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$(\odot, \oplus)$-Derivations and $(\ominus, \odot)$-Derivations on MV-algebras

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Abstract. In this paper, we introduce the notions of $(\odot, \oplus)$-derivations and $(\ominus, \odot)$-derivations for MV-algebras and discuss some related results. We study the connection between these derivations on an MV-algebra A and the derivations on its boolean center. We characterize the isotone $(\odot, \oplus)$-derivations and prove that $(\ominus, \odot)$-derivations are isotone. Finally we determine the relationship between $(\odot, \oplus)$-derivation and $(\ominus, \odot)$-derivation for MV-algebras.

Keywords: MV- algebra, Derivation, Boolean algebra, Fix point, Ideal.

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1. Introduction

The concept of an MV-algebra was introduced by C.C. Chang in [7] to prove the completeness theorem for the Łukasiewicz calculus. The properties of an MV-algebra were presented in [5], [8], [9],[15], [4]. The notion of derivation, introduced from the analytic theory, is helpful for the research of structures

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and properties in algebraic systems. Several authors [1], [2], [3], [14], [16] have studied derivations in rings and near rings. Also Jun and Xin [13] and J. Zhan and Y. Lin Liu [19] applied the notion of derivation in rings and near rings theory to BCI-algebras (some properties of BCI-algebra are in [12]). Szász introduced the concept of derivation on lattices in [17]. Recently some authors [6], [10] and [18] studied the properties of derivations for lattices.
In this paper, we define and study two kinds of derivations on MV-algebras which comes in analogy with Leibniz's formula for derivations in rings. The paper is organized as follows.
In section 2, the basic definitions and results are summarized. In section 3, we introduce $(\odot, \oplus)$-derivations on MV-algebras and study their properties. We show they are not isotone in general. Some conditions are obtained such that $(\odot, \oplus)$-derivations are isotone. In section 4, we introduce the notion of $(\ominus, \odot)$-derivation on MV-algebras and investigate some of its properties. Moreover, this derivation is isotone. We show that if d is a $(\ominus, \odot)$-derivation on an MV-algebra A, then the collection of fix points of this derivation($Fix_d(A)$) is an ideal of $B(A)$, the boolean center of A. In particular, we prove that if $d(1) = 1$, then $Fix_d(A) = B(A) = A$. Finally, we obtain some relations between these derivations.

2. Preliminaries

Definition 2.1. [7] An MV-algebra is an algebra $(A, \oplus, ^*, 0)$ of type $(2, 1, 0)$ satisfying the following equations:
$(MV1)$ $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,
$(MV2)$ $x \oplus y = y \oplus x$,
$(MV3)$ $x \oplus 0 = x$,
$(MV4)$ $x^{**} = x$,
$(MV5)$ $x \oplus 0^* = 0^*$,
$(MV6)$ $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$,
for all $x, y, z \in A$.

Example 2.2. $[7, 9]$ Consider the real unit interval $[0, 1]$ and for all $x, y \in [0, 1]$, define $x \oplus y = min\{1, x + y\}$ and $x^* = 1 - x$. Then $([0, 1], \oplus, ^*, 0)$ is an MV-algebra. The rational numbers in $[0, 1]$ and the n-element set $L_n = \{0, 1/(n-1), ..., (n-2)/(n-1), 1\}$, for each integer $n \geq 2$, yield examples of subalgebras of $[0, 1]$.

In the rest of this paper, we denote an MV-algebra $(A, \oplus, ^*, 0)$ by A.

On each MV-algebra A, we define the constant 1 and the operations $\odot$ and $\ominus$ as follows: $1 = 0^*$, $x \odot y = (x^* \oplus y^*)^*$, $x \ominus y = x \odot y^*$.

For any two elements x and y of A, define $x \leq y$ if and only if $x \odot y^* = 0$. Then $\leq$ is a partial order, called the natural order of A.

Proposition 2.3. [9] *On an MV-algebra A, the natural order determines a bounded distributive lattice structure. Specifically, the join $x \vee y$ and the meet $x \wedge y$ of the elements x and y are given by:*

$$x \vee y = (x \ominus y) \oplus y = (x \odot y^*) \oplus y$$

and

$$x \wedge y = (x^* \vee y^*)^* = x \odot (x^* \oplus y) = x \ominus (x \ominus y).$$

We denote this distributive lattice with 0 and 1 by $L(A)$.

Theorem 2.4. [5, 9] *Let A be an MV-algebra and $x, y, z \in A$. Then the following hold:*

(c_1) $x \oplus x^* = 1$, $x \odot x^* = 0$,
(c_2) $x \ominus 0 = x$, $0 \ominus x = 0$, $x \ominus x = 0$, $1 \ominus x = x^*$, $x \ominus 1 = 0$,
(c_3) $x \leq y$ *if and only if* $y^* \leq x^*$,
(c_4) *If* $x \leq y$, *then* $x \oplus z \leq y \oplus z$ *and* $x \odot z \leq y \odot z$,
(c_5) $x \odot y \leq x \wedge y \leq x, y \leq x \vee y \leq x \oplus y$,
(c_6) $x \oplus (y \wedge z) = (x \oplus y) \wedge (x \oplus z)$,
(c_7) $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$,
(c_8) $x \oplus y = y$ *if and only if* $x \wedge y^* = 0$,
(c_9) *If* $x \odot y = x \odot z$ *and* $x \oplus y = x \oplus z$, *then* $y = z$,
for all $x, y, z \in A$.

