Abstract. In this paper we define the $p$-analog of the restricted representations and the $p$-analog of the Fourier–Stieltjes algebras on inverse semigroups. Also we improve some results about Herz algebras on Clifford semigroups and we give a necessary and sufficient condition for amenability of these algebras on Clifford semigroups.

Keywords: Restricted fourier–Stieltjes algebras, Restricted inverse semigroup, Restricted representations, $QSL_p$-spaces, $p$-Analog of the Fourier–Stieltjes algebras.


1. Introduction and Preliminaries

An inverse semigroup $S$ is a discrete semigroup such that for each $s \in S$ there exists a unique element $s^* \in S$ such that $ss^*s = s, s^*ss^* = s^*$. The set $E(S)$ of idempotents of $S$ consists of elements of the form $ss^*, s \in S$. Actually for each abstract inverse semigroup $S$ there is a $*$-semigroup homomorphism from $S$ into the inverse semigroup of partial isometries on some Hilbert space[18].

Dunkl and Ramirez in [8] and T. M. Lau in [15] attempted to define a suitable substitution for Fourier and Fourier–Stieltjes algebras on semigroups. Each definition has its own difficulties. Amini and Medghalchi introduced and

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extensively studied the theory of restricted semigroups and restricted representations and restricted Fourier and Fourier–Stieltjes algebras, $A_{r,e}(S), B_{r,e}(S)$ in [2] and [3]. Also they studied the spectrum of the Fourier Stieltjes algebra for a unital foundation topological $*$-semigroup in [4]. In this section we mention some of their results.

Throughout this paper $S$ is an inverse semigroup. Given $x, y \in S$, the restricted product of $x, y$ is $xy$ if $x^*x = yy^*$, and undefined, otherwise. The set $S$ with its restricted product forms a groupoid [16, 3.1.4] which is called the associated groupoid of $S$. If we adjoin a zero element $0$ to this groupoid, and put $0^* = 0$, we will have an inverse semigroup $S_r$ with the multiplication rule

$$x \cdot y = \begin{cases} xy & \text{if } x^*x = yy^*, \\ 0 & \text{otherwise} \end{cases}$$

for $x, y \in S \cup \{0\}$, which is called the restricted semigroup of $S$. A restricted representation $\{\pi, H_\pi\}$ of $S$ is a map $\pi : S \to B(H_\pi)$ such that $\pi(x^*) = \pi(x)^*$ ($x \in S$) and

$$\pi(x)\pi(y) = \begin{cases} \pi(xy) & \text{if } x^*x = yy^*, \\ 0 & \text{otherwise} \end{cases}$$

for $x, y \in S$. Let $\Sigma_r = \Sigma_r(S)$ be the family of all restricted representations $\pi$ of $S$ with $\|\pi\| \leq 1$. Now it is clear that, via a canonical identification, $\Sigma_r(S) = \Sigma_0(S_r)$, consist of all $\pi \in \Sigma(S_r)$ with $\pi(0) = 0$, where the notation $\Sigma$ has been used for all $*$-homomorphism from $S$ into $B(H)$ [2]. One of the central concepts in the analytic theory of inverse semigroups is the left regular representation $\lambda : S \to B(\ell^2(S))$ defined by

$$\lambda(x)\xi(y) = \begin{cases} \xi(x^*y) & \text{if } xx^* \geq yy^*, \\ 0 & \text{otherwise} \end{cases}$$

for $\xi \in \ell^2(S), x, y \in S$. The restricted left regular representation $\lambda_r : S \to B(\ell^2(S))$ is defined in [2] by

$$\lambda_r(x)\xi(y) = \begin{cases} \xi(x^*y) & \text{if } xx^* = yy^*, \\ 0 & \text{otherwise} \end{cases}$$

for $\xi \in \ell^2(S), x, y \in S$. The main objective of [2] is to change the convolution product on the semigroup algebra to restore the relation with the left regular representation.

