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m-Ary Hypervector Space: Convergent Sequence and Bundle Subsets

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Abstract. In this paper, we have generalized the definition of the vector space by considering the group as a canonical m-ary hypergroup, the field as a krasner (m, n)-hyperfield and considering the multiplication structure of a vector by a scalar as hyperstructure. Also we will be consider a normed m-ary hypervector space and introduce the concept of convergent of sequence on m-ary hypernormed spaces and bundle subset.

Keywords: m-Ary hypervector space, Krasner (m, n)-hyperfield, Bundle subset, Hypernorm.

2000 Mathematics subject classification: 20N15, 46C50.

1. Introduction

Hypergroups were introduced in 1934 by a French mathematician Marty [19] Marty [19]at the 8^{th} Congress of Scandinavian Mathematicians. Since then, hundreds of papers and several books have been written on this topic. Nowadays, hyperstructures have a lot of applications to several domains of mathematics and computer science [1, 2, 3]. Algebraic hyperstructures are a suitable generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. More exactly, if V is a non-empty set and $\mathcal{P}^*(V)$ is the set of all non-empty subsets of V, then we

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consider maps $*: V \times V \longrightarrow \mathcal{P}^*(V)$. This maps are called *(binary) hyperoperations*. Sometimes, external hyperoperations are considered, which are maps $*: R \times V \longrightarrow \mathcal{P}^*(V)$, where $R \neq V$. An example of a hyperstructure, endowed both with an internal hyperoperation and an external hyperoperation is the so-called *hypermodule*.

n-Ary generalizations of algebraic structures is the most natural way for further development and deeper understanding of their fundamental properties. The notion of n-ary group was introduced by Dörnte [12]. Since then many papers concerning various n-ary algebras have appeared in the literature, for example see [8, 9, 10, 13, 14, 18, 22]. The concept of n-ary hypergroup is defined by Davvaz and Vougiouklis in [4], which is a generalization of the concept of hypergroup in the sense of Marty and a generalization of n-ary group, too. Then this concept was studied by Ghadiri and Waphare [15], Leoreanu-Fotea and Davvaz [17, 18], Davvaz et al. [5, 6] and others. Also Leoreanu-Fotea and Davvaz introduced and studied the notion of a partial n-hypergroupoid, associated with a binary relation and some important results, concerning Rosenberg partial hypergroupoids, induced by relations, are generalized to the case of n-hypergroupoids

Recently, the notation for (m, n)-hyperrings was defined by Mirvakili and Davvaz [20] and they obtained (m, n)-rings from (m, n)-hyperrings using fundamental relations. Moreover, they defined a certain class of (m, n)-hyperrings called Krasner (m, n)-hyperrings. Krasner (m, n)-hyperrings are a generalization of (m, n)-rings and a generalization of Krasner hyperrings. Also, several properties of Krasner (m, n)-hyperrings are presented.

The main purpose of this paper is to generalize and develop a few basic properties of the vector space and normed vector space. Also, we have established a few basic properties in m-ary hypervector space and several important properties obtained. Moreover, we introduced the notion of *bundle subspace* and we have established that the kernel of any linear functional is a bundle subset and for every bundle subset there exists a linear functional such that this bundle subset contained in the kernel of this lineal functional.

2. m-Ary Hypervector Space

Let R be a non-empty set and $n \in \mathbb{N}$, $n \geq 2$ and $f : R^n \longrightarrow \mathcal{P}^*(R)$, where $\mathcal{P}^*(R)$ is the set of all non-empty subsets of R. Then, f is called an *n-ary hyperoperation* on R and the pair (R, f) is called an *n-ary hypergroupoid.* If $R_1, ..., R_n$ are non-empty subsets of R, then we define

$$f(R_1, R_2, ..., R_n) = \bigcup \{f(x_1, x_2, ..., x_n) : x_i \in R_i,\ i \in 1, 2, ..., n\}.$$

The sequence $x_i, x_{i+1}, ..., x_j$ will be denoted by x_i^j. For $j < i$, x_i^j is the empty set. An n-ary hypergroupoid (R, f) will be called an *n-ary semihypergroup* if we have:

$$f\left(\overset{(i-1)}{x_1}, f\left(\overset{(n+i-1)}{x_i}\right), x_{n+i}^{(2n-1)}\right) = f\left(\overset{(j-1)}{x_1}, f\left(\overset{(n+j-1)}{x_j}\right), \overset{(2n-1)}{x_{n+j}}\right),$$

for every $i, j \in \{1, 2, ..., n\}$ and $x_1, x_2, ...x_{2n-1} \in R$. Suppose that the equation

$$y \in f\left(\overset{(i-1)}{x_1}, z_i, \overset{n}{x_{i+1}}\right),$$

has a solution $z_i \in R$ for every $x_1, x_2, ..., x_{i-1}, x_{i+1}, .., x_n, y \in R$. Then, R is called *n-ary hypergroup*. An n-ary hypergroupoid (R, f) is *commutative* if for all $\sigma \in S_n$, $f(x_1, x_2, ..., x_n) = f(x_{\sigma(1)}, x_{\sigma(2)}, ..., x_{\sigma(n)})$. A commutative n-ary hypergroupoid (R, f) is called *canonical n-ary hypergroup* if following axioms hold for all $1 \leq i, j \leq n$ and $x, x_i \in R$:

(i) There exists a unique element $0 \in R$ such that $x = f\left(\overset{(i-1)}{0}, x, \overset{(n-i)}{0}\right)$,

(ii) There exists a unique operation $-$ on R such that $x \in f(x_1^n)$ implies that $x_i \in f(-x_{i-1}, -x_{i-2}, ... - x_1, x, -x_n, ..., -x_{i+1})$.

Definition 2.1. A Krasner (m, n)-hyperfield is an algebraic hyperstructure (R, f, g) which satisfies the following axioms:

1. (R, f) is a canonical m-ary hypergroup,
2. (R, g) is an n-ary semigroup,
3. The n-ary operation is distributive with respect to the m-ary hyperoperation f, i.e, for every x_i^{i-1}, x_{i+1}^n, a_1^m, $1 \leq i \leq n$

$$g\left(\overset{(i-1)}{x_i}, f\left(a_1^m\right), \overset{n}{x_{i+1}}\right) = f\left(g\left(\overset{(i-1)}{x_1}, a_1, x_{i+1}^n\right), ..., g\left(\overset{(i-1)}{x_i}, a_m, \overset{n}{x_{i+1}}\right)\right).$$

4. 0 is a zero element (absorbing element) of the n-ary operation g, i.e., for every $x_2^n \in R$ we have

$$g(0, x_2^n) = g(x_1, 0, x_3^n) = ... = g(x_1^{(n-1)}, 0) = 0,$$

5. there exists an element $e \in R$, called the identity element such that $g(a, \underbrace{e, ..., e}_{n-1}) = a$, for every $a \in R$,
6. for each non-zero element $a \in R$ there exists, an element a^{-1} such that $g(a, a^{-1}, ..., a^{-1}) = e$,
7. g is a commutative operation.