For any MV-algebra A, write $B(A)$ as an abbreviation of set of all complemented elements of $L(A)$. Elements of $B(A)$ are called boolean center of A.

Theorem 2.5. [9] *For every element x in an MV-algebra A, the following conditions are equivalent:*

(i) $x \in B(A)$,
(ii) $x \vee x^* = 1$,
(iii) $x \wedge x^* = 0$,
(iv) $x \oplus x = x$,
(v) $x \odot x = x$,
(vi) $x \oplus y = x \vee y$, *for all* $y \in A$,
(vii) $x \odot y = x \wedge y$, *for all* $y \in A$.

Theorem 2.6. [5] *Let A be an MV-algebra, $a \in B(A)$ and $x, y \in A$. Then:*

(c_{10}) $a \wedge (x \oplus y) = (a \wedge x) \oplus (a \wedge y)$,
(c_{11}) $a \vee (x \oplus y) = (a \vee x) \oplus (a \vee y)$.

Definition 2.7. [9] An ideal of an MV-algebra A is a non-empty subset I of A satisfying the following conditions:

(I_1) If $x \in I$, $y \in A$ and $y \leq x$, then $y \in I$,
(I_2) If $x, y \in I$, then $x \oplus y \in I$.

Definition 2.8. [18] Let L be a lattice and $d : L \to L$ be a function. Then d is called a derivation on L, if $d(x \wedge y) = (d(x) \wedge y) \vee (x \wedge d(y))$.

3. $(\odot, \oplus)$-derivations

In this section, we define a $(\odot, \oplus)$-derivation on an MV-algebra A. Then we obtain some characterizations of a $(\odot, \oplus)$-derivation.

Definition 3.1. Let A be an MV-algebra and $d : A \to A$ be a function. We call d a $(\odot, \oplus)$-derivation on A, if it satisfies the following condition:

$$d(x \odot y) = (d(x) \odot y) \oplus (x \odot d(y))$$

for all $x, y \in A$.

Now we give some examples and present some properties of $(\odot, \oplus)$-derivations on MV-algebras.

Example 3.2. (1) Let A be an MV-algebra. We define a function $d : A \to A$ by $d(x) = 0$, for all $x \in A$. Then d is a $(\odot, \oplus)$-derivation on A, which is called the zero $(\odot, \oplus)$-derivation.

(2) Consider $L_3 = \{0, 1/2, 1\}$. We can see that identity function I on L_3 is not $(\odot, \oplus)$-derivation. But $d(0) = d(1) = 0$, and $d(1/2) = 1/2$ is a $(\odot, \oplus)$-derivation on L_3.

Proposition 3.3. *Let A be an MV-algebra and d be a $(\odot, \oplus)$-derivation on A. Then the following hold:*

(1) $d(0) = 0$,
(2) $d(x) \leq x$, *for all* $x \in A$,
(3) *if* $d(x) = 1$, *then* $x = 1$,
(4) $d(x^*) \leq (d(x))^*$,
(5) *if I is an ideal of A, then* $d(I) \subseteq I$,
(6) $d(x) \odot d(y) \leq d(x \odot y) \leq d(x) \oplus d(y) \leq x \oplus y$, *for all* $x, y \in A$
(7) $(d(x))^n \leq d(x^n)$, *for all* $n \geq 1$.

Proof. (1) Put $x = 0$ in Definition 3.1. Then we have

$$d(0) = d(0 \odot 0) = (d(0) \odot 0) \oplus (0 \odot d(0)) = 0.$$

(2) By part (1) and Definition 3.1, we get that

$$0 = d(0) = d(x \odot x^*) = (d(x) \odot x^*) \oplus (x \odot d(x^*)).$$

Then $d(x) \odot x^* = 0$ and $x \odot d(x^*) = 0$, and so $d(x) \leq x$.

(3) The proof is clear, by part (2).

(4) It can conclude by part (2) and (c_3).

(5) and (6) can be easily proved by part (2).

(7) The proof follows from part (6). □

In the following theorems, we will study the connection between the $(\odot, \oplus)$-derivations on an MV-algebra and the derivations on its boolean center. First, we will prove that the $(\odot, \oplus)$-derivations on the boolean center of an MV-algebra are lattice derivations.

Proposition 3.4. *Let A be an MV-algebra and d be a $(\odot, \oplus)$-derivation on A. Then*

(1) $d(B(A)) \subseteq B(A)$.

(2) $d(x \wedge y) = (d(x) \wedge y) \vee (x \wedge d(y))$, *for all* $x, y \in B(A)$.

Proof. (1) Let $x \in B(A)$. By Theorem 2.5, $d(x) = d(x \odot x) = (d(x) \odot x) \oplus (x \odot d(x))$. Since $x \in B(A)$ and $d(x) \leq x$, then $x \odot d(x) = x \wedge d(x) = d(x)$. Hence $d(x) = d(x) \oplus d(x)$. We obtain that $d(x) \in B(A)$.
(2) Let $x, y \in B(A)$. By Theorem 2.5 and (1),

$$d(x \wedge y) = d(x \odot y) = (d(x) \odot y) \oplus (x \odot d(y)) = (d(x) \wedge y) \vee (x \wedge d(y)).$$

□

By the above theorem part (2) we conclude that every $(\odot, \oplus)$-derivation on A is a $(\odot, \oplus)$-derivation on the boolean center of A.