For each $f, g \in \ell^1(S)$, define

$$(f \cdot g)(x) = \sum_{x^*x = yy^*} f(xy)g(y^*) \quad (x \in S),$$

and for all $x \in S$, $\hat{f}(x) = \overline{f(x^*)}$, $\ell^1_r(S) := (\ell^1(S), \cdot^*, \cdot)$ is a Banach $*$-algebra with an approximate identity. The left regular representation $\lambda_r$ lifts to a faithful representation $\hat{\lambda}$ of $\ell^1_r(S)$. We call the completion $C^*_{\hat{\lambda}}(S)$ of $\ell^1_r(S)$ with the
norm \| \cdot \|_{\lambda_r} := \| \tilde{\lambda}_r (\cdot) \| which is a C\textsuperscript{*}-norm on \ell_1^r (S), the restricted reduced C\textsuperscript{*}-algebra and its completion with the norm \| \cdot \|_{\Sigma_r} := \sup \{ \| \tilde{\pi}(\cdot) \|, \pi \in \Sigma(S_r) \} the restricted full C\textsuperscript{*}-algebra and show it by \( C^*_r(S) \). The dual space of \( C^*_r(S) \) is a unital Banach algebra which is called the restricted Fourier-Stieltjes algebra and is denoted by \( B_{r,e}(S) \). The closure of the set of finitely support functions in \( B_{r,e}(S) \) is called the restricted Fourier algebra and is denoted by \( A_{r,e}(S) \)\cite{2}.

In \cite{10}, Figà-Talamanca introduced a natural generalization of the Fourier algebra, for a compact abelian group \( G \), by replacing \( L_2(G) \) by \( L_p(G) \). In \cite{11}, Herz extended the notion to an arbitrary group, to get the commutative Banach algebra \( A_p(G) \), called the Figà–Talamanca–Herz algebra. Figà–Talamanca–Herz algebra and Eymard’s Fourier algebra have very similar behavior. For example, Leptin’s theorem is valid: \( G \) is amenable if and only if \( A_p(G) \) has a bounded approximate identity \cite{12}. The \( p \)-analogue, \( B_p(G) \) of the Fourier–Stieltjes algebra is defined as the multiplier algebra of \( A_p(G) \), by some authors, as mentioned in \cite{5} and \cite{19}. Runde in \cite{20} defined and studied \( B_p(G) \), the \( p \)-analogue of the Fourier–Stieltjes algebra on the locally compact group \( G \). He developed the theory of representations and defined the substitute coefficient functions on them.

For \( p \in (1, \infty) \), Medghalchi and Pourmahmood Aghababa developed the theory of restricted representations on \( \ell_p(S) \) and defined the Banach algebra of \( p \)-pseudomeasures \( PM_p(S) \) and the Figà–Talamanca–Herz algebras \( A_p(S) \). They showed that \( A_p(S)^* = PM_p(S) \) for dual pairs \( p, q \). They characterized \( PM_p(S) \) and \( A_p(S) \) for Clifford semigroups, in the sense of \( p \)-pseudomeasures and Figà–Talamanca–Herz algebras of maximal semigroups of \( S \), respectively\cite{17}.

Amini also worked on quantum version of Fourier transforms in \cite{1}.

In this paper we will combine what Medghalchi–Pourmahmood Aghababa and Runde have done. We will define the restricted representations on \( QSL_p^r \)-spaces and the \( p \)-analogue of the Fourier-Stieltjes algebra on the restricted inverse semigroup.

Section 2 is a review of the theory of \( QSL_p^r \)-spaces. In Section 3 we define the restricted representations on \( QSL_p^r \)-spaces and study their tensor product. In Sections 4 and 5 we construct the \( p \)-analogue of the restricted Fourier–Stieltjes algebra and study its order structure. The last section will be about Clifford semigroups and the \( p \)-analogue of their restricted Fourier–Stieltjes algebra. Some new results which improves the results of \cite{17} and \cite{22} will be given in Section 6.

2. REVIEW OF THE THEORY OF \( QSL_p^r \)-SPACES

This section is a review of the paper of Runde \cite{20}.

**Definition 2.1.** A Banach space \( E \) is called...
(i) an \( L_p \)-space if it is of the form \( L_p(X) \), for some measure space \( X \).

(ii) a \( QSL_p \)-space if it is isometrically isomorphic to a quotient of a subspace of an \( L_p \)-space (or equivalently, a subspace of a quotient of an \( L_p \)-space [20, Section 1, Remark 1]).

If \( E \) is a \( QSL_p \)-space and if \( p' \in (1, \infty) \) is such that \( \frac{1}{p} + \frac{1}{p'} = 1 \), the dual space \( E^* \) is an \( QSL_{p'} \)-spaces. In particular, every \( QSL_p \)-space is reflexive.

