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Application of Collocation Method in Finding Roots

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Abstract. In this paper we present a new method to find simple or multiple roots of functions in a finite interval. In this method using bisection method we can find an interval such that this function is one to one on it, thus we can transform problem of finding roots in this interval into an ordinary differential equation with boundary conditions. By solving this equation using collocation method we can find a root for given function in the special interval. We also present convergence analysis of the new method. Finally some examples are given to show efficiency of the presented method.

Keywords: Finding root, Collocation method, Jacobi polynomial, Boundary value equation, Convergence.

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1. Introduction

Finding roots of functions are one of the oldest problem in mathematics. In many problems such as digital filtering design, image filtering and etc. [18, 19, 12, 1, 21, 3, 22] we need to find the roots of special functions, therefore presenting new methods for solving the root finding problem has special importance. For this problem there exists some classical numerical methods

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such as bisection method [4], Newton method [4] and etc. Also some authors have presented new algorithms for solving this problem [5, 6]. In this paper we present a new method for finding roots of continuous function on a finite interval $[a,b]$. In this section we introduce some preliminaries and definitions. In Section 2 we present a new method for finding roots of continuous functions in $[a,b]$. In this method we transform the problem of finding roots to an ordinary differential equation and solve this equation using the collocation method. Also in this section we study convergence analysis of the new method. Finally in Section 3 we give some numerical examples to show the efficiency of the new method.

Definition 1.1. A root c is a multiple root of f with multiplicity m if f expressed as

$$f(x) = (x-c)^m r(x), \tag{1.1}$$

where r is a continuous function and $r(c) \neq 0$.

Theorem 1.2. *(Intermediate Value Theorem) Assume that the function $f(x)$ is continuous for $x \in [a,b]$, $f(a) \neq f(b)$, and k is between $f(a)$ and $f(b)$. Then there is a point $\zeta \in (a,b)$ such that $f(\zeta) = k$. In particular, if $f(a)f(b) < 0$, then the equation $f(x) = 0$ has at least one root in the interval (a,b).*

Proof. see [4]. □

1.1. **Jacobi polynomials.** The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, are defined as the orthogonal polynomials with respect to the weight function $\omega^{(\alpha,\beta)}(x) = (x-a)^\alpha (b-x)^\beta$, $(\alpha > -1, \beta > -1)$ on $[a,b]$. The Jacobi polynomials satisfy [10] the following three term recurrence relation on the interval $[a,b]$:

$$P_0^{(\alpha,\beta)}(x) = 1,$$

$$P_1^{(\alpha,\beta)}(x) = \tfrac{a(1+\beta)+b(1+\alpha)}{\alpha+\beta+1},$$

$$P_{i+1}^{(\alpha,\beta)}(x) = (x - A_i)\, P_i^{(\alpha,\beta)}(x) - B_i P_{i-1}^{(\alpha,\beta)}(x), \quad i = 1,2,3,...,$$

where

$$A_i = \tfrac{2i(i+1+\alpha+\beta)(a+b)+(a(\beta+1)+b(\alpha+1))(\alpha+\beta)}{(2i+\alpha+\beta)(2i+\alpha+\beta+2)},$$

$$B_i = \tfrac{i(i+\beta)(i+\alpha+\beta)(b-a)^2}{(2i+\alpha+\beta+1)(2i+\alpha+\beta+2)^2(2i+\alpha+\beta+1)}.$$

On the other hand the Jacobi polynomials have the following properties [10]:

$$P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} B_k^{(\alpha,\beta,n)}(x-b)^k, \tag{1.2}$$

where

$$B_k^{(\alpha,\beta,n)} = \frac{(b-a)^{n-k} n!(k+\beta+1)_{n-k}}{(n-k)!(n+k+\alpha+\beta+1)_{n-k} k!}, \tag{1.3}$$

and

$$(c)_0 = 1, \quad (c)_k = \prod_{i=0}^{k} (c+i-1); \quad k = 1, 2, 3, ...,$$

or

$$P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} E_k^{(\alpha,\beta,n)} (x-a)^k, \tag{1.4}$$

where

$$E_k^{(\alpha,\beta,n)} = \frac{(a-b)^{n-k} n!(k+\alpha+1)_{n-k}}{(n-k)!(n+k+\alpha+\beta+1)_{n-k} k!}. \tag{1.5}$$