Example 2.2. Let $\mathbb{R}$ be the set of all real numbers and G be a subgroup of $(\mathbb{R}, \cdot)$. We define $(a, b) \in \rho$ if and only if there exists $g \in G$ such that $a = bg^{-1}$. This is an equivalence relation on $\mathbb{R}$. Set $[\mathbb{R} : \rho] = \{\rho(a) : a \in \mathbb{R}\}$, where $\rho(a)$ is an equivalence class $a \in \mathbb{R}$, and define the m-ary hyperoperation f and n-ary multiplication g as follows:

$$\begin{aligned} f(\rho(a_1), \rho(a_2), ..., \rho(a_m)) &= \{\rho(x) : \rho(x) \subseteq \rho(a_1) + \rho(a_2) + ... + \rho(a_m)\}, \\ g(\rho(a_1), \rho(a_2), ..., \rho(a_n)) &= \rho(a_1 a_2 ... a_n), \end{aligned}$$

then $\mathbb{R}$ is a Krasner (m, n)-hyperring.

Definition 2.3. Let $\mathbb{R}$ be the set of all real numbers. The Krasner (m, n)-hyperfield denoted on $\mathbb{R}$ is called the *real Krasner (m, n)-hyperfield.*

Definition 2.4. Let (F, f, g) and (V, h) be a Krasner (m_1, n_1)-ary hyperfield and be a canonical m-ary hypergroup, respectively. Then, V is said to be *m-ary hypervector space* over Krasner (m_1, n_1)-hyperfield F, if there exists a hypermultiplication $\cdot : F \times V \longrightarrow \mathcal{P}^*(V)$(image to be denoted by $x \cdot v$ for $x \in F$ and $v \in V$) such that for all $x, x_1^{m_1}, x_1^{n_1} \in F$ and $v, v_1^m \in V$ satisfies the following axiom:

1. $x \cdot (h(v_1^m)) = h(x \cdot v_1, ..., x \cdot v_m)$,
2. $f(x_1^{m_1}) \cdot v = h(x_1 \cdot v, x_2 \cdot v, ..., x_{m_1} \cdot v)$,
3. $g(x_1^{n_1}) \cdot v = x_1 \cdot (x_2 \cdot (x_3 ... x_{n_1} \cdot v)$,
4. $(-x) \cdot v = x \cdot (-v) = -(x \cdot v)$,
5. $v \in 1_F \cdot v,\ 0 = 0 \cdot v$.

where 1_F is the identity element of F and $\mathcal{P}^*(V)$ is the set of all non-empty subset of V. In this definition if V is an m-ary group, then V is called *additive m-ary hypervector space.*

Throughout this paper, by an m-ary hypervector space V, we mean a hypervector space $(V, h, *)$ and by a Krasner (m, n)-hyperfield F, we mean a Krasner (m, n)-hyperfield (F, f, g).

Example 2.5. Let (F, f, g) be a Krasner $(m, 2)$-hyperfield and $V = F \times F$. We define m-ary hyperoperation h on V as follows:

$$h((a_1, b_1), (a_2, b_2), ..., (a_m, b_m)) = \{(x, y) : x \in f(a_1, a_2, ..., a_m), y \in f(b_1, b_2, ..., b_m)\},$$

then (V, h) is a canonical m-ary hypergroup. Now we define a scalar multiplication $* : F \times V \longrightarrow \mathcal{P}(V)$ by

$$c * (a, b) = (g(c, a), g(c, b)),$$

where $c \in F$ and $(a, b) \in V$. Then we easily verify that V is an m-ary hypervector space.

Proposition 2.6. *(Construction). Let $(V, +, \cdot)$ be a hypermodule over field F and m-ary hyperoperation h on V defined by $h(v_1^m) = \sum_{i=1}^{m} v_i$. Then, V is an m-ary hypermodule.*

Proof. We prove that V is a canonical m-ary hypergroup. Since $+$ is well-defined implies that h is well-defined. Let 0 be the zero element of $(V, +)$. Then, 0 is a zero element of (V, h). Now, let $v, v_1^m \in V$ and $1 \leq j \leq m$, such that $v \in h(v_1, v_2, ..., v_{j-1}, v_j, v_{j+1}, ..., v_m)$. Then, $v \in \sum_{i=1, i\neq j}^{m} v_i + v_j$. This implies that there exists $z \in \sum_{i=1, i\neq j}^{m} v_i$, such that $v \in z + v_j$. Hence $v_j \in -z + v$. But $-z \in -\left(\sum_{i=1, i\neq j}^{m} v_i\right) = \sum_{i=1, i\neq j}^{m} -v_i$. This implies that $v_j \in h(-v_{j-1}, ..., -v_1, v, -v_m, ..., v_{m+1})$. So, (V, h) is a canonical m-ary hypergroup. Since the multiplication $\cdot$ is distributive with respect to the hyperoperation $+$, it is not difficult to see that $(V, h, \cdot)$ is an m-ary hypermodule. □

A subset V_1 of an m-ary hypervector space V over F is called *m-ary hypervector space* if V_1 is an m-ary hypervector space over F. So a subset V_1 of V is an m-ary hypervector subspace if and only if following statements holds:

1. for every $v_1^m \in V_1$, $h(v_1^m) \subseteq V_1$,
2. for every $x \in F$ and $v_1 \in V_1$, $x \cdot v_1 \subseteq V_1$.

Definition 2.7. Let V_1 and V_2 be two m-ary hypervector space. We say that $T : V_1 \longrightarrow V_1$ is a homomorphism if

$$T(h(v_1, v_2, ..., v_m)) = h(T(v_1), T(v_2), ..., T(v_m)),\ T(\lambda \cdot v) = \lambda \cdot T(v),$$

where $v_1, v_2, .., v_m, v \in V_1$ and $\lambda \in F$.

Proposition 2.8. *Let V_1 be a non-empty subset of V. Then, V_1 is an m-ary hyper subspace if and only if $h(x_1 \cdot v_1, ..., x_m \cdot v_m) \subseteq V_1$, for every $x_1^m \in F$ and $v_1^m \in V_1$.*

Proof. Suppose that V_1 is an m-ary hyper subspace of V. So obviously, $h(x_1 \cdot v_1, x_2 \cdot v_2, ..., x_m \cdot v_m) \subseteq V_1$.

Conversely, let $v_1^m \in V_1$. Since $1_F \in F$, we have

$$h(v_1^m) \subseteq h(1_F \cdot v_1, 1_F \cdot v_2, ..., 1_F \cdot v_m) \subseteq V_1.$$

Let $x \in F$ and $v_1 \in V_1$. Hence $x \cdot v_1 = h(x \cdot v_1, \underbrace{0, 0, ..., 0}_{m-1}) = h(x \cdot v_1, 0 \cdot v_1, ..., 0 \cdot v_1) \subseteq V_1$. This complete the proof. □

Proposition 2.9. *Let V be an m-ary hypervector space over an (m, n)-hyperfield F. Then,*

1. *$x \cdot 0 = \{0\}$, for every $x \in F$,*
2. *$x \cdot v = \{0\}$, implies that $x = 0$ or $v = 0$.*

Proof. 1. Suppose that $x \in F$. By axiom (5), for every $v \in V$, $0 \cdot v = 0$. Then we have

$$x \cdot 0 = x \cdot (0 \cdot v) = x \cdot (0 \cdot (0 \cdot v)) = ... = x \cdot (\underbrace{0 \cdot (0 ... (0}_{n-1} \cdot v)) = g(x, \underbrace{0, 0, ..., 0}_{n-1}) \cdot v = 0 \cdot v = 0.$$

2. Let $0 \neq x \in F$ and $v \in V$ be such that $x \cdot v = 0$. Since $x^{-1} \in F$, implies that

$$0 = x{\cdot}v = x^{-1}{\cdot}(x{\cdot}v) = ... = \underbrace{x^{-1}(x^{-1}(...x^{-1}}_{n-1}(x.v)) = g(x, \underbrace{x^{-1}, x^{-1}, ..., x^{-1}}_{n-1}){\cdot}v = v.$$

□

3. Hypernorm Spaces

In this section we define a hypernorm on V and then we have established some important results. Then we introduce the notion of innerproduct and consider the relation between the structures of norm and innerproduct on hyperspaces. Moreover, we introduce the bundle subset and prove some important theorems.