By [14, Theorem 2], the \( QSL_p \)-spaces are precisely the \( p \)-spaces in the sense of [11], i.e. those Banach spaces \( E \) such that for any two measure spaces \( X \) and \( Y \) the amplification map

\[
B(L_p(X), L_p(Y)) \to B(L_p(X, E), L_p(Y, E)), T \to T \otimes id_E
\]

is an isometry. In particular, an \( L_q \)-space is a \( QSL_p \)-space if and only if \( 2 \leq q \leq p \) or \( p \leq q \leq 2 \). Consequently, if \( 2 \leq q \leq p \) or \( p \leq q \leq 2 \), then every \( QSL_q \)-space is a \( QSL_p \)-space.

Runde equipped the algebraic tensor product of two \( QSL_p \)-spaces with a suitable norm, which comes in the following.

**Theorem 2.2.** [20, Theorem 3.1] Let \( E \) and \( F \) be \( QSL_p \)-spaces. Then there exists a norm \( \| \cdot \|_p \) on the algebraic tensor product \( E \otimes F \) such that:

(i) \( \| \cdot \|_p \) dominates the injective norm;

(ii) \( \| \cdot \|_p \) is a cross norm;

(iii) the completion \( E \hat{\otimes}_p F \) of \( E \otimes F \) with respect to \( \| \cdot \|_p \) is a \( QSL_p \)-space.

The Banach space \( E \hat{\otimes}_p F \) will be called \( p \)-projective tensor product of \( E \) and \( F \).

### 3. Restricted Representation on a Banach space

In this section we give an analog of the theory of group representations on a Hilbert space for the restricted representations for an inverse semigroup on a \( QSL_p \)-space.

**Definition 3.1.** A representation of a discrete inverse semigroup \( S \) on a Banach space \( E \) is a pair \( (\pi, E) \) consisting of a map \( \pi : S \to B(E) \) satisfying \( \pi(x)\pi(y) = \pi(xy) \), for \( x, y \in S \) and \( \|\pi(x)\| \leq 1 \), for all \( x \in S \).

**Definition 3.2.** A restricted representation of a discrete inverse semigroup \( S \) on a Banach space \( E \) is a pair \( (\pi, E) \) consisting of a map \( \pi : S \to B(E) \) satisfying

\[
\pi(x)\pi(y) = \begin{cases} 
\pi(xy) & \text{if } x^*x = yy^*, \\
0 & \text{otherwise}
\end{cases}
\]

for \( x, y \in S \), and \( \|\pi(x)\| \leq 1 \), for all \( x \in S \).

**Definition 3.3.** Let \( S \) be an inverse semigroup, and let \( (\pi, E) \) and \( (\rho, F) \) be restricted representations of \( S \), then these restricted representations are said to be equivalent if there exists a surjective isometry \( T : E \to F \) such that

\[
T\pi(x)T^{-1} = \rho(x), \quad (x \in S).
\]
For any inverse semigroup $S$ and $p \in (1, \infty)$, we denote by $\Sigma_{p,r}(S)$ the collection of all (equivalence classes) of restricted representations of $S$ on a $QSL_p$-space.

**Remark 3.4.** By [17] for $p \in (1, \infty)$ the restricted left regular representation

$$\lambda_p : S \rightarrow B(\ell^p(S))$$

$$\lambda_p(s)(\delta_t) = \begin{cases} 
\delta_{st} & \text{if } s^*s = tt^*, \\
0 & \text{otherwise}
\end{cases}$$

for $s, t \in S$ is a restricted representation so it belongs to $\Sigma_{p,r}(S)$.

The following propositions are easy to check, similar to [2].

**Proposition 3.5.** For an inverse semigroup $S$ and its related restricted semigroup $S_r$, each restricted representation of $S$ on a Banach space is a representation on $S_r$ which is zero on $0 \in S_r$, i.e. it is multiplicative with respect to the restricted multiplication.

**Proposition 3.6.** For an inverse semigroup $S$, each restricted representation $\pi$ of $S$ on a Banach space lifts to a representation of $\ell^1_r(S)$, via

$$\tilde{\pi}(f) = \sum_{x \in S} f(x)\pi(x).$$

4. **Banach Algebra $B_{p,r}(S)$**

In this section we define the $p$-analog of the Fourier–Stieltjes algebra on a

inverse semigroup. We show that for $p = 2$ we get the known algebra $B_{r,e}(S)$, defined in [2].

**Theorem 4.1.** Let $(\pi, E), (\rho, F) \in \Sigma_{p,r}(S)$ then $(\pi \otimes \rho, E \overset{\circ}{\otimes}_p F) \in \Sigma_{p,r}(S)$.