1.2. **Operational matrix of derivatives for Jacobi polynomials.** Suppose :

$$\varphi = [\varphi_0(t), \varphi_1(t), ..., \varphi_{n-1}(t)]^T, \tag{1.6}$$

where the elements $\varphi_0(t), \varphi_1(t), ..., \varphi_{n-1}(t)$, are the basis functions on the interval $[a, b]$. The matrix $D_{n\times n}$ is the operational matrix of derivatives if and only if

$$\frac{d}{dx}\varphi = D\varphi. \tag{1.7}$$

Suppose in (6) we define $\varphi_i(t) = P_i^{(\alpha,\beta)}(t)$. Using properties of Jacobi polynomials we have:

$$\frac{d}{dx} P_i^{(\alpha,\beta)}(x) = \sum_{j=0}^{i-1} d_{i,j} P_j^{(\alpha,\beta)}(x), \tag{1.8}$$

where the coefficients $d_{i,j}$ are obtained from the following upper triangular system [7]:

$$\begin{bmatrix} B_0^{(\alpha,\beta,0)} & \cdots & B_0^{(\alpha,\beta,i-3)} & B_0^{(\alpha,\beta,i-2)} & B_0^{(\alpha,\beta,i-1)} \\ 0 & B_1^{(\alpha,\beta,1)} & \cdots & B_1^{(\alpha,\beta,i-2)} & B_1^{(\alpha,\beta,i-1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & B_{i-2}^{(\alpha,\beta,i-2)} & B_{i-2}^{(\alpha,\beta,i-1)} \\ 0 & 0 & \cdots & 0 & B_{i-1}^{(\alpha,\beta,i-1)} \end{bmatrix} \begin{bmatrix} d_{i,0} \\ d_{i,1} \\ \vdots \\ d_{i,i-2} \\ d_{i,i-1} \end{bmatrix}$$

$$= \begin{bmatrix} P_i{}'(b)/0! \\ P_i{}''(b)/1! \\ \vdots \\ P_i^{(i-1)}(b)/\,(i-2)! \\ P_i^{i}(b)/\,(i-1)! \end{bmatrix}, \tag{1.9}$$

where $P_j^{(\alpha,\beta)}(x) = P_j(x)$ and $\frac{d^k}{dx^k}P_j^{(\alpha,\beta)}(x) = P_j^{(k)}(x)$, $(1 \le k \le N)$ and $B_k^{(\alpha,\beta,j)}$ is defined in (1.3). Solving this linear system concludes:

$$d_{i,i-1} = \frac{P_i^{(i)}(1)}{(i-1)!B_{i-1}^{(\alpha,\beta,i-1)}} = \frac{\alpha+\beta+2i}{2(\alpha+\beta+i)}, \tag{1.10}$$

and

$$d_{i,j} = \frac{P_i^{j+1}(1) - j!\sum\limits_{k=i-1}^{j+1} B_j^{(\alpha,\beta,k)} d_{i,k}}{j!B_j^{(\alpha,\beta,j)}}, \quad j = 0,1,2,...,i-2. \tag{1.11}$$

Now it can be seen that:

$$D = \begin{bmatrix} 0 & d_{1,0} & d_{2,0} & \cdots & d_{n,0} \\ 0 & 0 & d_{2,1} & \cdots & d_{n,1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & d_{n,n-1} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{1.12}$$

2. Finding roots of continuous functions

Suppose $f : [a,b] \longrightarrow \mathbb{R}$ is a continuous function and $f(a)f(b) < 0$. Therefore using Theorem 1 we conclude there exists $c \in (a,b)$ such that $f(c) = 0$. In this section we present a new method for finding c. Suppose c is a simple root of f on $[a,b]$. Now we look for an interval $[a_1,b_1] \subset [a,b]$ such that f be one to one on $[a_1,b_1]$. Using bisection method we determine a sequence of intervals $I_k = (c_k, c_k)$, $k = 1,2,3,...,n$ such that

$$(c_1,d_1) \supset (c_2,d_2) \supset (c_3,d_3) \supset ... \supset (c_n,d_n),$$

where (c_j,d_j), $j = 1,2,...,n$ contain a root of $f(x)$ and f is a one to one function on (c_n,d_n). For this purpose we present the following algorithm:

Input. *end points a and b.*

Step 1. *While f is not one to one on $[a,b]$ do steps* $3-6$.

Step 3. *Set* $c_1 = a$,

Step 4. *Set* $p_1 = a + \frac{b-a}{2}$,
$R_1 = f(c_1)$,
$R_2 = f(p_1)$.

Step 5. *If* $R_2 = 0$,
$c = p_1$
Stop.

Step 6. *If* $R_1R_2 < 0$,
$a = c_1$ *and* $b = p_1$,
else $a = p_1$.

Step 7. *Set* $a_1 = a$ *and* $b_1 = b$

Stop.