Definition 3.1. Let V be an m-ary hypervector space over the real Krasner (m, n) hyperfield $\mathbb{R}$. A *hypernorm* on V is a mapping $\|\cdot\| : V \longrightarrow \mathbb{R}$, where $\mathbb{R}$ is a usual real space, such that for all $x \in \mathbb{R}$ and $v, v_1, v_2, ..., v_m \in V$ following conditions hold:

1. $\|v\| \geq 0$ and $\|v\| = 0$ if and only if $v = 0$,
2. $sup\|h\,(v_1^m)\,\| \leq \sum_{i=1}^{m} \|v_i\|$, where $\|h\,(v_1^m)\,\| = \{\|x\| : x \in h\,(v_1^m)\}$,
3. $sup\|x \cdot v\| \leq |x|\|v\|$, where $\|x \cdot v\| = \{\|y\| : y \in x \cdot v\}$.

Example 3.2. Let $V = \mathbb{Z}_4 \cup \{0\}$ and define 2-ary hyperoperation f as follows:

$$\begin{aligned} f(\bar{a}, 0) &= f(0, \bar{a}) = \{\bar{a}\} \text{ for all } \bar{a} \in V, \\ f(\bar{a}, \bar{a}) &= \{\bar{a}, 0\} \text{ for all } \bar{a} \in V, \\ f(\bar{a}, \bar{b}) &= f(\bar{b}, \bar{a}) = V \backslash \{\bar{a}, \bar{b}\}. \end{aligned}$$

Then, (V, f) is a canonical 2-ary hypergroup. If we define the 2-ary multiplication on $F = V$ by

$$\begin{aligned} g(\bar{a}, 0) &= g(0, \bar{a}) = 0 \text{ for all } \bar{a} \in V, \\ g(\bar{a}, \bar{b}) &= \overline{ab}. \end{aligned}$$

then the map $\|\ \bar{x}\ \| \longrightarrow x$ is a hypernorm on V. Then (F, f, g) is a Krasner $(2, 2)$- hyperfield. We define the scaler multiplication

$$\begin{aligned} * : F \times V &\longrightarrow V \\ (\bar{a}, \bar{b}) &\longmapsto g(\bar{a}, \bar{b}). \end{aligned}$$

It can be verified obviously that V is a 2-ary hypervector space. We define $\|\cdot\|: V \longrightarrow \mathbb{R}$, by $\bar{x} \longrightarrow x$. Then $(V, \|\cdot\|)$ is normed 2-ary hypervector space.

EXAMPLE 3.3. Let $(\mathbb{Z}_p, +, \cdot)$ be a field and $V = \mathbb{Z}_p$. We define a 2-ary hyperoperation f as follows:

$$\begin{aligned} f(\overline{a}, \overline{b}) &= \{\overline{a}, \overline{b}, \overline{a} + \overline{b}\}, \text{for all } \overline{a}, \overline{b} \in \mathbb{Z}_p \text{ and } \overline{a} \neq -\overline{b}, \\ f(\overline{a}, \overline{0}) &= f(\overline{0}, \overline{a}) = \overline{a}, \text{ for all } \overline{a} \in \mathbb{Z}_p, \\ f(\overline{a}, -\overline{a}) &= \mathbb{Z}_p, \text{ for all } \overline{a} \in \mathbb{Z}_p \backslash \overline{0}. \end{aligned}$$

Then (V, f) is a canonical 2-ary hypergroup. Let $F = \mathbb{Z}_p$ and scaler multiplication on $* : F \times V \longrightarrow V$ be defined by $(\overline{a}, \overline{b}) \longmapsto \overline{ab}$. Then, V is a 2-ary hypervector space. We define $\| \cdot \|: V \longrightarrow R$ by $\| \overline{x} \| \longrightarrow x$, for all $\overline{x} \in V$. Then $\| \cdot \|$ is a hypernorm on V.

Suppose that $\| \cdot \|$ is a hypernorm on V then the couple $(V, \| \cdot \|)$ is said to be a *normed m-ary hypervector space* or *hypernormed space*. In this section V will be consider as a hypernormed space.
Let V_1 and V_2 be two m-ary hypervector space. A linear transformation is a mapping $T : V_1 \longrightarrow V_2$ such that for every $v_1, v_2, ..., v_m, v \in V_1$ and $\lambda \in F$ the following hold:

1. $T(h(v_1, v_2, ..., v_m)) = h(T(x_1), T(x_2), ..., T(x_m))$,
2. $T(\lambda \cdot v) = \lambda \cdot T(v)$.

We define $ker\ T = \{v \in V_1 : T(v) = 0\}$. A linear transformation $T : V \longrightarrow F$ is called lineal functional, where V is an m-ary hypervector space over F.

Proposition 3.4. *Let V be an m-ary hypervector space and T_1, T_2 be two linear transformations such that $ker\ T_1 = ker\ T_2$. Then, there is $\lambda \in F$ such that $T_2 = \lambda T_1$.*

Proof. Suppose that $T_1 \neq 0$. Indeed, it is trivial if $T_1 = 0$. Let $v_0 \in V$ be such that $T_1(v_0) \neq 0$. This implies that $T_2(v_0) \neq 0$. Let $\lambda = \frac{T_2(v_0)}{T_1(v_0)}$, $v \in V$ and $\delta = \frac{T_1(v)}{T_1(v_0)}$. So $T_1(v) = \delta \cdot T_1(v_0) = T_1(\delta \cdot v_0)$. For every $w \in \delta \cdot v_0$, we have $T_1(v - w) = 0$. Hence $v - \delta \cdot v_0 \subseteq ker\ T_1 = ker\ T_2$. Therefore,

$$T_2(v) = T_2(\delta \cdot v_0) = \delta \cdot T_2(v_0) = \delta\lambda \cdot T_1(v_0) = \lambda \cdot T_1(v).$$

This completes the proof. □

Proposition 3.5. *Let V be a hypernormed space. Then, following assertions holds:*

1. $sup\ \| h(V_1, V_2, ..., V_m) \| \leq \sum_{i=1}^{m} sup\ \|V_i\|$,
   where $V_1, V_2, ..., V_m$ are subsets of V,
2. $\|v\| = \| - v\|$, *for every $v \in V$,*
3. $\left\| h\left(v_1, -v_2, \overset{(m-2)}{0}\right) \right\| = \left\| h\left(-v_1, v_2, \overset{(m-2)}{0}\right) \right\|$,
4. *if* $inf \left\| h\left(v_1, -v_2, \overset{(m-2)}{0}\right) \right\| = 0$, *then* $\|v_1\| = \|v_2\|$,

5. $|\ \|v_1\| - \|v_2\|\ | \leq inf \left\| h\left(v_1, -v_2, \overset{(m-2)}{0}\right) \right\|$.