**Proof.** By the definition of $\pi \otimes \rho$ we have $\pi \otimes \rho(x)(\xi \otimes \eta) = \pi(x)\xi \otimes \rho(x)\eta$. For $x, y \in S, x^*x = yy^*$,

$$\pi \otimes \rho(xy)(\xi \otimes \eta) = \pi(xy)\xi \otimes \rho(xy)\eta$$

$$= \pi(x)\pi(y)\xi \otimes \rho(x)\rho(y)\eta$$

$$= \pi(x)(\pi(y)\xi) \otimes \rho(x)(\rho(y)\eta)$$

$$= \pi \otimes \rho(x)(\pi(y)\xi \otimes \rho(y)\eta)$$

$$= \pi \otimes \rho(x)\pi \otimes \rho(y)(\xi \otimes \eta)$$

when $x^*x \neq yy^*$

$$\pi \otimes \rho(x)\pi \otimes \rho(y)(\xi \otimes \eta) = \pi \otimes \rho(x)(\pi(y)\xi \otimes \rho(y)\eta)$$

$$= \pi(x)(\pi(y)\xi) \otimes \rho(x)(\rho(y)\eta)$$

which is equal to zero. Now it is enough to show that $\pi(x) \in B(E)$ and $\rho(y) \in B(F)$, $\pi(x) \otimes \rho(y)$ could be extend to $E \overset{\circ}{\otimes}_p F$. This is shown as in the group case [20, Therem 3.1]. \qed
Definition 4.2. Let $S$ be an inverse semigroup, and let $(\pi, E) \in \Sigma_{p,r}(S)$. A coefficient function of $(\pi, E)$ is a function $f : S \to \mathbb{C}$ of the form

$$f(x) = <\pi(x)\xi, \phi> \quad (x \in S),$$

where $\xi \in E$ and $\phi \in E^*$.

Definition 4.3. Let $S$ be an inverse semigroup, let $p \in (1, \infty)$, and let $q \in (1, \infty)$ be the dual scalar to $p$, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. We define

$$B_{p,r}(S) := \{f : S \to \mathbb{C} : f \text{ is a coefficient function of some } (\pi, E) \in \Sigma_{q,r}(S)\}.$$

Proposition 4.4. Let $S$ be an inverse semigroup, let $p \in (1, \infty)$, and let $q \in (1, \infty)$ be the dual scalar to $p$, i.e. $\frac{1}{p} + \frac{1}{q} = 1$, and let $f : S \to \mathbb{C}$ be defined by

$$f(x) = \sum_{n=1}^{\infty} <\pi_n(x)\xi_n, \phi_n>, \quad (x \in S),$$

where $((\pi_n, E_n))_{n=1}^{\infty}$, $(\xi_n)_{n=1}^{\infty}$, and $(\phi_n)_{n=1}^{\infty}$ are sequences with $(\pi_n, E_n) \in \Sigma_{q,r}(S)$, $\xi_n \in E_n$, and $\phi_n \in E_n^*$, for $n \in \mathbb{N}$ such that

$$\sum_{n=1}^{\infty} \|\xi_n\|\|\phi_n\| < \infty.$$

Then $f$ lies in $B_{p,r}(S)$.

Proof. The proof is similar to [20]. Without loss of generality, we may suppose that

$$\sum_{i=1}^{\infty} \|\xi_i\|_q < \infty, \quad \text{and} \quad \sum_{i=1}^{\infty} \|\phi_i\|_p < \infty.$$

Then $E := \ell_q - \oplus_{n=1}^{\infty} E_n$ is a QSL$_q$--space and for $\xi := (\xi_1, \xi_2, \ldots)$ and $\phi := (\phi_1, \phi_2, \ldots)$, we have $\xi \in E$ and $\phi \in E^*$. Now the map $\pi : S \to B(E)$ with $\pi(x)\eta = (\pi_1(x)\eta_1, \pi_2(x)\eta_2, \ldots)$ is a restricted representation of $S$ on $E$, and $f$ is the coefficient function of $\pi$. \qed

Definition 4.5. [17, Definition 3.1]. Let $S$ be an inverse semigroup and let $p, q \in (1, \infty)$ be dual pairs. The space $A_q(S)$ consists of those $u \in c_0(S)$ such that there exist sequences $(f_n)_{n=1}^{\infty} \subseteq \ell_q(S)$ and $(g_n)_{n=1}^{\infty} \subseteq \ell_p(S)$ with $\sum_{n=1}^{\infty} \|f_n\|_q \|g_n\|_q \leq \infty$ and $u = \sum_{n=1}^{\infty} f_n \bullet g_n$. For $u \in A_q(S)$, let

$$\|u\| = \inf \left\{ \sum_{n=1}^{\infty} \|f_n\|_q \|g_n\|_p : \text{\emph{\textit{u}} = \sum_{n=1}^{\infty} f_n \bullet g_n} \right\}$$

Proposition 4.6. [17, Proposition 3.2]. Let $S$ be an inverse semigroup and let $p \in (1, \infty)$, then $A_p(S)$ is a Banach space and is the closure of finite support functions on $S$. 