Corollary 3.5. *Let A be an MV-algebra such that $B(A) = \{0, 1\}$ and d be a $(\odot, \oplus)$-derivation on A. Then $d(1) = 0$ or $d(1) = 1$.*

Proof. Since $B(A) = \{0, 1\}$, we have $d(1) \in B(A)$ by part (2) of the above proposition. Hence $d(1) = 0$ or $d(1) = 1$. □

In the following example , we will show that a $(\odot, \oplus)$-derivation is not a derivation for lattices in general.

Example 3.6. Consider $L_4 = \{0, 1/3, 2/3, 1\}$ and define a map d on L_4 by:

$$d(0) = d(1) = d(1/3) = 0, \ d(2/3) = 1/3.$$

Then d is a $(\odot, \oplus)$-derivation, and also $d(2/3 \wedge 1/3) = d(1/3) = 0$, $d(2/3) \wedge 1/3 = 1/3$ and $d(1/3) \wedge 2/3 = 0$. Thus

$$d(2/3 \wedge 1/3) \neq [d(2/3) \wedge 1/3] \vee [2/3 \wedge d(1/3)].$$

By Proposition 3.4, we can obtain a class of maps $d : A \to A$ on an MV-algebra A such that they are not $(\odot, \oplus)$-derivation on A. See the following proposition.

Proposition 3.7. *Let A be an MV-algebra and $a \in A - \{0, 1\}$. Then the function $d : A \to A$, by $d(x) = a$, for all $x \in A - \{0\}$ and $d(0) = 0$ is not a $(\odot, \oplus)$-derivation on A.*

Proof. Suppose that d is a $(\odot, \oplus)$-derivation on A. We have two cases:
case 1: $a \notin B(A)$. Since $d(1) \in B(A)$ by Proposition 3.4 part (1), then $a \in B(A)$, which is a contradiction.
case 2: $a \in B(A)$. Then

$$0 = d(a \odot a^*) = (d(a) \odot a^*) \oplus (a \odot d(a^*)) = (a \odot a^*) \oplus (a \odot a) = a^2 = a,$$

which is a contradiction. □

Lemma 3.8. *Let d be a $(\odot, \oplus)$-derivation on an MV-algebra A. Then $x \wedge d(1) \wedge (d(x))^* = 0$ and so $x \wedge d(1) \wedge x^* = 0$.*

Proof. Since $d(x) = d(x) \oplus (x \odot d(1))$, then by (c_8) of Theorem 2.4, we obtain

$$(x \odot d(1)) \wedge (d(x))^* = 0.$$

We have $d(1) \in B(A)$ by part (1) of Proposition 3.4, then

$$x \wedge d(1) \wedge (d(x))^* = 0.$$

Also $x^* \leq (d(x))^*$ implies that $x \wedge d(1) \wedge x^* = 0$. □

Proposition 3.9. *Let d be a $(\odot, \oplus)$-derivation on an MV-algebra A. Then the following hold:*
(1) *if $x \leq d(1)$, then $d(x) = x$ and $x \in B(A)$,*
(2) *if $d(1) \leq x$, then $d(1) \leq d(x)$,*
(3) *$d(d(1)) = d(1)$.*

Proof. (1) Let $x \leq d(1)$. Then by Lemma 3.8, we get that $x \wedge (d(x))^* = 0$ and so $x \leq d(x) \leq x$. Thus $d(x) = x$. Also by Lemma 3.8, $x \wedge x^* = 0$. Hence $x \in B(A)$, by Theorem 2.5.
(2) Let $d(1) \leq x$. Then by Lemma 3.8, $(d(x))^* \odot d(1) = 0$ and so $d(1) \leq d(x)$.
(3) The proof follows from part (1). □

Let A be an MV-algebra which is not a boolean-algebra. By the above theorem, we conclude that the map d on A where $d(1) = 1$ is not a $(\odot, \oplus)$-derivation on A. Also, we have the following corollary:

Corollary 3.10. *Let d be a $(\odot, \oplus)$-derivation on a boolean algebra A. Then $d(1) = 1$ if and only if d is identity function.*

Theorem 3.11. *Let A be an MV-algebra which is not a boolean algebra. Then there is no one to one or onto $(\odot, \oplus)$-derivation on A.*

Proof. Let d be a $(\odot, \oplus)$-derivation on A. If d is onto, then by Proposition 3.3 part (3), we get that $d(1) = 1$ which is a contradiction. If d is one to one, then by Proposition 3.9 part (3), $d(1) = 1$ which is a contradiction. □

Proposition 3.12. *Identity function I on an MV-algebra A is a $(\odot, \oplus)$-derivation if and only if A is a boolean algebra.*

Proof. Let identity function I on A be a $(\odot, \oplus)$-derivation. Then

$$x = I(x) = I(1 \odot x) = (I(1) \odot x) \oplus (I(x) \odot 1) = x \oplus x,$$

for all $x \in A$. Therefore $A = B(A)$. The proof of the converse is easy. □

Corollary 3.13. *Let d be a $(\odot, \oplus)$-derivation on a boolean algebra A. Then the following are equivalent:*
(a) *d is identity function,*
(b) *d is onto,*
(c) *d is one to one*

Proof. The proof follows from Proposition 3.3 part (3), Proposition 3.9 part (3), Proposition 3.12 and Corollary 3.10. □