Proof. 1. Let $v_i \in V_i$, for $1 \leq i \leq m$. Then, we have

$$sup\ \ \|h(v_1^m)\| \leq \sum_{i=1}^{m} \|v_i\| \leq \sum_{i=1}^{m} \sup\ \ \|V_i\|.$$

Hence,

$$sup\ _{v_i \in V_i} (\|sup\ h(v_1^m)\|) \leq \sum_{i=1}^{m} \|v_i\| \leq \sum_{i=1}^{m} \sup\ \ \|V_i\|.$$

Therefore, $sup\ \ \|h(V_1, V_2, ..., V_m)\| \leq \sum_{i=1}^{m} sup\ \|V_i\|$.

2. Suppose that $v \in V$. Then we have

$$-v \in -1 \cdot v \Longrightarrow \| - v\| \leq sup\| - 1 \cdot v\| \Longrightarrow \| - v\| \leq | - 1|\|v\| \Longrightarrow \| - v\| \leq \|v\|.$$

Also

$$v \in -1 \cdot -v \Longrightarrow \|v\| \leq sup\| - 1 \cdot -v\| \Longrightarrow \|v\| \leq | - 1|\| - v\| \Longrightarrow \|v\| \leq \| - v\|.$$

Hence, $\|v\| = \| - v\|$.

3. Suppose that $v \in h\left(v_1, -v_2, \overset{(m-2)}{0}\right)$. Then we have

$$\begin{aligned} v \in h\left(v_1, -v_2, \overset{(m-2)}{0}\right) &\Longleftrightarrow v_1 \in h\left(-(-v_2), v, \overset{(m-2)}{0}\right) \\ &\Longleftrightarrow v_1 \in h\left(v_2, v, \overset{(m-2)}{0}\right) \\ &\Longleftrightarrow v_2 \in h\left(v_1, -v, \overset{(m-2)}{0}\right) \\ &\Longleftrightarrow -v \in h\left(v_2, -v_1, \overset{(m-2)}{0}\right). \end{aligned}$$

This implies that $\left\|h\left(v_1,-v_2,\overset{(m-2)}{0}\right)\right\| = \left\|h\left(-v_1,v_2,\overset{(m-2)}{0}\right)\right\|$.

4. Let $v \in h\left(v_1,-v_2,\overset{(m-2)}{0}\right)$ and $w \in h\left(v_2,-v_3,\overset{(m-2)}{0}\right)$. Then, we have

$$
\begin{aligned}
&-v_2 \in h\left(v,-v_1,\overset{(m-2)}{0}\right),\ v_2 \in h\left(w,-(-v_3),\overset{(m-2)}{0}\right)\\
&\Longrightarrow -v_2 \in h\left(v,-v_1,\overset{(m-2)}{0}\right),\ v_2 \in h\left(w,v_3,\overset{(m-2)}{0}\right)\\
&\Longrightarrow h\left(v_2,-v_2,\overset{(m-2)}{0}\right) \subseteq h\left(h\left(w,v_3,\overset{(m-2)}{0}\right),h\left(v,-v_1,\overset{(m-2)}{0}\right),\overset{(m-2)}{0}\right)\\
&\Longrightarrow 0 \in h\left(h\left(v,w,\overset{(m-2)}{0}\right),h\left(-v_1,v_3,\overset{(m-2)}{0}\right),\overset{(m-2)}{0}\right)\\
&\Longrightarrow 0 \in h\left(h\left(v,w,\overset{(m-2)}{0}\right),z,\overset{(m-2)}{0}\right),\ \text{for some } z \in h\left(-v_1,v_3,\overset{(m-2)}{0}\right)\\
&\Longrightarrow -z \in h\left(v,w,\overset{(m-2)}{0}\right)\\
&\Longrightarrow \|-z\| \le sup\left\|h\left(v,w,\overset{(m-2)}{0}\right)\right\| \le \|v\|+\|w\|\\
&\Longrightarrow sup\left\|h\left(v_1,-v_3,\overset{(m-2)}{0}\right)\right\| \le sup\left\|h\left(v_1,-v_2,\overset{(m-2)}{0}\right)\right\| + sup\left\|h\left(v_2,-v_3,\overset{(m-2)}{0}\right)\right\|.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&sup\left\|h\left(v_1,\overset{(m-1)}{0}\right)\right\| \le sup\left\|h\left(v_1,-v_2,\overset{(m-2)}{0}\right)\right\| + sup\left\|h\left(v_2,\overset{(m-1)}{0}\right)\right\|\\
&\Longrightarrow \|v_1\| \le sup\left\|h\left(v_1,-v_2,\overset{(m-2)}{0}\right)\right\| + \|v_2\| \Longrightarrow \|v_1\| \le \|v_2\|.
\end{aligned}
$$

and

$$
\begin{aligned}
&sup\left\|h\left(v_2,\overset{(m-1)}{0}\right)\right\| \le sup\left\|h\left(v_2,-v_1,\overset{(m-2)}{0}\right)\right\| + sup\left\|h\left(v_1,\overset{(m-1)}{0}\right)\right\|\\
&\Longrightarrow \|v_2\| \le sup\left\|h\left(v_2,-v_1,\overset{(m-2)}{0}\right)\right\| + \|v_1\|\\
&\Longrightarrow \|v_2\| \le sup\left\|h\left(v_2,-v_1,\overset{(m-1)}{0}\right)\right\| + \|v_1\|\\
&\Longrightarrow \|v_2\| \le \|v_1\|.
\end{aligned}
$$

5. Suppose that $v \in h\left(v_1, -v_2, \overset{(m-2)}{0}\right)$. Then $v_1 \in h\left(v, v_2, \overset{(m-2)}{0}\right)$

$$\begin{aligned}
&\Longrightarrow \|v_1\| \leq sup\left\|h\left(v, v_2, \overset{(m-2)}{0}\right)\right\| \leq \|v\| + \|v_2\| \\
&\Longrightarrow \|v_1\| - \|v_2\| \leq \|v\| \\
&\Longrightarrow \|v_1\| - \|v_2\| \leq sup\left\|h\left(v, -v_2, \overset{(m-2)}{0}\right)\right\|.
\end{aligned}$$

Moreover $-v_2 \in h\left(v, -v_1, \overset{(m-2)}{0}\right)$. Then, we have

$$\begin{aligned}
&\Longrightarrow \|v_2\| \leq sup\left\|h\left(v, -v_1, \overset{(m-2)}{0}\right)\right\| \leq \|v\| + \|v_1\| \\
&\Longrightarrow \|v_2\| - \|v_1\| \leq \|v\| \\
&\Longrightarrow \|v_2\| - \|v_1\| \leq sup\left\|h\left(v_1, -v_2, \overset{(m-2)}{0}\right)\right\|.
\end{aligned}$$

This completes the proof. □

Definition 3.6. Let $\{a_n\}$ be a sequence in a normed hypervector space V. We say that this sequence is converge to a point a if for any $\epsilon > 0$; there exists a positive integer m such that $sup\left\|h\left(a_n, -a, \overset{(m-2)}{0}\right)\right\| < \epsilon$, for every $n \geq m$. If a sequence $\{a_n\}$ converges to a point a in V, then we write $lim_{n\longrightarrow\infty} = a$ and we call a is a limit of $\{a_n\}$ in V.