Proposition 4.7. Let $S$ be an inverse semigroup, let $p \in (1, \infty)$. Then $B_{p,r}(S)$ is a linear subspace of $c_0(S)$ containing $A_p(S)$. Moreover, if $2 \leq q \leq p$, then $B_{q,r}(S) \subseteq B_{p,r}(S)$.

Proof. Every thing is easy to check, and is similar to [20].

Definition 4.8. Let $S$ be an inverse semigroup, and let $(\pi, E)$ be a restricted representation of $S$ on the Banach space $E$. Then $(\pi, E)$ is called cyclic if there exists $x \in E$ such that $\pi(\ell_1^1(S))x$ is dense in $E$. For $p \in (1, \infty)$, we set $Cyc_{p,r}(S) := \{ (\pi, E) : (\pi, E)$ is a cyclic restricted representation on a $QSL_p$-space $E \}$.

Definition 4.9. Let $S$ be an inverse semigroup, let $p,q \in (1, \infty)$ be the dual scalars, and let $f \in B_{p,r}(S)$. We define $\|f\|_{B_{p,r}(S)}$ as the infimum over all expressions $\sum_{n=1}^{\infty} \|\xi_n\| \|\phi_n\|$, where, for each $n \in \mathbb{N}$, there is $(\pi_n, E_n) \in Cyc_{q,r}(S)$ with $\xi_n \in E_n$ and $\phi_n \in E_n^*$ such that $\sum_{n=1}^{\infty} \|\xi_n\| \|\phi_n\| < \infty$ and

$$f(x) = \sum_{n=1}^{\infty} <\pi_n(x)\xi_n, \phi_n>, \quad (x \in S).$$

The proof of the following theorem is similar to the group case.

Theorem 4.10. Let $S$ be an inverse semigroup, let $p \in (1, \infty)$, and let $f, g : S \to \mathbb{C}$ be coefficient functions of $(\pi, E)$ and $(\rho, F)$ in $\Sigma_{p,r}(S)$, respectively. Then the pointwise product of $f$ and $g$ is a coefficient function of $(\pi \otimes \rho, E \otimes_{\rho, F})$.

In the next theorem we give some result about our new constructed space and also the relation between semigroup restricted Herz algebra and our new space.

Theorem 4.11. Let $S$ be an inverse semigroup, let $p \in (1, \infty)$. Then:

(i) $B_{p,r}(S)$ is a commutative Banach algebra.

(ii) the inclusion $A_p(S) \subseteq B_{p,r}(S)$ is a contraction.

(iii) for $2 \leq p' \leq p$ or $p \leq p' \leq 2$, the inclusion $B_{p',r}(S) \subseteq B_{p,r}(S)$ is a contraction.

(iv) for $p = 2$, $B_{r,c}(S)$ is isometrically isomorphic to $B_{p,r}(S)$ as Banach algebras.

Proof. (i) Let $\frac{1}{p} + \frac{1}{q} = 1$. The space $B_{p,r}(S)$ is the quotient space of complete $q$-projective tensor product of $E \hat{\otimes}_q E^*$, for the universal restricted representation $(\pi, E)$, on $QSL_q$-space $E$. Also Theorem 4.10 shows it is an algebra. The submultiplicitive property for norm of $B_{p,r}(S)$ is similar to the group case in [20] and it is only based on characteristic property of infimum.

(ii) By the definition of semigroup Herz algebra in [17] for conjugate numbers $p,q$, each $f \in A_p(S)$ is a coefficient function of the restricted left regular representation on the $\ell_q$-space, $\ell_q(S)$. So $A_p(S) \subseteq B_{p,r}(S)$. By the definition of the norm of $f \in B_{p,r}(S)$, the infimum is taken on all expressions of $f$ as the coefficient function of some restricted representation on a $QSL_q$-space, and
the norm on the \(A_p(S)\) is the infimum only on expressions of \(f\) as the coefficient function of restricted left regular representation, so the inclusion map is a contraction.