Otuput. a_1 *and* b_1.

Remark 2.1. Because f has a simple root in $[a,b]$, therefore there exists $[a_1,b_1]$ such that f is one to one in this interval. Therefore the above algorithm could be ended after finite number of iterations.

If we know that multiplicity of f is m then we can use $g(x) = \frac{f(x)}{(x-c)^{m-1}}$, instead of $f(x)$ in the presented algorithm and if f be a function with multiplicity m in c where m is unknown, we can use $G(x) = \begin{cases} \frac{f(x)}{f'(x)} & x \neq 0 \\ 0 & x = 0 \end{cases}$ instead of f in the presented algorithm [13]. After finding $[a_1,b_1]$ using the above algorithm, we look for $c \in [a_1,b_1]$ such that $c = f^{-1}(0)$. Suppose

$$y(x) = f^{-1}(x), \quad x \in [f(a_1), f(b_1)], \tag{2.1}$$

therefore we have

$$f(y(x)) = x, \quad x \in [f(a_1), f(b_1)], \tag{2.2}$$

and it is easy to see that

$$y'(x) = \frac{1}{f'(y(x))}, \quad x \in [f(a_1), f(b_1)]. \tag{2.3}$$

Now we consider the following boundary value problem:

$$\begin{aligned} &y'(x) = \tfrac{1}{f'(y(x))}, \\ &y(f(a_1)) = a_1, \\ &y(f(b_1)) = b_1. \end{aligned} \tag{2.4}$$

We can check that solving (2.4) is equivalent to finding $y(x)$ such that satisfies in (2.1). Solving this equation and finding $y(x)$, we can determine c.

Lemma 2.2. *Suppose S_1 and S_2 are the sets of the solutions of the following equations respectively*

$$\begin{cases} y(x) = f^{-1}(x), \\ y\left(f(a_1)\right) = a_1, \\ y\left(f(b_1)\right) = b_1, \end{cases} \tag{2.5}$$

and

$$\begin{cases} y'(x) = \frac{1}{f'(y(x))}, \\ y(f(a_1)) = a_1, \\ y(f(b_1)) = b_1. \end{cases} \tag{2.6}$$

Therefore we conclude $S_1 = S_2$.

Proof. Suppose $y_1(x) \in S_1$, therefore y_1 satisfies in (17). It is easy to see that

$$\begin{cases} f\left(y_1(x)\right) = x, \\ y_1\left(f(a_1)\right) = a_1, \\ y_1\left(f(b_1)\right) = b_1, \end{cases} \tag{2.7}$$

therefore we conclude

$$\begin{cases} {y'}_1(x)f'(y_1(x)) = 1 \\ y_1\left(f(a_1)\right) = a_1, \\ y_1\left(f(b_1)\right) = b_1. \end{cases} \tag{2.8}$$

That is y_1 satisfy in (18), therefore we conclude $S_1 \subset S_2$. Now suppose $y_2(x) \in S_2$ is the solution of (18). Therefore we have

$$\int\limits_{f(a_1)}^{x} {y'}_2(t)f'(y_2(t))dt = \int\limits_{f(a_1)}^{x} 1dt. \tag{2.9}$$

Using (2.9) yields

$$\int\limits_{f(a_1)}^{x} (f(y_2(t)))'dt = x - f(a_1). \tag{2.10}$$

Finally one can conclude:

$$\begin{cases} f\left(y_2(x)\right) = x, \\ y_2\left(f(a_1)\right) = a_1, \\ y_2\left(f(b_1)\right) = b_1, \end{cases} \tag{2.11}$$

That is y_2 satisfy in (17), therefore we conclude $S_2 \subset S_1$. □

2.1. **The Jacobi Collocation method for solving (2.4).** Collocation method is a powerful tool for solving ordinary differential equations. In this section we use the Jacobi collocation method for solving (2.4). Suppose

$$y_N(x) = \sum_{i=0}^{N} c_i P_i^{(\alpha,\beta)}(x), \tag{2.12}$$

where c_i is unknown for $i = 0, 1, ..., N$. Let us define $\frac{d^k}{dx^k} P_i^{(\alpha,\beta)}(x) = P_i^{(k)}(x)$ for $k = 0, 1, ..., N$. Now it is clear that

$$y_N(x) = P(x)C^T, \tag{2.13}$$

$$y'_N(x) = P'(x)C^T, \tag{2.14}$$

where

$$P(x) = [P_0(x), P_1(x), ..., P_N(x)], \tag{2.15}$$

$$P'(x) = [P'_0(x), P'_1(x), ..., P'_N(x)], \tag{2.16}$$

$$C = [c_0, c_1, ..., c_N]. \tag{2.17}$$

The relation between the matrix $P(x)$ and it's derivative, $P'(x)$, is :