Proposition 3.14. *Let $a \in A$ and $d_a : A \to A$ be defined by $d_a(x) = a \odot x$, for all $x \in A$. Then d_a is a $(\odot, \oplus)$-derivation if and only if $d_a(A) \subseteq B(A)$.*

Proof. Let d_a be a $(\odot, \oplus)$-derivation. Then

$$a \odot x = d_a(x) = d_a(x \odot 1) = (d_a(x) \odot 1) \oplus (d_a(1) \odot x) = (a \odot x) \oplus (a \odot x),$$

for all $x \in A$ that is $d_a(A) \subseteq B(A)$. Conversely, let $d_a(A) \subseteq B(A)$. Now we show that d_a is $(\odot, \oplus)$-derivation.

$$\begin{aligned} d_a(x \odot y) &= a \odot (x \odot y) \\ &= (a \odot (x \odot y)) \oplus (a \odot (x \odot y)) \\ &= (d_a(x) \odot y) \oplus (x \odot d_a(y)). \end{aligned}$$

□

By Proposition 3.14, we get the following corollary.

Corollary 3.15. *Let A be a boolean algebra and $a \in A$. Then $d_a(x) = x \odot a$ is a $(\odot, \oplus)$-derivation on A.*

Let $d : A \to A$ be a function. If $x \leq y$,then $d(x) \leq d(y)$, we call d an isotone function.

Consider the $(\odot, \oplus)$-derivation d in Example 3.6, we can see that d is not isotone, because $d(2/3) \nleq d(1)$.

Example 3.16. Let $A = \{0, a, b, c, e, 1\}$. Define

$\odot$	0	a	b	c	e	1
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	0	0	b	b
c	0	a	0	a	b	c
e	0	0	b	b	e	e
1	0	a	b	c	e	1

$*$	0	a	b	c	e	1
	1	e	c	b	a	0

Then A is an MV-algebra and there is an isomorphism between A and $L_2 \times L_3$. Define the function d on A by:

$$d(1) = d(a) = d(c) = a,\ d(0) = d(b) = d(e) = 0.$$

Then d is a $(\odot, \oplus)$-derivation and also d is isotone.

In the following theorem we obtain some equivalent conditions such that a $(\odot, \oplus)$-derivation is isotone.

Proposition 3.17. *Let d be a $(\odot, \oplus)$-derivation on an MV-algebra A. Then The following are equivalent:*
(0) *d is isotone,*
(1) $d(x) \leq d(1)$,
(2) $d(x) = d(1) \odot x$,
(3) $d(x \wedge y) = d(x) \wedge d(y)$,
(4) $d(x \vee y) = d(x) \vee d(y)$,
(5) $d(x \oplus y) = d(x) \oplus d(y)$,
(6) $d(x \odot y) = d(x) \odot d(y)$,
for all $x, y \in A$.

Proof. (0) $\Rightarrow$ (1) It is clear.
(1) $\Rightarrow$ (2) Since $d(x) \leq x$, then $d(1) \odot d(x) \leq x \odot d(1)$. Also $d(x) \leq d(1)$ and $d(1) \in B(A)$ imply that $d(x) \odot d(1) = d(x) \wedge d(1) = d(x)$. Thus

$$x \odot d(1) \leq d(x) \oplus (x \odot d(1)) = d(x) \leq x \odot d(1),$$

and so $d(x) = x \odot d(1)$, for all $x \in A$.
(2) $\Rightarrow$ (0) Let $x \leq y$. Then $x \odot d(1) \leq y \odot d(1)$, and so $d(x) \leq d(y)$.
(2) $\Rightarrow$ (3) By $d(1) \in B(A)$, we have

$$\begin{aligned} d(x \wedge y) &= d(1) \odot (x \wedge y) \\ &= d(1) \wedge (x \wedge y) \\ &= (d(1) \wedge x) \wedge (d(1) \wedge y) = d(x) \wedge d(y). \end{aligned}$$

(3) $\Rightarrow$ (0) Let $x \leq y$. Then $x \wedge y = x$, hence

$$d(x) = d(x \wedge y) = d(x) \wedge d(y),$$

it follows that $d(x) \leq d(y)$.

(2) $\Rightarrow$ (4) Since $L(A)$ is a distributive lattice and $d(1)\in B(A)$, then

$$\begin{aligned} d(x\vee y) &= d(1)\odot(x\vee y)\\ &= d(1)\wedge(x\vee y)\\ &= (d(1)\wedge x)\vee(d(1)\wedge y)=d(x)\vee d(y).\end{aligned}$$

(4) $\Rightarrow$ (0) The proof is similar to the proof of (3) $\Rightarrow$ (0).

(2) $\Rightarrow$ (5) By (c_{10}), we have $d(x\oplus y)=d(1)\odot(x\oplus y)=(d(1)\odot x)\oplus(d(1)\odot y)=d(x)\oplus d(y)$.

(5) $\Rightarrow$ (1) We have $d(1)=d(x\oplus 1)=d(x)\oplus d(1)$. So $d(x)\leq d(1)$, for all $x\in A$.