(iii) For \(2 \leq p' \leq p\) or \(p \leq p' \leq 2\) and \(q, q'\) conjugate scalars to \(p\) and \(p'\) respectively. Then each restricted representation on a \(QSL_q\)-space is a restricted representation on a \(QSL_{q'}\)-space.

(iv) By the definition, each element of \(B_{r,e}(S)\) is a coefficient function of a 2-restricted representation [3].

\[\square\]

**Remark 4.12.** A very natural question is that when \(A_p(S)\) is an ideal in \(B_{p,r}(S)\). Even in \(p = 2\) this question is not studied. If we want to go along the proof of the group case, a difficulty to prove this is that in general for \(p \in (1, \infty)\), and \((\pi, E) \in \Sigma_{p,r}(S)\), the representations \((\lambda_p \otimes \pi, \ell_p(S, E))\) and \((\lambda_p \otimes \text{id}_E, \ell_p(S, E))\) are not equivalent. In fact we can not find a suitable substitution for representation \(\text{id} : S \to B(E)\), \(\text{id}(s) = \text{id}_E\) in the class of restricted representations. But in a special case, such as Clifford semigroups, we can give a better result.

5. **Order Structure of the \(p\)-Analog of the Semigroup Fourier–Steiltjes Algebras \(B_{p,r}(S)\)**

Studying the ordered spaces and order structures has a long history. The natural order structure of the Fourier-Stieltjes algebras was favorite in 80s. In [21] the authors studied the order structure of Figà-Talamanca–Herz algebra and generalized results on Fourier algebras. In this section, we consider the \(p\)-analog of the restricted Fourier–Stieltjes algebra, \(B_{p,r}(S)\), introduced in Section 4, and study its order structure given by the \(p\)-analog of positive definite continuous functions.

A compatible couple of Banach spaces in the sense of interpolation theory (see [3]) is a pair \((\mathcal{E}_0, \mathcal{E}_1)\) of Banach spaces such that both \(\mathcal{E}_0\) and \(\mathcal{E}_1\) are embedded continuously in some (Hausdorff) topological vector space. In this case, the intersection \(\mathcal{E}_0 \cap \mathcal{E}_1\) is again a Banach space under the norm \(\|\cdot\|_{(\mathcal{E}_0, \mathcal{E}_1)} = \max\{|\cdot|_{\mathcal{E}_0}, |\cdot|_{\mathcal{E}_1}\}\). For example, for a locally compact group \(G\), the pairs \((A_p(G), A_q(G))\) and \((L_p(G), L_q(G))\) are compatible couples.

**Definition 5.1.** Let \((\pi, E)\) be a restricted representation of \(S\) on a Banach space \(E\), such that \((\mathcal{E}, \mathcal{E}^*)\) is a compatible couple. We mean by a \(\pi_r\)-positive definite function on \(S\), a function which has a representation as \(f(x) = \langle \pi(x) \xi, \xi \rangle\), \((x \in S)\), where \(\xi \in \mathcal{E} \cap \mathcal{E}^*\). For dual scalars \(p, q \in (1, \infty)\), we call each element in the closure of the set of all \(\pi_r\)-positive definite functions on \(S\) in \(B_{p,r}(S)\), where \(\pi\) is a restricted representation of \(S\) on an \(L_q\)-space, a restricted \(p\)-positive definite function on \(S\) and the set of all restricted \(p\)-positive definite functions on \(S\), will be denoted by \(F_{p,r}(S)\).
It follows from [21] and the definition of \( P_{p,r}(S) \), that for each \( f \in P_{p,r}(S) \), associated to a representation \((\pi,E)\), for a \( QSL_q \)-space \( E \), there exists a sequence \((\pi_n,\mathcal{E}_n)_{n=1}^{\infty}\) of cyclic restricted representations of \( S \) on closed subspaces \( E_n \) of \( E \cap E^* \), and \( \{\xi_n\} \) in \( \mathcal{E}_n \), such that
\[
f(x) = \sum_{n=1}^{\infty} (\pi_n(x)\xi_n,\xi_n) \quad (x \in S).
\]

**Proposition 5.2.** The linear span of all finite support elements in \( P_{p,r}(S) \) is dense in \( A_p(S) \), and \( A_p(S) \) is an ordered space.