Proposition 3.7. *Let $\{a_n\}$ be a sequence in a normed hypervector space V such that $lim_{n\longrightarrow\infty}a_n = a$ and $lim_{n\longrightarrow\infty}a_n = b$. Then, $a = b$.*

Proof. Suppose that $\epsilon > 0$. Then there exists a positive integer m such that

$$sup\left\|h\left(a_n, -a, \overset{(m-2)}{0}\right)\right\| < \frac{\epsilon}{2},\ sup\left\|h\left(a_n, -b, \overset{(m-2)}{0}\right)\right\| < \frac{\epsilon}{2},$$

for every $n \geq m$. By the theorem 3.5, we have

$$\begin{aligned}
sup\left\|h\left(a, -b, \overset{(m-2)}{0}\right)\right\| &= sup\left\|h\left(h\left(a, \overset{(m-1)}{0}\right), -b, \overset{(m-2)}{0}\right)\right\| \\
&\leq sup\left\|h\left(h\left(a, h\left(a_n, -a_n, \overset{(m-2)}{0}\right), \overset{(m-2)}{0}\right), -b, \overset{(m-2)}{0}\right)\right\| \\
&\leq sup\left\|h\left(h\left(h\left(a, -a_n, \overset{(m-2)}{0}\right), a_n, \overset{(m-2)}{0}\right), -b, \overset{(m-2)}{0}\right)\right\| \\
&\leq sup\left\|h\left(h\left(a, -a_n, \overset{(m-2)}{0}\right), h\left(a_n, -b, \overset{(m-2)}{0}\right), \overset{(m-2)}{0}\right)\right\| \\
&\leq sup\left\|h\left(a_n, -a, \overset{(m-2)}{0}\right)\right\| + sup\left\|h\left(a_n, -b, \overset{(m-2)}{0}\right)\right\|.
\end{aligned}$$

Therefore,

$$sup\left\|h\left(a,-b,\overset{(m-2)}{0}\right)\right\|<\epsilon,$$

for every $\epsilon>0$. This implies that $h\left(a,-b,\overset{(m-2)}{0}\right)=0$.

$$\begin{aligned} a=h\left(a,\overset{(m-1)}{0}\right) &\subseteq h\left(h\left(b,-b,\overset{(m-2)}{0}\right),a,\overset{(m-3)}{0}\right)\\ &=h\left(b,h\left(a,-b,\overset{(m-2)}{0}\right),\overset{(m-3)}{0}\right)\\ &=h\left(b,\overset{(m-1)}{0}\right)=b. \end{aligned}$$

This completes the proof. □

Proposition 3.8. *Let $\{a_n\}$ be a sequence in V and $lim_{n\longrightarrow\infty}=a$. Then, this sequence is bonded.*

Proof. Suppose that the sequence a sequence $\{a_n\}$ converges to a point a in V. Then there exists a positive number m such that

$$sup\left\|h\left(a_n,-a,\overset{(m-2)}{0}\right)\right\|<1,$$

for every $n\geq m$. Let $x\in h\left(a_n,-a,\overset{(m-2)}{0}\right)$. Then, $a_n\in h\left(x,a,\overset{(m-2)}{0}\right)$. This implies that

$$\|a_n\|\leq sup\ \left\|h\left(x,a,\overset{(m-2)}{0}\right)\right\|\leq\|x\|+\|a\|.$$

So

$$\begin{aligned} \|a_n\|\leq\|x\|+\|a\| &\Longrightarrow\|a_n\|\leq sup\left\|h\left(a_n,-a,\overset{(m-2)}{0}\right)\right\|+\|a\|\\ &\Longrightarrow\|a_n\|\leq 1+\|a\| \end{aligned}$$

Let $M=max\{\|a_1\|,\|a_2\|,...,\|a_{m-1}\|,1+\|a\|\}$. Therefore, $\|a_n\|\leq M$ for all positive integer n.
This completes the proof. □

Theorem 3.9. *Let $\{a_n\}$ and $\{b_n\}$ be sequences in V such that $lim_{n\longrightarrow\infty}a_n=a$ and $lim_{n\longrightarrow\infty}b_n=b$, respectively and $c\in h\left(a,b,\overset{(m-2)}{0}\right)$. Then,there there exists a sequence $\{c_n\}$ such that $c_n\in h\left(a_n,b_n,\overset{(m-2)}{0}\right)$ and $lim_{n\longrightarrow\infty}c_n=c$.*

Proof. Suppose that $\{a_n\}$ and $\{b_n\}$ be two convergent sequences which are convergent to a and b, respectively. There is a positive integer m such that

$$sup\left\|h\left(a_n,-a,\overset{(m-2)}{0}\right)\right\|<\frac{\epsilon}{2},\ sup\left\|h\left(b_n,-b,\overset{(m-2)}{0}\right)\right\|<\frac{\epsilon}{2},$$

for every $n \geq m$. Let $x \in h\left(a_n, -a, \overset{(m-2)}{0}\right)$ and $y \in h\left(b_n, -b, \overset{(m-2)}{0}\right)$. Then, $a_n \in h\left(x, a, \overset{(m-2)}{0}\right)$ and $b_n \in h\left(y, b, \overset{(m-2)}{0}\right)$. This implies that $a \in h\left(a_n, -x, \overset{(m-2)}{0}\right)$ and $b \in h\left(b_n, -y, \overset{(m-2)}{0}\right)$. Hence

$$\begin{aligned} h\left(a, b, \overset{(m-2)}{0}\right) &\subseteq h\left(h\left(a_n, -x, \overset{(m-2)}{0}\right), h\left(b_n, -y, \overset{(m-2)}{0}\right), \overset{(m-2)}{0}\right) \\ &= h\left(h\left(h\left(a_n, -x, \overset{(m-2)}{0}\right), b_n, \overset{(m-2)}{0}\right), -y, \overset{(m-2)}{0}\right) \\ &= h\left(h\left(h\left(a_n, b_n, \overset{(m-2)}{0}\right), -x, \overset{(m-2)}{0}\right), -y, \overset{(m-2)}{0}\right) \\ &= h\left(h\left(a_n, b_n, \overset{(m-2)}{0}\right), h\left(-x, -y, \overset{(m-2)}{0}\right), \overset{(m-2)}{0}\right). \end{aligned}$$

Hence for every n there exist $x_n \in h\left(a_n, b_n, \overset{(m-2)}{0}\right)$ and $y_n \in h\left(-x, -y, \overset{(m-2)}{0}\right)$ such that $c \in h\left(x_n, y_n, \overset{(m-2)}{0}\right)$. So $y_n \in h\left(c, -x_n, \overset{(m-2)}{0}\right)$.