$$P'(x) = P(x)D, \tag{2.18}$$

where D is the operational matrix of derivative that is defined in Subsection 1.3. Therefore we can write

$$y'_N(x) = P(x)DC^T. \tag{2.19}$$

Suppose

$$Z(y_N(x)) = \frac{1}{f'(y_N(x))}. \tag{2.20}$$

Using (2.19) we conclude:

$$\begin{cases} P(x)DC^T - Z(P(x)C^T) \simeq 0, \\ P(f(a_1))C^T \simeq a_1, \\ P(f(b_1))C^T \simeq b_1. \end{cases} \tag{2.21}$$

Now we use the collocation method for solving (2.21). Suppose $\{x_i\}_{i=0}^N$ is the set of $(N+1)$ Jacobi-Gauss or Jacobi-Gauss-Radau or Jacobi-Gauss-Lobatto quadrature nodes [2]. We substitute these nodes in (2.21), therefore we have the following system of nonlinear equations:

$$\begin{cases} P(f(a_1))C^T = a_1, \\ P(x_k)DC^T - Z(P(x_k)C^T) = 0, \quad k = 1, 2, ..., N-1, \\ P(f(b_1))C^T = b_1. \end{cases} \tag{2.22}$$

This system can be solved by Newton's iterative method [4].

2.1.1. *Convergence analysis.* In this section we prove the convergence of the Jacobi collocation method that is presented in Section 2.2. Before proving the main theorem of this section, we present some preliminaries and notations.

Definition 2.3. Suppose $I = (a, b)$ and $L^2_{\omega^{\alpha,\beta}}(I)$ is the space of square integrable functions in I. Now we can define the following inner product and norm on $L^2_{\omega^{\alpha,\beta}}(I)$:

$$(u, v)_{\omega^{\alpha,\beta},I} = \int_a^b \omega^{\alpha,\beta}(x)u(x)v(x)dx, \quad \forall u, v \in L^2_{\omega^{\alpha,\beta}}(I),$$

$$\|u\|_{\omega^{\alpha,\beta},I} = \left(\int_a^b \omega^{\alpha,\beta}(x)(u(x))^2dx\right)^{\frac{1}{2}}, \quad \forall u \in L^2_{\omega^{\alpha,\beta}}(I).$$

For any u, v continuous on $[a,b]$, we set

$$(u,v)_N = \sum_{i=0}^{N} u(x_i)v(x_i)w_i,$$

where x_j, $(0 \leq j \leq N)$, are the Gauss- Jacobi or Gauss-Radau or Gauss-Lobatto quadrature nodes and w_j, $(0 \leq j \leq N)$, are the Gauss- Jacobi or Gauss-Radau or Gauss- Lobatto quadrature weights. The Gauss integration formulas imply that:

$$(u,v)_{N,\omega^{\alpha,\beta}} = (u,v)_{\omega^{\alpha,\beta}}, \quad if \quad uv \in \mathbb{P}_{2N+\delta},$$

where $\delta = 1, 0, -1$ for Gauss, Gauss-Radau or Gauss-Lobatto integration rules respectively.

Definition 2.4. [2] Suppose $I = (a,b)$, therefore we define:

$$H^k_{\omega^{\alpha,\beta}}(I) = \left\{ u | \, \partial^l_x u \in L^2_{\omega^{\alpha,\beta}}(I), \ \ 0 \leq l \leq k \right\},$$

where $\partial^l_x u = \frac{\partial^l u}{\partial x^l}$. $H^k_{\omega^{\alpha,\beta}}(I)$ is a Hilbert space with respect to the inner product :

$$(u,v)_{k,\omega^{\alpha,\beta},I} = \sum_{m=0}^{k} \left(\partial^m_x u, \partial^m_x v\right)_{\omega^{\alpha,\beta},I},$$

which induces the norm:

$$\|u\|_{k,\omega^{\alpha,\beta},I} = \left(\sum_{j=0}^{k} \left\| \partial^j_x u \right\|^2_{\omega^{\alpha,\beta},I} \right)^{\frac{1}{2}},$$

also it is easy to see that :

$$\|\partial^m_x u\|_{\omega^{\alpha,\beta},I} \leq \|u\|_{k,\omega^{\alpha,\beta},I}, \quad 0 \leq m \leq k. \tag{2.23}$$

