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Cuts and overspill properties in models of bounded arithmetic

Morteza Moniri* and S. Hosein Sajjadi
Department of Mathematics, Shahid Beheshti University, Evin, Tehran, Iran

E-mail: ezmoniri@gmail.com
E-mail: s.hoseinsajadi@gmail.com

Abstract. In this paper we are concerned with cuts in models of Samuel Buss' theories of bounded arithmetic, i.e. theories like S_2^i and T_2^i. In correspondence with polynomial induction, we consider a rather new notion of cut that we call p-cut. We also consider small cuts, i.e. cuts that are bounded above by a small element. We study the basic properties of p-cuts and small cuts. In particular, we prove some overspill and underspill properties for them.

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1. Introduction

First-order arithmetic has had an important role in the development of mathematical logic. Peano arithmetic is the most important logical theory of arithmetic. This theory has a small number of basic axioms together with the axiom scheme of induction.

Let M be a nonstandard model of Peano arithmetic. A non-empty subset I of M is a cut if I is closed under successor and also is downward closed. Studying cuts has an important place in the model theory of arithmetic, specially the issue of definability and undefinability of cuts. Definable cuts are discussed mostly in the contexts of Gödel's incompleteness theorem, consistency and inconsistency (see [5]). Undefinable cuts are studied in the context of overspill properties and their consequences. For this type of results see [6].

*Corresponding Author

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Bounded arithmetic is obtained from Peano arithmetic by restricting the axiom scheme of induction to bounded formulas. The importance of bounded arithmetic is essentially based on its connection to computational complexity (see [2]).

In this paper we consider undefinable cuts in models of bounded arithmetic theories. In [4] and [3] overspill properties for models of bounded arithmetic in the language of PA are studied. We work in the special language of bounded arithmetic introduced by Buss in [2]. In correspondence with polynomial induction $PIND$ in these theories, we define a notion of p-cut for models of S_2^i and T_2^i. We also consider small cuts, i.e. cuts bounded above by a small element of the form $|a|$. We investigate basic properties of p-cuts and small cuts. In particular, we prove some overspill and underspill properties and their converses.

2. Some backgrounds

In this paper we consider some well-known fragments of bounded arithmetic like S_2^i and T_2^i. In this section we review some basic properties of these theories (see [2] for the details).

The language of the theories S_2^i and T_2^i extends the usual language of first-order arithmetic by adding function symbols $\llcorner \frac{x}{2} \lrcorner$ (= $\frac{x}{2}$ rounded down to the nearest integer), $|x|$ (= the length of binary representation for x) and $\#$ ($x\#y = 2^{|x||y|}$).

The base theory $BASIC$ is a finite set of quantifier-free formulas expressing basic properties of the relation and function symbols. $\Sigma_0^b = \Pi_0^b$ is the class of all sharply bounded formulas. A sharply bounded formula is a bounded formula in which all quantifiers are sharply bounded, i.e. of the form $\exists x \leqslant |t|$ or $\forall x \leqslant |t|$ where t is a term which does not contain x.

The syntactic classes $\Sigma_{i+1}^b, \Pi_{i+1}^b$ of bounded formulas are defined by counting alternations of bounded quantifiers ignoring sharply bounded quantifiers. A formula φ is in Δ_i^b with respect to a model (resp., theory), if there is a Σ_i^b-formula and a Π_i^b-formula such that φ is equivalent to both of them in the model (resp., theory).

The theory S_2^i is axiomatized by adding the $\Sigma_i^b - PIND$ axioms to $BASIC$, i.e.

$$[\varphi(0) \wedge \forall x(\varphi(\lfloor x/2 \rfloor) \rightarrow \varphi(x)] \rightarrow \forall x \varphi(x),$$

where $\varphi(x)$ is a Σ_i^b-formula that can have more free variables besides x. The theory T_2^i is defined by adding the $\Sigma_i^b - IND$ axioms to $BASIC$.