(2) $\Rightarrow$ (6)

$$\begin{aligned} d(x\odot y) &= d(1)\odot(x\odot y)\\ &= d(1)\odot d(1)\odot x\odot y\\ &= d(1)\odot x\odot d(1)\odot y=d(x)\odot d(y).\end{aligned}$$

(6) $\Rightarrow$ (1) $d(x)=d(x\odot 1)=d(x)\odot d(1)\leq d(1)$, for all $x\in A$. □

Theorem 3.18. *Let $d:A\to A$ be a function on an MV-algebra A. Then d is an isotone $(\odot,\oplus)$-derivation if and only if d is an isotone $(\wedge,\vee)$-derivation and $d(A)\subseteq B(A)$.*

Proof. It follows from Proposition 3.14, 3.17 and Theorem 3.18 in [16]. □

Remark 3.19. By the above theorem, we conclude that if d is an isotone $(\odot,\oplus)$-derivation, then d is a lattice derivation i.e.

$$d(x\wedge y)=(d(x)\wedge y)\vee(x\wedge d(y)),$$

but the converse is not true, even if d is an isotone. Consider L_4 and define $d(x)=x\wedge 2/3$, for all $x\in L_4$. Then d is an isotone lattice derivation, while it is not a $(\odot,\oplus)$-derivation, since $d(1)=2/3\notin B(A)$.

By proposition 3.17 and Corollary 3.5 we have the following:

Corollary 3.20. *The only isotone $(\odot,\oplus)$-derivation on L_n and $[0,1]$ is zero $(\odot,\oplus)$-derivation.*

Corollary 3.21. *Let d be a $(\odot,\oplus)$-derivation on an MV-algebra A. If d is isotone, then $d(d(x))=d(x)$, for all $x\in A$. Moreover $d(A)\subseteq B(A)$.*

The following example shows that the converse of the above corollary is not true in general.

Example 3.22. (1) Consider derivation d in Example 3.6. We can see that

$$d(d(2/3)) = d(1/3) = 0 \neq 1/3 = d(2/3).$$

Hence $d(d(2/3)) \neq d(2/3)$.

(2) Consider derivation d in Example 3.2 part (2). We get that $d(d(x)) = d(x)$, for all $x \in A$, but d is not isotone.

Let A be an MV-algebra and $d : A \to A$ be a function. Define

$$Fix_d(A) = \{x \in A : d(x) = x\}.$$

If d is an isotone $(\odot, \oplus)$-derivation on an MV-algebra A, then $d(A) \subseteq Fix_d(A)$.

Proposition 3.23. *Let d be a $(\odot, \oplus)$-derivation on an MV-algebra A. Then we have*

(1) *$d(1) \in Fix_d(A)$,*

(2) *if $x \leq y$ and $y \in Fix_d(A)$, then $x \in Fix_d(A)$,*

(3) *if d is isotone, then $Fix_d(A)$ is an ideal of A.*

Proof. (1) This part is clear by Proposition 3.9 part (3).

(2) Let $x \leq y$ and $d(y) = y$. Then

$$\begin{aligned} d(x) = d(x \wedge y) &= d((x \oplus y^*) \odot y) \\ &= (d(x \oplus y^*) \odot y) \oplus ((x \oplus y^*) \odot d(y)) \\ &= (d(x \oplus y^*) \odot y) \oplus ((x \oplus y^*) \odot y) \\ &= (d(x \oplus y^*) \odot y) \oplus (x \wedge y) \\ &= (d(x \oplus y^*) \odot y) \oplus x. \end{aligned}$$

Hence $x \leq d(x) \leq x$, and so $d(x) = x$.

(3) Let $x, y \in Fix_d(A)$. By Proposition 3.17 part (2), $x = d(x) = d(1) \odot x$ and $y = d(y) = d(1) \odot y$. Thus by Proposition 3.17 part (5), we get that $x \oplus y \in Fix_d(A)$. Therefore by part (2), $Fix_d(A)$ is an ideal of A. □

Theorem 3.24. *Let d_1 and d_2 be two isotone $(\odot, \oplus)$-derivations on an MV-algebra A. Then $d_1 = d_2$ if and only if $Fix_{d_1}(A) = Fix_{d_2}(A)$.*

Proof. Let $Fix_{d_1}(A) = Fix_{d_2}(A)$ and $x \in A$. By Corollary 3.21, $d_1(d_1(x)) = d_1(x)$ and $d_2(d_2(x)) = d_2(x)$. Then $d_1(x) \in Fix_{d_1}(A)$ and $d_2(x) \in Fix_{d_2}(A)$. So $d_2(d_1(x)) = d_1(x)$ and $d_1(d_2(x)) = d_2(x)$. By $d_1(x) \leq x$ and d_2 is isotone, we get that $d_1(x) = d_2(d_1(x)) \leq d_2(x)$. Similarly, $d_2(x) \leq d_1(x)$. Therefore, $d_1(x) = d_2(x)$, for all $x \in A$. The proof of the converse is trivial. □

4. $(\ominus,\odot)$-DERIVATIONS

Definition 4.1. Let A be an MV-algebra. A $(\ominus,\odot)$-derivation on A is a function $d : A \to A$ such that

$$d(x \ominus y) = (d(x) \ominus y) \odot (x \ominus d(y)),$$

for all $x, y \in A$.

Example 4.2. (i) Let A be an arbitrary MV-algebra. Define a function $d : A \to A$ by $d(x) = 0$, for all $x \in A$. Then d is a $(\ominus,\odot)$-derivation on A, which is called the zero $(\ominus,\odot)$-derivation.
(ii) Let A be MV-algebra L_n, for $n \geq 2$, in Example 2.2. Define function d on A by $d(1) = 1/(n-1)$ and $d(x) = 0$, for all $x \in L_n - \{1\}$. Then we can see that d is a $(\ominus,\odot)$-derivation on A.
(iii) Consider MV-algebra A in Example 3.16. Define function $d : A \to A$ by

$$d(0) = d(b) = 0,\ d(a) = d(c) = a,\ d(e) = b,\ d(1) = c.$$

Then d is a $(\ominus,\odot)$-derivations on MV-algebra A.