**Proof.** From [17, Proposition 3.2] \( A_p(S) \) is a norm closure of the set of elements of the form \( \sum_{i=1}^{n} f_i \cdot \hat{g}_i \) where \( f_i, g_i \) are finite support functions on \( S \), \( i = 1, \ldots \, n \). Also \( f_i \cdot \hat{g}_i(x) = (\lambda_r(x^*)f_i,g_i) \). Now by Polarization identity, we have the statement. \( \square \)

Since \( A_p(S) \) is the set of coefficient functions of the restricted left regular representation of \( S \) on \( \ell_p(S) \), we define the positive cone of \( A_p(S) \) as the closure in \( \ell_p(S) \), of the set of all function of the form \( f = \sum_{i=1}^{n} \xi_i \cdot \hat{\xi}_i \), for a sequence \( (\xi_i) \) in \( \ell_p(S) \cap \ell_q(S) \), and denote it by \( A_p(S)_+ \).

This order structure, in the case where \( p = 2 \), is the same as the order structure of \( A_{r,e}(S) \), induced by the set \( P_{r,e}(S) \cap A_{r,e}(S) \), as a positive cone. Because in the case \( p = 2 \), the extensible restricted positive definitive functions are exactly the closed linear span of \( h \cdot \hat{h} \), for \( h \in \ell^2(S) \).

6. \( p \)-ANALOG OF THE FOURIER–STILETJES ALGEBRAS ON CLIFFORD SEMIGROUPS

Let \( S \) be a semigroup. Then, by [13, Chapter 2], there is an equivalence relation \( D \) on \( S \) by \( sDt \) if and only if there exists \( x \in S \) such that
\[
Ss \cup \{s\} = Sx \cup \{x\} \quad \text{and} \quad tS \cup \{t\} = xS \cup \{x\}.
\]
If \( S \) is an inverse semigroup, then by [13, Proposition 5.1.2(4)], \( sDt \) if and only if there exists \( x \in S \) such that \( s^*s = xx^* \) and \( tt^* = x^*x \).

**Proposition 6.1.** [17, Proposition 4.1]. Let \( S \) be an inverse semigroup,
(i) and let \( D \) be a \( D \)-class of \( S \). Then \( \ell_p(D) \) is a closed \( \ell_p^1 \)-submodule of \( \ell_p(S) \).

(ii) and let \( \{D_\lambda; \lambda \in \Lambda\} \) be the family of \( D \)-classes of \( S \) indexed by some set \( \Lambda \). Then there is an isometric isomorphism of Banach \( \ell_p^1 \)-bimodules
\[
\ell^p(S) \cong \ell^p - \mathop{\bigoplus}_{\lambda \in \Lambda} \ell_p(D_\lambda). \tag{6.1}
\]

**Corollary 6.2.** Let \( S \) be an inverse semigroup, and let \( \{D_\lambda; \lambda \in \Lambda\} \) be the family of \( D \)-classes of \( S \) indexed by some set \( \Lambda \). Then for a \( QSL_p \)-space \( E \) of functions on \( S \), there is a family of \( QSL_p \)-spaces \( \{E_\lambda\}_{\lambda \in \Lambda} \), where for each \( \lambda \in \Lambda \), \( E_\lambda \) consists of functions on \( D_\lambda \), and \( E \cong \ell^p - \mathop{\bigoplus}_{\lambda \in \Lambda} E_\lambda \).
Proof. This is clear by the definition of a $QSL_{p}$-space, and the fact that the isomorphism 6.1 is compatible with taking quotients and subspaces of $\ell_p(D_\lambda)s$.

An inverse semigroup $S$ is called a Clifford semigroup if $s^*s = ss^*$ for all $s \in S$. For $e \in E(S)$ define $G_e := \{ s \in S | s^*s = ss^* = e \}$. Then $G_e$ is a group with identity $e$. Here each $D$-class $D$ contains a single idempotent (say $e$) and we have $D = G_e$.

We modified the isometrical isomorphism derived in [17, Section 5.3] in the following theorem.

**Theorem 6.3.** Let $S$ be a Clifford semigroup with the family of $D$-classes $\{G_e\}_{e \in E(S)}$, and let $p \in (1, \infty)$. Then

$$B_{p,r}(S) \cong \ell^1 - \bigoplus_{e \in E(S)} B_p(G_e)$$

**Proof.** Let $p, q$ are conjugate scalars. Fix $e \in E(S)$, assume that $G_e = \{ x \in S; x^*x = e \}$. Define $\pi : S \to B(\ell_q(G_e))$.