The $PLIND$ and $LIND$ axioms are defined as

$$PLIND: \; [\varphi(0) \wedge \forall x(\varphi(\lfloor x/2 \rfloor) \rightarrow \varphi(x)] \rightarrow \forall x \varphi(|x|),$$

$$LIND: \; [\varphi(0) \wedge \forall x(\varphi(x) \rightarrow \varphi(x+1)] \rightarrow \forall x \varphi(|x|).$$

The theories L_2^i and R_2^i are axiomatized by adding the $\Sigma_i^b - LIND$ and $\Sigma_i^b - PLIND$ axioms to $BASIC$, respectively.

Throughout this paper, i denotes a natural number greater than or equal to 1, unless it is explicitly stated that it can also be 0.

Let M be a nonstandard countable model of $BASIC$. A non-empty subset I of M is said to be a cut if I is closed under successor and whenever $a \in I$ and $M \models b < a$, then $b \in I$. If I is a cut of M, we write $I \subseteq_e M$. I is small if there is an $a \in M$ such that $I < |a|$, i.e. every element of I is less than $|a|$.

Let Ψ be a class of formulas. It is well-known and easy to see that, if $M \models \Psi - IND$, then no proper cut I of M can be Ψ-definable, that is to say, there is no $\psi \in \Psi$ such that $M \models \psi(a)$ if and only if $a \in I$. This is a simple version of the overspill property for models of arithmetic. It is not hard to prove that the overspill property for cuts is equivalent to the induction-scheme (see [6, Page 72]).

3. New proofs for old results

In this section we give relatively new proofs for some well-known facts concerning bounded arithmetic theories using the notion of cut.

Definition 3.1. Let $M \models BASIC$ and $\emptyset \neq I \subseteq M$. We say that I is a p-cut if $\lfloor x/2 \rfloor \in I$ implies $x \in I$ and if $a \in I$ and $M \models b \leq a$ then $b \in I$.

Note that p-cuts are those cuts that are closed under addition. For an example of a cut which is not a p-cut, let $a \in M \models BASIC$ be a non-standard element. Then

$$I_a = \{x \in M : \exists n \in \mathbb{N}\ x \leq a + n\}$$

is a cut but not a p-cut ($a \in I$ and $2a \notin I$). If I is a p-cut then $|I| = \{|x| : x \in I\}$ is a cut, because from $BASIC$ we have $S(|x|) = |2x|$ and if $|b| \leq |a|$ then $b < 2a$. Moreover, if I is proper, then $|I|$ is a small cut.

From the above definition, it is clear that no model of S_2^i contains a proper Σ_i^b-definable p-cut, and no model of T_2^i contains a proper Σ_i^b-definable cut.

Proposition 3.2. *We have* $BASIC + \Pi_i^b - IND \vdash \Pi_i^b - PIND$.

Proof. Let $\varphi(x) \in \Pi_i^b$ and $M \models BASIC + \Pi_i^b - IND$ and

$$M \models \varphi(0) \wedge \forall x\,(\varphi(\lfloor x/2 \rfloor) \rightarrow \varphi(x)).$$

Then the set

$$I = \{x \in M : M \models \forall y \leq x\, \varphi(y)\}$$

is a p-cut, and it is defined by $\forall y \leq x\ \varphi(y) \in \Pi_i^b$. Therefore, I is a cut and $I = M$. This means that $M \models \forall x\ \varphi(x)$. So $M \models \Pi_i^b - PIND$. □

In the proof of the following two propositions we use an idea due to Solovay (see [8]). It shows how to construct a p-cut in a cut.

Proposition 3.3. *Let M be a model of $BASIC$. Let $i \geq 0$. Then in every small Δ_i^b-definable cut there is a small Δ_i^b-definable p-cut.*

Proof. Assume that I is a small Δ_i^b-definable cut of M, $a \in M$ and $|a| > I$. Let $\varphi(x) \in \Delta_i^b$ define I. Then the formula

$$\forall y < |a| \ (\varphi(y) \to \varphi(x+y))$$

defines a small p-cut. For this, note that if $M \models \varphi(\lfloor x/2 \rfloor + y)$, then $\lfloor x/2 \rfloor + y < |a|$. □

The following proposition can be proved similarly.