Proposition 4.3. *Let d be a $(\ominus,\odot)$-derivation on an MV-algebra A. Then the following hold:*
(1) $d(0) = 0$,
(2) $d(x) = d(x) \odot x$,
(3) $d(x) \leq x$,
(4) $d(x^*) \leq (d(x))^*$,
(5) $x \leq y$ *implies* $d(x) \leq d(y)$,
(6) $d(x) \leq d(1)$,
(7) $d(x) = d(1) \odot x \odot (d(x^*))^*$,
(8) *if* $d(x^*) = 0$, *then* $d(x) = d(1) \odot x$,
for all $x, y \in A$.

Proof. (1) $d(0) = d(0 \ominus 0) = (d(0) \ominus 0) \odot (0 \ominus d(0)) = 0$.
(2) Let $x \in A$. By Definition 4.1 and part (1), we have

$$d(x) = d(x \ominus 0) = (d(x) \ominus 0) \odot (x \ominus d(0)) = d(x) \odot x.$$

(3) It follows from part (2).
(4) Let $x \in A$. By Definition 4.1, we get that

$$d(x^*) = d(1 \ominus x) = (d(1) \ominus x) \odot (1 \ominus d(x)) \leq (d(x))^*.$$

(5) If $x \leq y$, then $x = x \wedge y = y \ominus (y \ominus x)$. So

$$\begin{aligned} d(x) &= d(y \ominus (y \ominus x)) \\ &= (d(y) \ominus (y \ominus x)) \odot (y \ominus d(y \ominus x)) \\ &\leq d(y) \ominus (y \ominus x) \leq d(y). \end{aligned}$$

(6) It follows from part (5).
(7) We have $d(x) = d(1 \ominus x^*) = (d(1) \ominus x^*) \odot (1 \ominus d(x^*)) = d(1) \odot x \odot (d(x^*))^*$.
(8) follows from (7). □

Proposition 4.4. *Let d be a $(\ominus, \odot)$-derivation on an MV-algebra A. Then*
(1)$Fix_d(A) \subseteq B(A)$,
(2) if $x \in B(A)$, then $d(x) = d(1) \odot x$,
for all $x, y \in A$.

Proof. (1) Let $x \in Fix_d(A)$. Since $d(x) = d(x) \odot x$, by part (2) of Proposition 4.3 and $d(x) = x$, we have $x \odot x = x$. Hence we get that $x \in B(A)$.
(2) By part (3) of Proposition 4.3, we have $x \leq (d(x^*))^*$. So

$$d(x) = d(1) \odot x \odot (d(x^*))^* = d(1) \odot x \wedge (d(x^*))^* = d(1) \odot x.$$

□

Remark 4.5. The converse of parts (1) and (2) of the above proposition is not true in general. Consider Example 4.2 part (iii), we have $e \in B(A)$, while $d(e) \neq e$. Also $d(b) = d(1) \odot b$, but $b \notin B(A)$.

By the above proposition we can conclude that if d is a $(\ominus, \odot)$-derivation on A such that $d(1) \in B(A)$, then d is a $(\ominus, \odot)$-derivation on the boolean center of A.

Corollary 4.6. *Identity function I on an MV-algebra A is a $(\ominus, \odot)$-derivation if and only if $A = B(A)$.*

Corollary 4.7. *Let d be a $(\ominus, \odot)$-derivation on an MV-algebra A and $x \in B(A)$. Then*
(1) $x \leq d(1)$ if and only if $d(x) = x$,
(2) $d(1) \leq x$ if and only if $d(x) = d(1)$.

Corollary 4.8. *Let d be a $(\ominus, \odot)$-derivation on an MV-algebra A. Then $d(B(A))$ is a sublattice of $(A, \wedge, \vee, 0)$.*

By Proposition 4.3 parts (2), (3) and (7) and Proposition 4.4 part (1), we can see that the only $(\ominus, \odot)$-derivations on L_n are zero $(\ominus, \odot)$-derivation and $(\ominus, \odot)$-derivation on Example 4.2 (ii). Also the only $(\ominus, \odot)$-derivation on $[0, 1]$ is zero $(\ominus, \odot)$-derivation.

Proposition 4.9. *Let d be a map on an MV-algebra A such that $d(x)=a\odot x$, for all $x\in A$ and $a\in A$. If $d(A)\subseteq B(A)$, then d is a $(\ominus,\odot)$-derivation.*

Proof. Let $x,y\in A$ be arbitrary. By Theorem 2.5 parts (vii) and (v) and Proposition 4.3 part (2),

$$\begin{aligned}(d(x)\ominus y)\odot(x\ominus d(y)) &= (d(x)\odot x)\odot(y^*\odot d(y)^*)\\ &= (d(x)\wedge x)\odot(y\vee d(y))^*\\ &= d(x)\odot y^*\\ &= a\odot x\odot y^*\\ &= a\odot(x\ominus y)=d(x\ominus y).\end{aligned}$$

Hence d is a $(\ominus,\odot)$-derivation. ☐

Theorem 4.10. *Let d be a $(\ominus,\odot)$-derivation on an MV-algebra A. Then $Fix_d(A)$ is an ideal of A.*