$$\begin{aligned} &\|y_n\| \leq sup\left\|h\left(-x, -y, \overset{(m-2)}{0}\right)\right\| \leq \|-x\| + \|-y\| = \|x\| + \|y\| \\ &\Longrightarrow sup\left\|h\left(-x, -y, \overset{(m-2)}{0}\right)\right\| \leq \|x\| + \|y\| \\ &\Longrightarrow sup\left\|h\left(x_n, -c, \overset{(m-2)}{0}\right)\right\| \leq \|x\| + \|y\| \\ &\Longrightarrow sup\left\|h\left(h\left(a_n, b_n, \overset{(m-2)}{0}\right), -c, \overset{(m-2)}{0}\right)\right\| \leq \|x\| + \|y\| \\ &\Longrightarrow sup\left\|h\left(h\left(a_n, b_n, \overset{(m-2)}{0}\right), -c, \overset{(m-2)}{0}\right)\right\| \leq \left\{\|x\| : x \in h\left(a_n, -a, \overset{(m-2)}{0}\right)\right\} \\ &+ \left\{\|y\| : y \in h\left(b_n, -b, \overset{(m-2)}{0}\right)\right\} \\ &\Longrightarrow sup\left\|h\left(h\left(a_n, b_n, \overset{(m-2)}{0}\right), -c, \overset{(m-2)}{0}\right)\right\| \leq sup\left\|h\left(a_n, -a, \overset{(m-2)}{0}\right)\right\| \\ &+ sup\left\|h\left(b_n, -b, \overset{(m-2)}{0}\right)\right\|. \end{aligned}$$

Therefore for every n, there exists a sequence c_n such that

$$sup\left\|h\left(c_n, -c, \overset{(m-2)}{0}\right)\right\| \leq sup\left\|h\left(a_n, -a, \overset{(m-2)}{0}\right)\right\| + sup\left\|h\left(b_n, -b, \overset{(m-2)}{0}\right)\right\|$$

Therefore there exists a sequence c_n which converges to c.
This completes the proof. □

Proposition 3.10. *Let a sequence $\{a_n\}$ converges to a in V and a sequence t_n converges to t in $\mathbb{R}$. Then for every $b \in t.a$ there exists a sequence $\{b_n\}$ in $t_n \cdot a_n$ such that $\{b_n\}$ converges to b in V.*

Proof. Suppose that $\{a_n\}$ and t_n are convergent sequence in V. Then there exist positive integer M_1 and M_2 such that $\|a_n\| < M_1$ and $|t_n| < M_2$. Since $\{a_n\}$ converges to a and $\{t_n\}$ is converges to t, for every $\epsilon > 0$ there exists a positive number m such that for every $n \geq m$

$$sup\left\|h\left(a_n, -a, \overset{(m-2)}{0}\right)\right\| < \frac{\epsilon}{M_1+M_2},\ sup\left\|f\left(t_n, -t, \overset{(m-2)}{0}\right)\right\| < \frac{\epsilon}{M_1+M_2}.$$

Let $x \in h\left(a_n, -a, \overset{(m-2)}{0}\right)$ and $y \in f\left(t_n, -t, \overset{(m-2)}{0}\right)$. This implies that

$$\begin{aligned}
& a \in h\left(a_n, -x, \overset{(m-2)}{0}\right) \text{ and } t \in f\left(t_n, -y, \overset{(m-2)}{0}\right) \\
& \Longrightarrow t \cdot a \subseteq f\left(t_n, -y, \overset{(m-2)}{0}\right) \cdot h\left(a_n, -x, \overset{(m-2)}{0}\right) \\
& = \left\{z_1 \cdot z_2 : z_1 \in f\left(t_n, -y, \overset{(m-2)}{0}\right), z_2 \in h\left(a_n, -x, \overset{(m-2)}{0}\right)\right\} \\
& = \left\{f\left(t_n, -y, \overset{(m-2)}{0}\right) \cdot z_2 : z_2 \in h\left(a_n, -x, \overset{(m-2)}{0}\right)\right\} \\
& = \left\{h\left(t_n \cdot z_2, -y.z_2, \overset{(m-2)}{0}\right) : z_2 \in h\left(a_n, -x, \overset{(m-2)}{0}\right)\right\} \\
& \subseteq h\left(t_n.h\left(a_n, -x, \overset{(m-2)}{0}\right), -y \cdot h\left(a_n, -x, \overset{(m-2)}{0}\right), \overset{(m-2)}{0}\right) \\
& \subseteq h\left(h\left(t_n \cdot a_n, t_n \cdot (-x), \overset{(m-2)}{0}\right), h\left(-y.a_n, (-y).(-x), \overset{(m-2)}{0}\right), \overset{(m-2)}{0}\right) \\
& \subseteq h\left(t_n \cdot a_n, h\left(t_n \cdot (-x), h\left(-y \cdot a_n, (-y) \cdot (-x), \overset{(m-2)}{0}\right), \overset{(m-2)}{0}\right), \overset{(m-2)}{0}\right).
\end{aligned}$$

Let b be any element of $t.a$. Then, there exists $c_n \in t_n.a_n$ and

$$d_n \in h\left(t_n.(-x), h\left(-y.a_n, (-y).(-x), \overset{(m-2)}{0}\right), \overset{(m-2)}{0}\right),$$

such that $b \in h\left(c_n, d_n, \overset{(m-2)}{0}\right)$. Hence

$$
\begin{aligned}
& d_n \in h\left(b, -c_n, \overset{(m-2)}{0}\right) \\
& \Longrightarrow \|d_n\| \leq sup\left\|h\left(t_n \cdot (-x), h\left(-y.a_n, (-y)\cdot(-x), \overset{(m-2)}{0}\right), \overset{(m-2)}{0}\right)\right\| \\
& \leq sup\, \|t_n \cdot (-x)\| + sup\left\|h\left(-y \cdot a_n, (-y)\cdot(-x), \overset{(m-2)}{0}\right)\right\| \\
& \leq sup\, \|t_n \cdot (-x)\| + sup\, \|-y \cdot a_n\| + sup\|(-y)\cdot(-x)\| \\
& \Longrightarrow sup\left\|h\left(b, -c_n, \overset{(m-2)}{0}\right)\right\| \leq \mid t_n \mid \|x\| + \mid y \mid \|a_n\| + \mid y \mid \|x\| \\
& \leq M_2\|x\| + \mid y \mid M_1 + \mid y \mid \|x\|,
\end{aligned}
$$

this is true for every $x \in h\left(a_n, -a, \overset{(m-2)}{0}\right)$ and $y \in f\left(t_n, -t, \overset{(m-2)}{0}\right)$. This implies that

$$
\begin{aligned}
sup\left\|h\left(c_n, -b, \overset{(m-2)}{0}\right)\right\| &\leq \frac{\epsilon}{M_1+M_2}M + \frac{\epsilon}{M_1+M_2}N + \frac{\epsilon}{M_1+M_2}\frac{\epsilon}{M_1+M_2} \\
&\leq \{1 + \frac{\epsilon}{(M_1+M_2)^2}\}\epsilon < 2\epsilon.
\end{aligned}
$$