Definition 2.5. [20] We define:

$$H^k_{\omega^{\alpha,\beta},*}(I) = \left\{ u | \, \partial^j_x u \in L^2_{\omega^{\alpha+j,\beta+j}}(I), \ \ 0 \leq j \leq k, \quad k \in \mathbb{N} \right\},$$

where $\partial^j_x u = \frac{\partial^j u}{\partial x^j}$. $H^k_{\omega^{\alpha,\beta},*}(I)$ is a Hilbert space with respect to the inner product

$$(u,v)_{k,\omega^{\alpha,\beta},*} = \sum_{j=0}^{k} \left(\partial^j_x u, \partial^j_x v\right)_{\omega^{\alpha+j,\beta+j}},$$

which induces the norm:

$$\|u\|_{k,\omega^{\alpha,\beta},*} = \left(\sum_{j=0}^{k} \left\| \partial^j_x u \right\|^2_{\omega^{\alpha+j,\beta+j}} \right)^{\frac{1}{2}}.$$

Definition 2.6. [2] Suppose $\mathbb{P}_N$ is the space of all polynomials of degree at most N. $\Pi_{N,\omega^{\alpha,\beta}} : L^2_{\omega^{\alpha,\beta}}(I) \to \mathbb{P}_N$ is an orthogonal projection if and only if for any $u \in L^2_{\omega^{\alpha,\beta}}(I)$, we have :

$$\left(\Pi_{N,\omega^{\alpha,\beta}}(u(x)) - u(x), v(x)\right)_{\omega^{\alpha,\beta},I} = 0, \quad \forall v \in \mathbb{P}_N. \tag{2.24}$$

Lemma 2.7. *(The Poincare Inequality) Suppose $u \in H^1_{\omega^{\alpha,\beta}}(I)$ and there exists a point $x_0 \in [a,b]$, $u(x_0) = 0$ therefore we have:*

$$\|u\|_{\omega^{\alpha,\beta},I} \leq C\|\partial_x u\|_{\omega^{\alpha,\beta},I}, \tag{2.25}$$

where C is a constant.

Proof. See [2]. □

Lemma 2.8. *Suppose $f \in H^1_{\omega^{0,0}}(I)$, therefore we conclude:*

$$\Pi_{N-1,\omega^{0,0}}\left(\frac{d}{dx}f\right) = \left(\Pi_{N,\omega^{0,0}}(f)\right)'. \tag{2.26}$$

Proof. See [8]. □

Definition 2.9. [2] Suppose $I = (c_1, c_2)$, we define $\Pi_{1,N} : H^1_{\omega^{0,0}}(I) \to \mathbb{P}_N$ such that

$$\Pi_{1,N}u(x) = u(c_1) + \int_{c_1}^{x} \Pi_{N-1,\omega^{0,0}}\left(\frac{d}{dx}u\right)dx, \tag{2.27}$$

where $\Pi_{N,\omega^{0,0}}$ are defined in (2.24). Using (2.26) we conclude:

$$\Pi_{1,N}y(c_1) = y(c_1),$$

$$\Pi_{1,N}y(c_2) = y(c_2).$$

Lemma 2.10. *Suppose $y \in H^\sigma_{\omega^{0,0}}(I)$, $\sigma \geq 1$, therefore there exists a constant C independent of N such that:*

$$\|\Pi_{1,N}y - y\|_{l,\omega^{0,0},I} \leq CN^{l-\sigma}\|y\|_{\sigma,\omega^{0,0},I}, \quad l = 0, 1. \tag{2.28}$$

Proof. See [11]. □

Lemma 2.11. *Suppose $I = (a,b)$ and x_j, $(0 \leq j \leq N)$, be the Gauss-Jacobi or Gauss-Radau or Gauss-Lobatto quadrature nodes and w_j, $(0 \leq j \leq N)$, be the Gauss-Jacobi or Gauss-Radau or Gauss-Lobatto quadrature weights. The Jacobi interpolating polynomial is denoted by $I^{\alpha,\beta}_N(u)$. For any $u \in H^k_{\omega^{\alpha,\beta}}(I)$, $k \geq 1$, there exists a constant C_2 independent of N, such that:*

$$\left\|u - I^{\alpha,\beta}_N(u)\right\|_{\omega^{\alpha,\beta}} \leq C_2 N^{-k}\|u\|_{k,\omega^{\alpha,\beta}}, \tag{2.29}$$

and for any $u \in H^m_{\omega^{\alpha,\beta},*}(I)$, $m > k$, there exist constants C_3 and C_4 independent of N, such that:

$$\left\|\partial^k_x\left(u - \Pi_{N,\omega^{\alpha,\beta}}(u)\right)\right\|_{\omega^{\alpha+k,\beta+k}} \leq C_3 N^{k-m}\|\partial^m_x u\|_{\omega^{\alpha+m,\beta+m}}, \tag{2.30}$$

$$\left\|\partial_x^k\left(u-I_N^{\alpha,\beta}(u)\right)\right\|_{\omega^{\alpha+k,\beta+k}}\leq C_4N^{k-m}\|\partial_x^m u\|_{\omega^{\alpha+m,\beta+m}}. \tag{2.31}$$

