max}(M) \setminus T} P' = \emptyset.$$

*Then every vertex $N$ of $G^*(M)$ where $\mathcal{V}(N) = S$ has eccentricity 2. Therefore, if such an $S$ exists then $r(G^*(M)) = 2$ and any element $N$ such that $\mathcal{V}(N) = S$ belongs to the center.*

*Proof.* Let $\mathcal{V}(N) = S$ and let $L$ be a vertex of $G^*(M)$ not adjacent to $N$. Since $M$ is multiplication, $\mathcal{V}(N) \cap \mathcal{V}(L) \neq \emptyset$. If $N \cap L \nsubseteq \mathrm{Rad}(M)$, then there exists a maximal submodule $P$ of $M$ such that $N \cap L \nsubseteq P$. Therefore $(N \cap L) + P = M$. Hence $P$ is adjacent to $N$ and $L$. This implies that $d(N, L) = 2$. If $N \cap L \subseteq \mathrm{Rad}(M)$, then $\mathrm{Max}(M) \setminus \mathcal{V}(L) \subset \mathcal{V}(N)$, since $M$ is multiplication. If the set $S$ is a singleton, this cannot happen and we are done. Otherwise, since $L \nsubseteq \mathrm{Rad}(M)$, there exists a maximal submodule $Q = \mathfrak{m}M$, where $\mathfrak{m} \in \mathrm{Max}(R)$, such that $L \nsubseteq Q$. Thus $L + Q = M$. There exists an element $y \in \mathfrak{m}$ such that $L + yM = M$. Hence $\mathcal{V}(yM) \cap \mathcal{V}(L) = \emptyset$. We conclude that $\mathcal{V}(yM) \subset \mathcal{V}(N) = S$ and

$$yM \subseteq \bigcap_{P \in \mathcal{V}(yM)} P \setminus \bigcup_{P' \in \mathrm{Max}(M) \setminus \mathcal{V}(yM)} P'.$$

This is a contradiction. □

## Acknowledgments

The first author was supported by grants of University of Guilan. The authors are indebted to an anonymous referee for his/her suggestions and helpful remarks.

## References

1. A. Abbasi, Sh. Habibi, Zero-divisor graph in commutative rings with decomposable zero ideal, *Far East J. Math. Sci. (FJMS)*, **59** (1), (2011), 65–72.
2. A. Abbasi, Sh. Habibi, The total graph of a commutative ring with respect to proper ideals, *J. Korean Math. Soc.*, **49** (1), (2012), 85–98.
3. A. Abbasi, D. Hassanzadeh-Lelekaami, Modules and spectral spaces, *Comm. Algebra*, to appear.
4. S. Akbari, H. R. Maimani, S. Yassemi, When a zero-divisor graph is planar or a complete $r$-partite graph, *J. Algebra*, **270**, (2003), 169–180.
5. S. Akbari, A. Mohammadian, Zero-divisor graphs of non-commutative rings, *J. Algebra*, **296**, (2006), 462–479.
6. D. D. Anderson, M. Naseer, Beck's coloring of a commutative ring, *J. Algebra*, **159**, (1993), 500–514.
7. D. F. Anderson, A. Badawi, On the zero-divisor graph of a ring, *Comm. Algebra*, **36**, (2008), 3073–3092.
8. D. F. Anderson, P. S. Livingston, The zero-divisor graph of a commutative ring, *J. Algebra*, **217**, (1999), 434–447.
9. D. F. Anderson, S. B. Mulay, On the diameter and girth of a zero-divisor graph, *J. Pure Appl. Algebra*, **210** (2), (2007), 543–550.
10. F. W. Anderson, K. R. Fuller, *Rings and categories of modules*, Springer-Verlag, New York, (13), Graduate Texts in Math., 1992.
11. M. Axtell, J. Coykendall, J. Stickles, Zero-divisor graphs of polynomial and power series over commutative rings, *Comm. Algebra*, **33**, (2005), 2043–2050.
12. F. Azarpanah, M. Motamedi, Zero-divisor graph of C(X), *Acta Math. Hungar*, **108**, (2005), 25–36.
13. A. Barnard, Multiplication modules, *J. Algebra*, **71** (1), (1981), 174–178.
14. I. Beck, Coloring of commutative rings, *J. Algebra*, **116** (1), (1988), 208–226.
15. M. P. Brodmann, R. Y. Sharp, *Local cohomology an algebraic introduction with geometric applications*, Cambridge Univercity Press, 1998.
16. Z. A. El-Bast, P. F. Smith, Multiplication modules, *Comm. Algebra*, **16** (4), (1988), 755–779.
17. J. Hausen, Supplemented modules over dedekind domains, *Pacific J. Math.*, **100** (2), (1982), 387–402.
18. J. D. LaGrange, Complemented zero-divisor graphs and boolean rings, *J. Algebra*, **315**, (2007), 600–611.
19. Chin-Pi Lu, Prime submodules of modules, *Comment. Math. Univ. St. Pauli*, **33** (1), (1984), 61–69.
20. Chin-Pi Lu, Spectra of modules, *Comm. Algebra*, **23** (10), (1995), 3741–3752.
21. Chin-Pi Lu, A module whose prime spectrum has the surjective natural map, *Houston J. Math.*, **33** (1), (2007), 125–143.
22. H. R. Maimani, Median and center of zero-divisor graph of commutative semogroups, *Iranian J. Math. Sci. and Inf.*, **3** (2), (2008), 69-76.

23. H. R. Maimani, M. Salimi, A. Sattari, S. Yassemi, Comaximal graph of commutative rings, *J. Algebra*, **319**, (2008), 1801–1808.
24. R. L. McCasland, M. E. Moore, P. F. Smith, On the spectrum of a module over a commutative ring, *Comm. Algebra*, **25** (1), (1997), 79–103.
25. P. K. Sharma, S. M. Bhatwadekar, A note on graphical representation of rings, *J. Algebra*, **176**, (1995), 124–127.
26. A. Tehranian, H. R. Maimani, A study of total graph, *Iranian J. Math. Sci. and Inf.*, **6** (2), (2011), 75-80.
27. H. J. Wang, Graphs associated to co-maximal ideals of commutative rings, *J. Algebra*, **320**, (2008), 2917–2933.
28. R. Wisbauer, *Foundations of module and ring theory*, Gordon and Breach, 1991.
29. H. Zöschinger, Koatomare moduln, *Math. Z.*, **170**, (1980), 221–232.