Therefore $\{c_n\}$ converges to b.
This completes the proof. □

Proposition 3.11. *Let $\{a_n\}$ be a convergent sequence in V. Then, every subsequence of $\{a_n\}$ is convergent to V.*

Proof. Suppose that $\{a_n\}$ converges to a in V. Then for any $\epsilon > 0$ there exists a positive integer k such that

$$
sup\left\|h\left(a_n, -a, \overset{(m-2)}{0}\right)\right\| < \frac{\epsilon}{2},\ sup\left\|h\left(a_n, -a_m, \overset{(m-2)}{0}\right)\right\| < \frac{\epsilon}{2}.
$$

for every $n, m > k$. Let $\{a_{n_k}\}$ be a subsequence of $\{a_n\}$. Now we have

$$
\begin{aligned}
& sup\left\|h\left(a_{n_k}, -a, \overset{(m-2)}{0}\right)\right\| \\
& \leq sup\left\|h\left(a_{n_k}, h\left(-a, h\left(a_n, -a_n, \overset{(m-2)}{0}\right), \overset{(m-2)}{0}\right), \overset{(m-2)}{0}\right)\right\| \\
& = sup\left\|h\left(a_{n_k}, h\left(h\left(a_n, -a, \overset{(m-2)}{0}\right), -a_n, \overset{(m-2)}{0}\right), \overset{(m-2)}{0}\right)\right\| \\
& = sup\left\|h\left(h\left(a_{n_k}, -a_n, \overset{(m-2)}{0}\right), h\left(a_n, -a, \overset{(m-2)}{0}\right), \overset{(m-2)}{0}\right)\right\| \\
& \leq sup\left\|h\left(a_{n_k}, -a_n, \overset{(m-2)}{0}\right)\right\| + sup\left\|h\left(a_n, -a, \overset{(m-2)}{0}\right)\right\|.
\end{aligned}
$$

Hence $sup \left\| h\left(a_{n_k}, -a, \overset{(m-2)}{0}\right) \right\| < \epsilon$. For every $n_{n_k} > m$. This implies that $\{a_{n_k}\}$ converges to a.
This completes the proof. □

Definition 3.12. Let V be a hypervector space over F and C be a subspace of V. We say that C is a *bundle subspace* if for every $x \in V$ there exists $\lambda \in F$, such that $x \in h\left(\lambda \cdot y, C, \underbrace{0, ..., 0}_{m-2}\right)$, for every y such that $1 \cdot y \cap C = \emptyset$.

Definition 3.13. In the Example 3.2, $C = \{\overline{0}, \overline{1}, 0\}$ is a bundle subspace.

Proposition 3.14. *Let C be a bundle subspace of additive m-ary hypervector space V and $y \in V$ such that $1 \cdot y \cap C = \emptyset$. Then, for every $x \in V$ there exists a unique $\lambda \in F$ such that*

$$x \in h\left(\lambda \cdot x, C, \underbrace{0, ..., 0}_{m-2}\right).$$

Proof. Suppose that $\lambda_1, \lambda_2 \in F$ such that $\lambda_1 \neq \lambda_2$ such that $x \in h\left(\lambda_1 \cdot y, C, \underbrace{0, ..., 0}_{m-2}\right)$ and $x \in h\left(\lambda_2 \cdot y, C, \underbrace{0, ..., 0}_{m-2}\right)$. So there exist $z_1 \in \lambda_1 \cdot y$, $z_2 \in \lambda_2 \cdot y$ and $c_1, c_2 \in C$ such that $x = h\left(z_1, c_1, \underbrace{0, ..., 0}_{m-2}\right)$ and $x = h\left(z_2, c_2, \underbrace{0, ..., 0}_{m-2}\right)$. Hence

$$h\left(z_1, -z_2, \underbrace{0, ..., 0}_{m-2}\right) \in h\left(\lambda_1 \cdot y, -\lambda_2 \cdot y, \underbrace{0, ..., 0}_{m-2}\right) = f\left(\lambda_1, -\lambda_2, \underbrace{0, ..., 0}_{m-2}\right) \cdot y.$$

On the other hand

$$\begin{aligned} h\left(z_1, -z_2, \overset{(m-2)}{0}\right) &= h\left(h\left(x, -c_1, \overset{(m-2)}{0}\right), h\left(-x, c_2, \overset{(m-2)}{0}\right), \overset{(m-2)}{0}\right) \\ &= h\left(h\left(h\left(x, -c_1, \overset{(m-2)}{0}\right), -x, \overset{(m-2)}{0}\right), c_2, \overset{(m-2)}{0}\right) \\ &= h\left(h\left(h\left(-x, x, \overset{(m-2)}{0}\right), -c_1, \overset{(m-2)}{0}\right), c_2, \overset{(m-2)}{0}\right) \\ &= h\left(h\left(-x, x, \overset{(m-2)}{0}\right), h\left(c_2, -c_1 \overset{(m-2)}{0}\right), \overset{(m-2)}{0}\right) \\ &= h\left(c_2, -c_1 \overset{(m-2)}{0}\right). \end{aligned}$$

Since C is a vector space $C \cap 1_F \cdot y \neq \emptyset$, and this is contradiction. This completes the proof. □

Proposition 3.15. *Let V be an additive m- ary hypervector space over $\mathbb{R}$ and $T : V \longrightarrow \mathbb{R}$ be a linear functional. Then, $Kerl\ T$ is a bundle subspace.*

Proof. One can see that $Kerl\ T$ is a subspace of V. Suppose that $1_F \cdot x_0 \cap Kerl\, T = \emptyset$ and $x \in V$. Let $\lambda = \frac{T(x)}{T(x_0)}$. We prove that $x \in h\left(\lambda \cdot x_0, KerlT, \overset{(m-2)}{0}\right)$. Let $y \in h\left(x, -\frac{T(x)}{T(x_0)} \cdot x_0, \overset{(m-2)}{0}\right)$. Then

$$\begin{aligned}
T(y) &\in T\left(h\left(x, -\frac{T(x)}{T(x_0)} \cdot x_0, \overset{(m-2)}{0}\right)\right) = \left\{T\left(x, -z\ \overset{(m-2)}{0}\right) : z \in \frac{T(x)}{T(x_0)}\right\} \\
&= h\left(T(x), -\left\{T(z) : z \in \frac{T(x)}{T(x_0)} \cdot x_0\right\}, \overset{(m-2)}{0}\right) \\
&= h\left(T(x), -T\left(\frac{T(x)}{T(x_0)} \cdot x_0\right), \overset{(m-2)}{0}\right) \\
&= h\left(T(x), -\frac{T(x)}{T(x_0)} T(x_0), \overset{(m-2)}{0}\right) = 0.
\end{aligned}$$

So $y \in kerl\ T$. Since $h\left(x, -y, \overset{(m-2)}{0}\right) \in \frac{T(x)}{T(x_0)} \cdot x_0$, then

$$x = h\left(h\left(x, -y, \overset{(m-2)}{0}\right), y, \overset{(m-2)}{0}\right) \in h\left(\frac{T(x)}{T(x_0)} \cdot x_0, Kerl\ T, \overset{(m-2)}{0}\right).$$