Also for any $u\in H^m_{\omega^{\alpha,\beta}}(I)$, $1\leq k\leq m$, there exist constants C_5 and C_6 independent of N, such that:

$$\left\|u-\Pi_{N,\omega^{0,0}}(u)\right\|_{k,\omega^{0,0}}\leq C_5N^{2k-\frac{1}{2}-m}\|u\|_{m,\omega^{0,0}},$$

$$\left\|u-I_N^{0,0}(u)\right\|_{k,\omega^{0,0}}\leq C_6N^{2k-\frac{1}{2}-m}\|u\|_{m,\omega^{0,0}}.$$

Proof. See [20, 2, 9]. □

Theorem 2.12. *Suppose y and y_N are the solutions of* (2.4) *and* (2.22) *respectively. Assume that $y\in H^r_{\omega^{0,0},I}$. Also suppose $Z\in H^r_{\omega^{0,0},I,*}$, $r>1$ which is defined in* (2.20) *satisfy in the following relation:*

$$\left\|\frac{d^m}{dx^m}Z(f_1)-\frac{d^m}{dx^m}Z(f_2)\right\|_{\omega^{m,m},I}\leq K_m\|f_1-f_2\|_{\omega^{m,m},I},\ \ \forall f_1,f_2\in L^2_{\omega^{m,m}}(I),\ \ m=0,1,...,r, \tag{2.32}$$

where $k_m<1$. Suppose $\{x_k\}_{k=0}^N$ and $\{\omega_k\}_{k=0}^N$ are the Legendre-Gauss-Lobatto nodes and weights, respectively. Therefore we conclude:

$$\|y-y_N\|_{\omega^{0,0},I}\leq\lambda_0N^{-r}\|y\|_{r,\omega^{0,0},I}+\left(\lambda_1N^{1-r}+\lambda_2N^{-2r}\right)\|y\|_{\omega^{0,0},I}, \tag{2.33}$$

where λ_0, λ_1 and λ_2 are constants independent of N.

Proof. Suppose $e_N(x)=y_N(x)-\Pi_{1,N}y(x)$ where $\Pi_{1,N}$ is defined in (2.27). We know that y_N satisfies in the following equation:

$$(y_N(x_i)-\Pi_{1,N}y(x_i))'=Z_N(y_N(x_i))+(\Pi_{1,N}y(x_i))',\quad i=1,2,...,N-1. \tag{2.34}$$

It is easy to see that $e_N(c_1)=0$ and $e_N(c_2)=0$ where $c_1=f(a_1)$ and $c_2=f(b_1)$. Therefore we can write:

$$(y_N(x_i)-\Pi_{1,N}y(x_i))'e'_N(x_i)=Z(y_N(x_i))e'_N(x_i)+(\Pi_{1,N}y(x_i))'e'_N(x_i),\quad i=0,1,2,...,N. \tag{2.35}$$

Multiplying both sides of (2.35), in ω_k, $0\leq k\leq N$ and summing up these equations we conclude:

$$(e'_N,e'_N)_{N,\omega^{0,0}}=(Z(y_N),e'_N)_{N,\omega^{0,0}}+\left((\Pi_{1,N}y)',e'_N\right)_{N,\omega^{0,0}}. \tag{2.36}$$

We know that the N-point Legendre-Gauss-Lobato quadrature is exact for all polynomials of degree at most $2N-1$, therefore we have:

$$(e'_N,e'_N)_{\omega^{0,0},I}=(Z(y_N),e'_N)_{N,\omega^{0,0}}+\left((\Pi_{1,N}y)',e'_N\right)_{\omega^{0,0},I}. \tag{2.37}$$

Multiplying both sides of (2.4) in $e'_N(x)\omega^{0,0}(x)$ and integrating over I we conclude:

$$(y',e'_N)_{\omega^{0,0},I}=(Z(y),e'_N)_{\omega^{0,0},I}. \tag{2.38}$$

Subtracting (2.38) from (2.37) yields:

$$(e'_N, e'_N)_{\omega^{0,0},I} = (Z(\Pi_{1,N}y) - Z(y_N), e'_N)_{\omega^{0,0},I} + \left((\Pi_{1,N}y)' - y', e'_N\right)_{\omega^{0,0},I} +$$
$$(Z(y) - Z(\Pi_{1,N}y), e'_N)_{\omega^{0,0},I} + (Z(y_N),, e'_N)_{\omega^{0,0},I} - (Z(y_N),, e'_N)_{N,\omega^{0,0}}. \tag{2.39}$$