$$\pi(s)(\delta_t) = \begin{cases} \delta_t & \text{if } s^*s = e, \\ 0 & \text{otherwise} \end{cases}$$

for $s \in S$. Then $\pi$ is a restricted representation and $\chi_{G_e}(s) = \langle \pi(s)\delta_t, \delta_t \rangle$. Hence $\chi_{G_e}$ is in $B_{p,r}(S)$, and indeed $\chi_{G_e}$ is a restricted positive definite function. Now for each $u \in B_{p,r}(S)$, $u \cdot \chi_{G_e}$ is in $B_{p,r}(S)$. In fact the set $\{ u \in B_{p,r}(S); u(s) = 0 \text{ for all } s \in S \setminus G_e \}$ is a closed subspace of $B_{p,r}(S)$ and it is isometrically isomorphic to $B_p(G_e)$. This follows from the fact that, each coefficient function of a restricted representation of $S$ on a $QSL_q$-space that is zero on $G_e^c$, is a coefficient function of a representation on a $QSL_q$-space of $G_e$, using Corollary 6.2.

Let $u \in B_{p,r}(S)$, then we could decompose $u$ to $(ue)_{e \in E(S)}$, for some $ue \in B_p(G_e)$, by the above paragraph. Now for all $e \in E(S)$ and all explanations of $ue$ as $ue = \langle \pi_e(\cdot)\xi_e, \eta_e \rangle$, where $\pi_e \in \Sigma_q(G_e)$, $\xi_e$ in some $QSL_q$ and $\eta_e$ in some $QSL_p$-space for dual scalars $p, q$ we have $\|ue\| \leq \|\xi_e\|\|\eta_e\|$ and also $u = (ue)_{e \in E(S)} = \langle \oplus \pi_e(\cdot) \oplus \xi_e, \oplus \eta_e \rangle$ and then $\oplus \pi_e$ is a restricted representation of $S$ on a $QSL_q$-space and $\sum_{e \in E(S)} \|ue\| \leq \sum_{e \in E(S)} \|\xi_e\|\|\eta_e\| \leq (\sum_{e \in E(S)} \|\xi_e\|)^{\frac{1}{q}}(\sum_{e \in E(S)} \|\eta_e\|)^{\frac{1}{p}} = \|\langle \xi_e \rangle\|\|\langle \eta_e \rangle\|$, the last equality comes from Proposition 6.1. Now we have:

$$\sum_{e \in E(S)} \|ue\| \leq \|(ue)_{e \in E(S)}\| = \sum_{e \in E(S)} \|ue\|$$

by the definition of the norm in the Fourier–Stieltjes algebras. So we have an isometric isomorphism of Banach algebras.

The following corollary improves [22, Proposition 2.6].
Corollary 6.4. Let $S$ be a Clifford semigroup with the family of $D$-classes \( \{ G_e \}_{e \in E(S)} \), and let $p \in (1, \infty)$. Then \( A_p(S) \) is an ideal of \( B_{p,r}(S) \), and \( B_{p,r}(S) \) is amenable if and only if \( B_p(G_e) \) is amenable for all $e \in E(S)$.

Theorem 6.5. Let $S$ be a Clifford semigroup with the family of amenable $D$-classes \( \{ G_e \}_{e \in E(S)} \) and let $p \in (1, \infty)$. Then \( A_p(S) \) is equal to the closure of \( B_{p,r}(S) \cap F(S) \) in the norm of \( A_p(S) \).

Proof. Since for each $e \in E(S)$ the group $G_e$ is amenable, the natural embedding \( i_e : A_p(G_e) \to B_p(G_e) \) is an isometry by [20, Corollary 5.3]. Now let $f \in B_{p,r}(S) \cap F(S)$. Then by Theorem 6.3, \( f = \sum_{e \in E(S)} f_e \), where \( f_e \in B_p(G_e) \cap F(G_e) \), so \( f_e \) belongs to \( A_p(G_e) \) by [20]. Now since

\[
A_p(S) \cong \ell_1 - \bigoplus_{e \in E(S)} A_p(G_e),
\]

[17, Equation 5.1], we conclude that \( f \in A_p(S) \). On the other hand let \( f \in A_p(S) \cap F(S) \). Then for each $e \in E(S)$ the function \( f_e \), which has been defined in Theorem 6.3, belongs to \( A_p(G_e) \cap F(G_e) \) and so to \( B_p(G_e) \cap F(G_e) \) with the same norm because of amenability of \( G_e \). Now, by Theorem 6.3 we have \( f \in B_{p,r}(S) \cap F(S) \). Since \( F(S) \) is dense in \( A_p(S) \) with norm of \( A_p(S) \), [17, Proposition 3.2 vi], result follows.

\[ \square \]

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References