Proposition 3.4. *Let M be a model of $BASIC$. Then in every proper Δ_i^b-definable cut there is a proper Π_i^b-definable p-cut.*

Proposition 3.5. *Let $M \models BASIC$. If there exists a proper Δ_i^b-definable subset of M containing 0 and closed under successor then there is a proper Π_i^b-definable p-cut in M.*

Proof. Let $\varphi(x)$ be the Δ_i^b formula such that

$$M \models \varphi(0) \wedge \forall x(\varphi(x) \to \varphi(x+1)).$$

Consider the formula

$$\psi(x) : \forall y < a \ (\varphi(y) \to \varphi(x+y)).$$

As in 3.3, one can show that

$$M \models \psi(0) \ \wedge \forall x \, (\psi(\lfloor x/2 \rfloor) \to \psi(x)).$$

Now notice that if $M \not\models \varphi(a)$, then the formula $\forall z \leq x \ \psi(z)$ defines a proper Π_i^b-definable p-cut. □

In the rest of this section we add two binary functions $a \dot{-} b$ and $MSP(a,i) (= \lfloor a/2^i \rfloor)$ to the language and also add the following primary properties of them as new axioms to $BASIC$:

i) $a \dot{-} 0 = a$

ii) $[b < a \to a \dot{-} (b+1) + 1 = a \dot{-} b] \wedge [b \geq a \to a \dot{-} b = 0]$

iii) $y \leq x \to a \dot{-} x \leq a \dot{-} y$

iv) $MSP(a,0) = a$

v) $MSP(a, i+1) = \lfloor 1/2 \ MSP(a,i) \rfloor$

vi) $b \geq |a| \to MSP(a,b) = 0$

vii) $y \geq x \to MSP(a,y) \leq MSP(a,x)$.

We represent this modified set of axioms with $BASIC^*$ and redefine fragments of bounded arithmetic in this language. Note that in the presence of S_2^1 the results in this extended language can be stated and proved in the original one.

Proposition 3.6. *Let $M \models BASIC^*$. Then the following statements are equivalent.*

i) There is no proper Σ_i^b-definable cut (resp., p-cut) in M,

ii) There is no proper Π_i^b-definable cut (resp., p-cut) in M.

Proof. Let $A(x)$ define a proper cut (resp., p-cut) I and $a > I$. Then $\neg A(a \dot{-} x)$ (resp. $\neg A(MSP(a, |x|))$) defines a proper cut (resp., p-cut) as well. □

Below by asserting that a cut I is Ψ-undefinable we mean that there is no formula in Ψ that defines I.

Proposition 3.7. *Let M be a model of $BASIC^*$ and Ψ be the one of the classes Σ_i^b or Π_i^b of formulas. We have*

1. $\Psi - IND$ is equivalent to Ψ-undefinability of proper cuts,

2. $\Psi - PIND$ is equivalent to Ψ-undefinability of proper p-cuts,

3. $\Psi - LIND$ is equivalent to Ψ-undefinability of small cuts,

4. $\Psi - PLIND$ is equivalent to Ψ-undefinability of small p-cuts.

Proof. In each case, the left to right part is straightforward. Also, the right to left parts for Π_i^b-formulas are easy. For Σ_i^b-formulas we only prove case 1. The other cases can be proved similarly.

Let $\varphi(x)$ be a Σ_i^b-formula and

$$M \models \varphi(0) \wedge \forall x(\varphi(x) \rightarrow \varphi(x+1)) \wedge \neg\varphi(a).$$

Then the formula $\exists y < a\ (y \geq x \wedge \varphi(y))$ is Σ_i^b and defines a proper cut of M which is impossible. □

Now we can give simple and uniform proofs for the following well-known facts.

Corollary 3.8. *We have the following.*

i) $BASIC^* + \Pi_i^b - IND \equiv BASIC^* + \Sigma_i^b - IND$,

ii) $BASIC^* + \Pi_i^b - PIND \equiv BASIC^* + \Sigma_i^b - PIND$,

iii) $BASIC^* + \Pi_i^b - LIND \equiv BASIC^* + \Sigma_i^b - LIND$,

iv) $BASIC^* + \Pi_i^b - PLIND \equiv BASIC^* + \Sigma_i^b - PLIND$.

Proof. Use Proposition 3.6 and 3.7. □

From Corollary 3.8 and Proposition 3.2, we have $T_2^i \vdash S_2^i$. Also, Proposition 3.4 and Corollary 3.8(ii) imply $S_2^{i+1} \vdash T_2^i$. Using Proposition 3.5 and Corollary 3.8(ii), one can see $S_2^i \vdash I\Delta_i^b$.