Proof. First, we will show that if $x\leq y$, $x\in A$ and $y\in Fix_d(A)$, then $x\in Fix_d(A)$. Since $d(y)=y$, then $y=d(1)\odot y$, by Proposition 4.4. By Proposition 4.3 parts (2) and (7) we have

$$x\odot d(1)\odot(d(x^*))^*=d(x)=x\odot d(x) \tag{4.1}$$

So

$$x\odot d(1)\odot(d(x^*))^*=x\odot x\odot d(1)\odot(d(x^*))^*$$

$$\Rightarrow x\odot y\odot d(1)\odot(d(x^*))^*=x\odot x\odot y\odot d(1)\odot(d(x^*))^*$$

$$\Rightarrow x\odot y\odot(d(x^*))^*=x\odot x\odot y\odot(d(x^*))^*,\quad \text{by } y=d(1)\odot y,$$

$$\Rightarrow x\odot(d(x^*))^*=x\odot x\odot(d(x^*))^*,\quad \text{by } x\leq y,\ y\in B(A)$$

Thus $x\odot(d(x^*))^*\leq x\odot x$. Also, since $x\leq(d(x^*))^*$, then we can get that $x\odot x\leq x\odot(d(x^*))^*$. Hence $x\odot(d(x^*))^*=x\odot x$. By hypothesis and (1), we have
$x\odot(d(x^*))^*=x\odot y\odot(d(x^*))^*=x\odot d(1)\odot y\odot((d(x^*))^*=x\odot y\odot d(x)=x\odot d(x)$.
Hence

$$x\odot x=x\odot d(x). \tag{4.2}$$

By Proposition 4.3 part (2) and (7) we have $x^*\odot d(1)\odot(d(x))^*=d(x^*)=x^*\odot d(x^*)$. Then similar to above we have $x^*\odot x^*\odot y\odot(d(x))^*=x^*\odot y\odot(d(x))^*$ and so

$$(x\oplus x\oplus d(x))^*\odot y=(x\oplus d(x))^*\odot y. \tag{4.3}$$

Since $x\leq y$ and $y\in B(A)$, then $x\oplus d(x)\leq x\oplus x\oplus d(x)\leq y$, so $(x\oplus x\oplus d(x))^*\oplus y=(x\oplus d(x))^*\oplus y$. Thus by (3) and (c_9), we get that $x\oplus x\oplus d(x)=x\oplus d(x)$

, similar to the above argument we can obtain that

$$x \oplus x = x \oplus d(x). \tag{4.4}$$

Therefore by (2), (4) and (c_9) we get that $d(x) = x$ and then $x \in Fix_d(A)$. Now let $x, y \in Fix_d(A)$. Then by Proposition 4.4, $x = d(1) \odot x$ and $y = d(1) \odot y$. Since $x, y \in B(A)$, we get that

$$d(x \vee y) = (x \vee y) \odot d(1) = (x \odot d(1)) \vee (y \odot d(1)) = x \vee y.$$

Therefore $x \oplus y = x \vee y \in Fix_d(A)$. □

Proposition 4.11. *Let d be a $(\ominus, \odot)$-derivation on an MV-algebra A. If d is one to one or onto, then d is identity and A is a boolean algebra.*

Proof. Suppose that d is one to one. Since d is a $(\ominus, \odot)$-derivation, we have

$$d(d(1)^*) = d(1 \ominus d(1)) = (d(1) \ominus d(1)) \odot (1 \ominus d(d(1))) = 0 = d(0),$$

that is $d(1) = 1$.
Now, suppose that d is onto. Then there exists $x \in A$ such that $d(x) = 1$. Since $1 = d(x) \leq x$, then $x = 1$. Hence $d(1) = 1$. Therefore by Theorem 4.10 and Proposition 4.4, we get that $A = Fix_d(A) = B(A)$. □

Corollary 4.12. *Let d be a $(\ominus, \odot)$-derivation on an MV-algebra A and $d(1) \in B(A)$. Then we have*
(1) $d(1) \in Fix_d(A)$,
(2) $d(A) = Fix_d(A)$.

Proof. (1) follows from Corollary 4.7 part (1).
(2) It is clear that $Fix_d(A) \subseteq d(A)$. Let $x \in d(A)$. Then there is $y \in A$ such that $x = d(y)$. Since $x = d(y) \leq d(1)$ and $d(1) \in Fix_d(A)$, then $x \in Fix_d(A)$, by Theorem 4.10. Therefore $d(A) \subseteq Fix_d(A)$. □

Theorem 4.13. *Let I be a finite ideal of an MV-algebra A such that $I \subseteq B(A)$. Then there exists a $(\ominus, \odot)$-derivation d such that $Fix_d(A) = I$.*

Proof. Since I is a finite ideal of A, then $b = \bigvee_{a \in I} a \in I$. Consider the map d defined on A, by $d(x) = x \odot b$, for all $x \in A$. Since $b \in I$ and $d(x) \leq b$, for all $x \in A$, we get that $d(A) \subseteq I \subseteq B(A)$. Thus by Proposition 4.9, d is $(\ominus, \odot)$-derivation on A. Now we show that $Fix_d(A) = I$. Let $x \in I$. Then $d(x) = x \wedge b = x$, and so $x \in Fix_d(A)$. Since $d(1) \in B(A)$, then by Corollary 4.12 part (2), $Fix_d(A) = d(A)$. Thus by hypothesis we get that $Fix_d(A) \subseteq I$. Therefore $Fix_d(A) = I$. □

In the following proposition, we determine the relationship between $(\ominus, \odot)$-derivations and $(\odot, \oplus)$-derivations on MV-algebras.