This completes the proof. □

Proposition 3.16. *Let V be an additive m-ary hypervector space and C be a bundle subset of V. Then, there exists a linear functional T such that $C \subseteq Kerl\ T$.*

Proof. Suppose that $x_0 \in V$, such that $1_F \cdot x_0 \cap C \neq \emptyset$. By Proposition 3.14 for every $x \in V$, there exists a unique $\lambda_x \in F$ such that $x \in h\left(\lambda_x \cdot x_0, C, \overset{(m-2)}{0}\right)$. We define $T : V \longrightarrow F$ by $T(x) = \lambda_x$, then T is linearly functional. Indeed, for every $x \in V$, there exist $\lambda_x \in F$, such that $x \in h\left(\lambda_x \cdot x_0, C, \underbrace{0, ..., 0}_{m-2}\right)$. Then we have

$$\begin{aligned} h(x_1, x_2, ..., x_m) &\in h\left(h\left(\lambda_{x_1} \cdot x_0, C, \overset{(m-2)}{0}\right), \cdots, h\left(\lambda_{x_m} \cdot x_0, C, \overset{(m-2)}{0}\right)\right) \\ &= h\left(h\left(\lambda_{x_1} \cdot x_0, \lambda_{x_2} \cdot x_0, ..., \lambda_{x_m} \cdot x_0\right), h\left(C, C, ..., C\right), \underbrace{0, ..., 0}_{m-2}\right) \\ &\subseteq h\left(h\left(\lambda_{x_1} \cdot x_0, \lambda_{x_2} \cdot x_0, ..., \lambda_{x_m} \cdot x_0\right), C, \underbrace{0, ..., 0}_{m-2}\right). \end{aligned}$$

Hence

$$T(h(x_1, x_2, ..., x_m)) = h\left(\lambda_{x_1}, \lambda_{x_2}, ..., \lambda_{x_m}\right) = h\left(T(x_1), T(x_2), ..., T(x_m)\right).$$

Also,

$$\lambda.x \subseteq \lambda.h\left(\lambda_x \cdot x_0, C, \overset{(m-2)}{0}\right) = h\left((\lambda \cdot \lambda_x) \cdot x_0, \lambda \cdot C, \overset{(m-2)}{0}\right) \subseteq h\left((\lambda \cdot \lambda_x) \cdot x_0, C, \overset{(m-2)}{0}\right).$$

Hence

$$T(\lambda \cdot x) = \lambda \cdot \lambda_x = \lambda \cdot T(x).$$

Now, let $x \in C$. Then, we have

$$0 \in h\left(0 \cdot x_0, C, \overset{(m-2)}{0}\right).$$

It means that $T(x) = 0$, and the proof is completes. □

Definition 3.17. Let V be an m-ary hypervector space and $V_1 \subseteq V$. We say that V_1 is called *closed if for every sequence* $\{x_n\}$ *in* V_1 *in such that* $lim_{n\to\infty} x_n = x$ *implies that* $x \in V_1$.

Definition 3.18. Let V_1 and V_2 be two normed hypervector space and $T : V_1 \to V_2$ be homomorphism. We define

$$\|T\| = sup\left\{sup\left\|T\left(\frac{1}{\|v\|} \cdot v\right)\right\| : 0 \neq v \in V\right\}.$$

Theorem 3.19. *Let V be an additive m-ary hypervector space on $\mathbb{R}$ and $T : V \to \mathbb{R}$ linear functional. Then, kerl T is closed subspace of V if and only if T is continuous.*

Proof. Suppose that $kerlT$ is a closed subspace of V and T is not continues. This implies that for every $n \in \mathbb{N}$ there exists $v_n \in V$ such that

$$sup\left\|T\left(\frac{1}{\|v_n\|} \cdot v_n\right)\right\| = \frac{|T(v_n)|}{\|v_n\|} > n,$$

for every $n \in \mathbb{N}$. Hence there exists $x_n \in \frac{1}{\|v_n\|} \cdot v_n$ such that $|T(x_n)| > n$. Let $x \in h\left(x_1, -\frac{T(x_1)}{T(x_n)} \cdot x_n, \overset{(m-2)}{0}\right)$. Then there exists $y \in \frac{T(x_1)}{T(x_n)} \cdot x_n$ such that

$x = h\left(x_1, -y, \overset{(m-2)}{0}\right)$. This implies that

$$T(x) = h\left(T(x_1), -T(y), \overset{(m-2)}{0}\right) \in h\left(T(x_1), -T\left(\frac{T(x_1)}{T(x_n)} \cdot x_n\right), \overset{(m-2)}{0}\right) = 0.$$

This implies that $h\left(x_1, -\frac{T(x_1)}{T(x_n)} \cdot x_n, \overset{(m-2)}{0}\right) \subseteq Kerl\ T$. For every $n \in \mathbb{N}$, let $t_n \in h\left(x_1, -\frac{T(x_1)}{T(x_n)} \cdot x_n, \overset{(m-2)}{0}\right)$. Then,

$$\left\| h\left(t_n, -x_1, \overset{(m-2)}{0}\right)\right\| \leq sup \left\| h\left(h\left(x_1, -\frac{T(x_1)}{T(x_n)} \cdot x_n, \overset{(m-2)}{0}\right), -x_1, \overset{(m-2)}{0}\right)\right\|$$

$$= sup \left\| \frac{T(x_1)}{T(x_n)} \cdot x_n \right\| \leq \frac{T(x_1)}{n}$$

So $lim_{n\to\infty} t_n = x_1$. This is contradiction. Hence T is continuous.

Conversely, let $\{x_n\}$ be a sequence in $kerlT$. For any $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that

$$|T(x) - T(x_n)| = |T(x)| < \varepsilon.$$

This completes the proof. □

Theorem 3.20. *Let V_1 and V_2 be two normed m-ary hypervector space and $\psi : V_1 \longrightarrow V_2$ be a homomorphism such that for every convergent sequence $\{x_n\}$ in V_1, the sequence $\{\psi(x_n)\}$ is a convergent sequence in V_1. Then ψ is continues.*

Proof. Suppose that ψ is not continues. So for every $n \in \mathbb{N}$, there is $x_n \in V_1$ such that

$$sup \left\| \psi\left(\frac{1}{\|x_n\|_1} \cdot x_n\right)\right\|_2 = sup \left\| \frac{1}{\|x_n\|_1} \cdot \psi(x_n)\right\|_2 > n.$$

Hence there exists $b_n \in \frac{1}{\|x_n\|_1} \cdot x_n$ such that $\|\psi(b_n)\| > n$, for every $n \in \mathbb{N}$. Thus,

$$sup \left\| \frac{1}{\sqrt{n}} \cdot \psi(b_n)\right\|_2 > \frac{n}{\sqrt{n}} = \sqrt{n}.$$

This implies that $\{\psi(b_n)\}$ is not convergent. Moreover,

$$sup \left\| \frac{1}{\sqrt{n}} \cdot b_n\right\|_1 \leq \frac{1}{\sqrt{n}}.$$

So $\{b_n\}$ is a convergent sequence in V_1 but $\{\psi(b_n)\}$ is not convergent. Therefore, ψ is continuous. □

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