In [9], using a proof theoretic method, Takeuti proved that $R_2^i \vdash S_2^{i-1}$. Below we give a model theoretic proof for this fact. Actually, we prove a stronger result.

Proposition 3.9. $R_2^i \vdash \Delta_i^b - PIND$*, for* $i \geq 0$.

Proof. Assume that $M \vDash R_2^i$. Let $\varphi(x)$ be a Δ_i^b-formula and

$$M \models \varphi(0) \wedge \forall x(\varphi(\lfloor x/2 \rfloor) \rightarrow \varphi(x)) \wedge \neg\varphi(a).$$

Let $\psi(x)$ be the formula $\neg\varphi(MSP(a,x))$. We have

$$M \models \psi(0) \wedge \forall x(\psi(x) \rightarrow \psi(x+1)) \wedge \neg\psi(|a|).$$

Then the formula $\exists y < |a|\ (y \geqslant x \wedge \psi(y))$ is Δ_i^b and defines a small cut of M. But as by Proposition 3.3, in every small proper Δ_i^b-definable cut there is a small Δ_i^b-definable p-cut, which is impossible. □

Note that in the presence of MSP, $\Delta_i^b - PIND$ is equivalent to $\Delta_i^b - LIND$. So Proposition 3.9 can be stated as $R_2^i \vdash \Delta_i^b - LIND$. This result appears as Theorem 2.7.7 of [1] where it is shown that R_2^i proves $\Delta_i^b - LIND$, over a weak theory defining MSP. Also, from Proposition 2.11, we get $R_2^i \vdash S_2^{i-1}$.

Corollary 3.10. *$BASIC^* + \Delta_i^b - PIND \equiv BASIC^* + \Delta_i^b - PLIND$, for $i \geq 0$.*

Proof. The Left to right part is obvious by the definition of $PIND$ and $PLIND$. For the inverse side note that in the proof of Proposition 3.9 one can assume that $\varphi(x)$ is a Δ_i^b-formula. □

4. Overspill and underspill properties

In this section we study overspill properties for the theories S_2^i and T_2^i. In [4] and [3], the authors studied such properties for the classes E_n and U_n of bounded formulas in the language of PA.

Theorem 4.1. *(Overspill) Let $M \models T_2^i$ and $I \subseteq_e M$ be a proper cut and $\phi(x) \in \Pi_i^b$ such that for all $x \in I$, $M \models \phi(x)$. Then there exists an element $c > I$ such that $M \models \forall x \leq c\ \phi(x)$.*

Proof. If not, then I would be definable by the Π_i^b-formula $\forall x \leq y\ \phi(x)$ which is impossible. □

When a model M has the properties of Theorem 4.1, we say that it has the overspill property.

Theorem 4.2. *Let $M \models BASIC^*$ enjoying the overspill property for Π_i^b-formulas. Then $M \models T_2^i$.*

Proof. By Corollary 3.8(i), it is enough to show M satisfies the induction scheme for Π_i^b-formulas. Let $\phi(x) \in \Pi_i^b$ and

$$M \models \phi(0) \wedge \forall x(\phi(x) \rightarrow \phi(x+1)).$$

Then

$$I_\phi = \{x \in M : M \models \forall y\ \leq x\ \phi(y)\}$$

is a Π_i^b- definable cut. If $I_\phi \neq M$, then by overspill there is $c > I_\phi$ such that

$$M \models \forall x \leq c \; \forall y \leq x \; \phi(y).$$

Now, by putting $x = c$, we get $M \models \forall y \leq c \; \phi(y)$. So $c \in I_\phi$, which is a contradiction. □

There is a overspill property for Π_i^b-formulas in models of S_2^i (Theorem 4.3 below) which can be proved analogously.

Theorem 4.3. *Let M be a model of $BASIC^*$. We have $M \models S_2^i$ if and only if the following condition holds: for every proper p-cut I of M and $\phi \in \Pi_i^b$ for which for all $x \in I$, $M \models \phi(x)$, there exists an element $c > I$ such that $M \models \forall x \leq c \; \phi(x)$.*

Now we state another overspill property for models of S_2^i (and hence T_2^i). For this we need the notion of small cut.