Proposition 4.14. (i) *If* d *is an isotone* $(\odot, \oplus)$*-derivation on* MV*-algebra* A*, then* d *is a* $(\ominus, \odot)$*-derivation,*

(ii) *If* d *is a* $(\ominus, \odot)$*-derivation on boolean algebra* A*, then* d *is a* $(\odot, \oplus)$*-derivation.*

Proof. (i) The proof follows from Propositions 3.17, 3.14 and 4.9.

(ii) It is clear by Proposition 4.4 and Corollary 3.15. □

5. Conclusion and Future Research

In this paper, we applied the notion of derivation in rings to MV-algebras and we have introduced the notions of $(\odot, \oplus)$- derivations and $(\ominus, \odot)$- derivations. We have also presented many important properties of these derivations on MV-algebras. Furthermore, we discussed the relations among $(\ominus, \odot)$-derivations and $(\odot, \oplus)$-derivations. We proved that if d is an isotone $(\odot, \oplus)$-derivation then d is a $(\ominus, \odot)$-derivation and $d(A) \subseteq B(A)$. Conversely, if d is a $(\ominus, \odot)$-derivation on a boolean algebra A, then d is an isotone $(\odot, \oplus)$-derivation.

In [11], B. Gerla introduced a pair of semirings $A^{\vee} = (A, \vee, \odot, 0, 1)$ and $A^{\wedge} = (A, \wedge, \oplus, 0, 1)$ on MV-algebra $(A, \oplus, ^*, 0)$ such that * is semirings isomorphism between $A^{\vee}$ and $A^{\wedge}$. Hence in future , we will use the other operations to define different derivations on an MV-algebra and we will obtain their properties. Also, we will study the relationship between them. We hope that the above work would serve as a foundation for further on study the structure of various derivations.

Since MV-algebras and BL-algebras are closely related, we will use the results of this paper to study derivations on BL-algebras and related algebraic systems. Some important issues for future work are: (i) developing the properties of a derivation, (ii) defining new derivations which are related to given derivations on MV-algebras, (iii) finding useful results on the other algebraic structures.

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References

[1] E. Albas, On ideals and orthogonal generalized derivations of semiprime ring, *Math. j. Okayama Uni.*, **49**, (2007), 53-58.

[2] H.E. Bell, L.C. Kappe, Rings in which derivations satify certain algebraic conditions, *Acta Math. Hungar.*, **53** (1989), 339-346.

[3] H.E. Bell, G. Mason, On derivations in near-rings and near-fields, *North-Holland Math. Studies*, **137**, (1987), 31-35.

[4] R.A. Borzooei, G.R. Rezaei and N. Kouhestani, On (semi) topological BL-algebras, *Iranian Journal of Mathematical Sciences and Informatics*, **6**(1), (2011), 59-72.

[5] D. Busneag and D. Piciu, Localization of MV-algebras and lu-groups, *Algebra univers.*, **50**, (2003), 359-380.

[6] Y. Ceven and M.A. Ozturk, On f-derivations of lattices, *Bull. Korean Math. Soc.*, **45**(4), (2008), 701-707.

[7] C.C. Chang, Algebraic analysis of many-valud logic, *Trans. Amer. Math. Soc.*, **88**, (1958), 467-490.

[8] C.C. Chang, A near proof of completeness of the Lukasiewicz axioms, *Trans. Amer. Math. Soc.*, **93**, (1959), 74-80.

[9] R. Cignoli, I.M.L. D'ottaviano and D. Mundici, *Algebraic foundations of many-valud reasoning*, Trends in Logic-Studia Logic Library, vol.7, Kluwer Academic Publishers, Dordrechet, 2000.

[10] L. Ferrari, On derivations of lattices, *Pure Math. and Appl.*, **12**, (2001), 365-382.

[11] B. Gerla, Many valued logics and semirings, *Neural Networks World*, **5**, (2003), 467-480.

[12] Sh. Ghorbani, Qutient BCI-algebras introduced by pseudo-valuation, *Iranian Journal of Mathematical Sciences and Informatics*, **5**(2), (2010), 13-24.

[13] Y.B. Jun and X.L. Xin, On derivations of BCI-algebras, *Information Sciences*, **159**, (2004), 167-176.

[14] Y.S. Jung and I.S. Chang, On approximately higher ring derivations, *J. Math. Anal. Appl.*, **343**, (2008), 638-643.

[15] D. Mundici, Interpretation of AFC^*-algebras in Lukasiewicz sentential calculus, *J.Functional Analysis*, **65**, (1986), 15-63.

[16] E. Posner, Derivations in prime rings, *Proc. Am. Math. Soc.*, **8**, (1957), 1093-1100.

[17] G. Szász, Derivations of lattices, *Acta Sci. Math(Szeged)*, **37**, (1975), 149-154.

[18] X.L. Xin, T.Y. Li and J.H. Lu, On derivations of lattices, *Information sciences* , **178**, (2008), 307-316.

[19] J. Zhan and Y. Lin Liu, On f-derivations of BCI-algebras, *International journal of Mathematices and mathematical sciences*, **11**, (2005), 1675-1684.