It is easy to see that:

$$\|e'_N\|^2_{\omega^{0,0},I} \le \|Z(y_N) - Z(y)\|_{\omega^{0,0},I}\|e'_N\|_{\omega^{0,0},I} + \left\|(\Pi_{1,N}y)' - y'\right\|_{\omega^{0,0},I}\|e'_N\|_{\omega^{0,0},I} +$$
$$\|Z(\Pi_{1,N}y) - Z(y)\|_{\omega^{0,0},I}\|e'_N\|_{\omega^{0,0},I} + \|I_N\left(Z(y_N)\right) - Z(y_N)\|_{\omega^{0,0},I}\|e'_N\|_{\omega^{0,0},I}. \tag{2.40}$$

Therefore we can write:

$$\|e'_N\|_{\omega^{0,0},I} \le \|Z(y_N) - Z(\Pi_{1,N}y)\|_{\omega^{0,0},I} + \|Z(\Pi_{1,N}y) - Z(y)\|_{\omega^{0,0},I} +$$
$$\|I_N\left(Z(y_N)\right) - Z(y_N)\|_{\omega^{0,0},I} + + \left\|(\Pi_{1,N}y)' - y'\right\|_{\omega^{0,0},I}. \tag{2.41}$$

Using (2.32) and Lemma 3 and (2.23) respectively we conclude:

$$\|Z(y_N) - Z(y)\|_{\omega^{0,0},I} \le K_0\|e_N\|_{\omega^{0,0},I}, \tag{2.42}$$

and

$$\|Z(\Pi_{1,N}y) - Z(y)\|_{\omega^{0,0},I} \le K_0\|\Pi_{1,N}y - y\|_{\omega^{0,0},I} \le C_1N^{-r}\|y\|_{r,\omega^{0,0},I}, \tag{2.43}$$

and

$$\left\|(\Pi_{1,N}y)' - y'\right\|_{\omega^{0,0},I} \le \|\Pi_{1,N}y - y\|_{1,\omega^{0,0},I} \le C_2N^{1-r}\|y\|_{1,\omega^{0,0},I}. \tag{2.44}$$

Now for simplicity we use $I_N(f)$ instead of $I_N^{0,0}(f)$, therefore employing Lemma 4, it can be seen

$$\|I_N\left(Z(y_N)\right) - Z(y_N)\|_{\omega^{0,0},I} \le C_3N^{-r}\left\|\tfrac{d^r}{dx^r}Z(y_N)\right\|_{\omega^{r,r},I} \le$$
$$C_3N^{-r}\left(\left\|\tfrac{d^r}{dx^r}Z(y_N) - \tfrac{d^r}{dx^r}Z(\Pi_{1,N}y)\right\|_{\omega^{r,r},I} + \left\|\tfrac{d^r}{dx^r}Z(y)\right\|_{\omega^{r,r},I} + \tag{2.45}$$
$$\left\|\tfrac{d^m}{dx^m}Z(y) - \tfrac{d^m}{dx^m}Z(\Pi_{1,N}y)\right\|_{\omega^{m,m},I}\right).$$

We know that Z satisfies in (2.32), therefore we conclude:

$$\left\|\tfrac{d^r}{dx^r}Z(y_N) - \tfrac{d^r}{dx^r}Z(\Pi_{1,N}y)\right\|_{\omega^{r,r},I} \le L_r\|y_N - \Pi_{1,N}y\|_{\omega^{r,r},I} \le$$
$$C_4\|e_N\|_{\omega^{0,0},I}. \tag{2.46}$$

Also using (2.32) and Lemma 3 yield

$$\left\|\tfrac{d^r}{dx^r}Z(y) - \tfrac{d^r}{dx^r}Z(\Pi_{1,N}y)\right\|_{\omega^{r,r},I} \le L_r\|y - \Pi_{1,N}y\|_{\omega^{r,r},I} \le$$
$$C_5N^{-r}\|y\|_{\omega^{0,0},I}. \tag{2.47}$$

Therefore substituting (2.46) and (2.47) in (2.45) we have

$$\|I_N(Z(y_N)) - Z(y_N)\|_{\omega^{0,0},I} \le C_6 N^{-r} \left\| \frac{d^r}{dx^r} Z(y) \right\|_{\omega^{r,r},I} + C_7 N^{-r} \|e_N\|_{\omega^{0,0},I} + C_8 N^{-2r} \|y\|_{\omega^{0,0},I}. \tag{2.48}$$