Theorem 4.4. *(Sharp overspill). Let $M \models S_2^i$ and $I \subseteq_e M$ be a small cut and φ be a $(\Sigma_i^b \bigcup \Pi_i^b)$-formula such that for all $x \in I$, $M \models \varphi(x)$. Then there exists an element $a \in M$ such that $|a| > I$ and $M \models \forall x \leq |a| \; \varphi(x)$.*

Proof. Otherwise, $\{z : M \models \forall x \leq |z| \; \varphi(x)\} = \{z : |z| \in I\}$ would be a proper $(\Sigma_i^b \bigcup \Pi_i^b)$-definable p-cut which is impossible. □

Theorem 4.5. *Let $M \models BASIC$ enjoying the sharp overspill property for Σ_i^b (resp. Π_i^b)-formulas. Then $M \models L_2^i$ (and hence $M \models S_2^i$).*

Proof. Let φ be a Σ_i^b (resp. Π_i^b))-formula. Assume that

$$M \models \phi(0) \wedge \forall x \; (\varphi(x) \rightarrow \phi \; (x+1)).$$

If $M \not\models \forall x \varphi(|x|)$, then there exists $c \in M$ such that $M \not\models \varphi(|c|)$. Define

$$\psi(z) : \exists x < |c| \; (z \leq x \wedge \varphi(x)).$$

This formula defines a cut I of M and since $|c|$ is not in this cut, I is a small cut. Now using sharp overspill, there is $|d| > I$ such that $M \models \forall z \leq |d| \; \psi(z)$. Now use $z = |d|$ to get a contradiction. □

Let $M \models S_2^i$ be nonstandard and let I be a proper p-cut of M. Suppose $\phi(x)$ is a $(\Sigma_i^b \bigcup \Pi_i^b)$-formula such that $M \models \phi(|a|)$ for all $a \in I$. Since in this case $|I|$ is a small cut, by Theorem 4.4, there is $c \in M$ such that $c > I$ and $M \models \forall x \leq |c| \; \phi(x)$.

Theorem 4.6. *Let $M \models R_2^i$. If I is a very small cut of M (i.e. it is bounded by an element of the form $||a||$) and φ is a Σ_i^b (resp. Π_i^b)-formula such that $M \models \phi(x)$, for all $x \in I$, then there exists an element $c \in M$ such that $|c| > I$ and $M \models \forall x \leq |c| \; \phi(x)$.*

Proof. This can be proved as Theorem 4.4. □

Below we prove an underspill property for models of S^i_2.

Lemma 4.7. *Let $M \models S^i_2$ and $\phi(x) \in \Pi^b_i$. Assume that I is a p-cut of M and for each $x \in I$, there exists $y \in |I|$ with $M \models y \geq |x| \wedge \phi(y)$. Then for each $c \in M$ with $|c| > |I|$, there exists an element $b \in M$ such that $|I| < b < |c|$ and $M \models \phi(b)$.*

Proof. Let $c \in M$ such that $|c| > |I|$. Apply Theorem 4.3 to the formula

$$\psi(x) := \exists y < |c| \, (y \geq |x| \wedge \phi(y)).$$

By the assumption, for all $x \in I$, we have $M \models \psi(x)$. Thus there is $d > I$ such that $M \models \forall x \leq d \; \psi(x)$. Now let $x = d$. There is $b < |c|$ such that $b \geq |d| > |I|$ and $M \models \phi(b)$. This completes the proof. □

Theorem 4.8. *(Underspill) Let $i \geq 0$ and $M \models S^i_2$. Let I be a proper p-cut of M and $\phi(x) \in \Sigma^b_i$. If $M \models \phi(c)$ for all $c > |I|$, then there exists an element $b \in I$ such that $M \models \forall x \geq |b| \; \phi(x)$.*

Proof. Suppose that for all $x \in I$ there exists $y \in |I|$ such that $M \models y \geq |x| \wedge \neg\phi(y)$.

Since $\neg\varphi(y) \in \Pi^b_i$, by the previous lemma, there is $c > |I|$ such that $M \models \neg\varphi(c)$ which contradicts the assumption of the theorem. □

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