Now using Lemma 1 we can write:

$$\|e_N\|_{\omega^{0,0},I} \le C_9 \|e'_N\|_{\omega^{0,0},I}. \tag{2.49}$$

By substituting (2.42), (2.43), (2.44), (2.48) and (2.49) in (2.41) we have:

$$\|e_N\|_{\omega^{0,0},I} \le \left(C_{10} N^{1-r} + C_{11} N^{-2r}\right) \|y\|_{\omega^{0,0},I}. \tag{2.50}$$

We know that :

$$\|y - y_N\|_{\omega^{0,0},I} = \|y - y_N \pm \Pi_{1,N} y\|_{\omega^{0,0},I} \le \|y - \Pi_{1,N} y\|_{\omega^{0,0},I} + \|y_N - \Pi_{1,N} y\|_{\omega^{0,0},I}. \tag{2.51}$$

Therefore using Lemma 3 and (2.50) we conclude:

$$\|y - y_N\|_{\omega^{0,0},I} \le \lambda_0 N^{-r} \|y\|_{r,\omega^{0,0},I} + \left(\lambda_1 N^{1-r} + \lambda_2 N^{-2r}\right) \|y\|_{\omega^{0,0},I}. \tag{2.52}$$

□

3. Numerical results

In this section we present some examples and solve them using presented algorithm.

Example 3.1. *We want to find roots of* [14]

$$f(x) = \left(x - \frac{\pi}{3} e^{\frac{\pi}{3} - x}\right)^3 \sin^2\left(\frac{x}{2} - \frac{\pi}{6}\right), \quad x \in [0, 2], \tag{3.1}$$

the root of this function is $\frac{\pi}{3}$ of multiplicity $m = 5$. In Table 1 the numerical results of finding root of this function using the presented method is shown. $E(\alpha, \beta)$ is the absolute error of finding root of f using the method presented in Section 2 with the parameters (α, β).

Table 1: The absolute error for roots of f in Example 1 using presented method.

Number of iterations	E(-0.8,-0.4)	E(-0.2,0.8)	E(-0.8,0.5)	E(0.5,0.5)	E(0.5,0.8)
4	1.7400(-1)	5.3163(-2)	9.8963(-2)	8.5130(-2)	9.3013(-2)
6	5.2701(-2)	9.7430(-3)	7.2306(-2)	2.9246(-2)	7.3941(-2)
8	3.2859(-2)	7.3820(-3)	9.173(-3)	9.1403(-3)	1.8316(-2)
10	9.45029(-3)	1.8361(-3)	7.9174(-3)	8.062(-3)	7.9409(-3)
12	6.3051(-3)	9.0418(-4)	8.6290(-4)	6.6273(-4)	2.9163(-3)

Example 3.2. *In this example, we want to find roots of* [17]

$$f(x) = (x-1)^2 tan(\frac{\pi x}{4}), \quad 0 < x < 2. \tag{3.2}$$

The root of this function is $x = 1$ of multiplicity $m = 2$. Table 2 shows the numerical solution of this equation using presented method. $E(\alpha, \beta)$ is the absolute error of finding root of f using the method is presented Section 2 with the parameters (α, β).

Table 2: The absolute error for roots of f in Example 2 using method is presented in this paper.

Number of iterations	E(-0.5,0.5)	E(-0.5,0)	E(0,0)	E(0,0.5)	E(0.5,0.5)
5	8.0030(-3)	9.0915(-3)	4.7631(-3)	6.4298(-3)	8.2065(-3)
7	7.6249(-3)	7.4056(-3)	4.6643(-3)	3.2908(-3)	5.3906(-3)
9	3.9501(-3)	3.0351(-3)	8.5920(-4)	1.7928(-3)	6.1649(-3)
11	9.1643(-4)	6.8120(-4)	4.9159(-4)	8.6418(-4)	5.9041(-4)
13	6.2875(-4)	4.8252(-4)	2.9517(-4)	7.1849(-4)	6.0269(-4)

4. Conclusion

In this paper we presented a method for finding roots of continuous function f. In this method we find an interval $[a_1, b_1]$, that f is one to one on it. Therefore we can construct a boundary value equation on $[a_1, b_1]$, where f^{-1} is the solution of it. Solving this equation using Jacobi collocation method we can find roots of function f. We also studied convergence analysis of the new method. Finally some examples are presented to show the efficiency of the new method.

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