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T-Stability Approach to the Homotopy Perturbation Method for Solving Fredholm Integral Equations

Hossein Jafari*, Maryam Alipour and Maryam Ghorbani
Department of Mathematics, University of Mazandaran, Babolsar, Iran

E-mail: jafari@umz.ac.ir

Abstract. The homotopy perturbation method is a powerful device for solving a wide variety of problems arising in many scientific applications. In this paper, we investigate several integral equations by using T-stability of the Homotopy perturbation method investigates for solving integral equations. Some illustrative examples are presented to show that the Homotopy perturbation method is T-stable for solving Fredholm integral equations.

Keywords: T-Stability, Homotopy perturbation method, Fredholm integral equation, Volterra integral equation.

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1. Introduction

Various kinds of analytical methods and numerical methods were used to solve integral equations [2,14]. Recently the Homotopy perturbation method [8] has been used by many authors to handle a wide variety of scientific and engineering applications to solve various functional equations. In this method, the solution is considered as the sum of an infinite series, which converges rapidly to exact solutions. Using the homotopy technique in topology, a homotopy is constructed with an embedding parameter $p \in [0, 1]$ which is considered as a "small parameter". It can be said that He's homotopy perturbation method is a universal one, and it is able to solve various kinds of nonlinear functional equations [1, 3, 5–7, 9, 11, 13, 15]. In this paper several integral equations are

*Corresponding Author

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solved by Homotopy perturbation method after that we show T-Stability of Homotopy perturbation method for solving integral equations.

First we recall T-Stability of Picards iteration and a theorem from [10]: Let $(X, \|.\|)$ be a Banach space and T a self-map of X. Let $x_{n+1} = f(T, x_n)$ be some iteration procedure. Suppose that $F(t)$ the fixed point set of T, is nonempty and that x_n converges to a point $q \in F(t)$. Let $\{y_n\} \subseteq X$ and define $\epsilon_n = \|y_{n+1} - f(T, y_n)\|$. If $\lim \epsilon = 0$ implies that $\lim y_n = q$, then the iteration procedure $x_{n+1} = f(T, x_n)$ is said to be T-stable. Without loss of generality, we may assume that $\{y_n\}$ is bounded, if $\{y_n\}$ is not bounded, then it cannot possibly converge. If these conditions hold for $x_{n+1} = Tx_n$, that is, Picards iteration, then we will say that Picards iteration is T-stable.

Theorem 1.1. *[10] Let $(X, \|.\|)$ be a Banach space and T a self-map of X satisfying*

$$\|Tx - Ty\| \leq L \|x - Tx\| + \alpha \|x - y\|, \tag{1.1}$$

for all $x, y \in X$, where $L \geq 0, 0 \leq \alpha \leq 1$. Suppose that T has a fixed point p. Then, T is Picard T-stable.

2. Homotopy perturbation method

To convey an idea of the Homotopy perturbation method (HPM), we consider a general equation of the type

$$L(u) = 0, \tag{2.1}$$

where L is an integral or differential operator. We define a convex homotopy $H(u, p)$ by

$$H(u, p) = (1 - p)F(u) + pL(u) = 0, \tag{2.2}$$

where $F(u)$ is a functional operator with known solutions v_0 which can be obtained easily. It is clear that, when

$$H(u, p) = 0, \tag{2.3}$$

we have $H(u, 0) = F(u)$ and $H(u, 1) = L(u)$. This shows that $H(u, p)$ continuously traces an implicitly defined curve from a starting point $H(v_0, 0)$ to a solution $H(u, 1)$. The embedding parameter p monotonously increases from zero to one as the trivial problem $F(u) = 0$, continuously deforms to original problem $L(u) = 0$. The embedding parameter $p \in [0, 1]$ can be considered as an expanding parameter to obtain

$$u = u_0 + pu_1 + p^2 u_2 + \cdots . \tag{2.4}$$

When $p \to 1$, Eq.(2.3) corresponds to Eq.(2.1) and becomes the approximate solution of Eq.(2.1), i.e.,

$$U = \lim_{p \longrightarrow 1} u = u_0 + u_1 + u_2 + \cdots . \tag{2.5}$$

The series (2.5) is convergent for most cases and the rate of convergence depends on $L(u)$ [8].

3. Fredholm integral equation of the second kind

Now we consider the Fredholm integral equation of the second kind in general case, which reads

$$u(x) = f(x) + \lambda \int_a^b k(x,t)u(t)dt, \tag{3.1}$$

where $k(x,t)$ is the kernel of the integral equation. There is a simple iteration formula for (3.1) in the form

$$u_{n+1}(x) = f(x) + \lambda \int_a^b k(x,t)u_n(t)dt. \tag{3.2}$$

Now, we show that the nonlinear mapping T, defined by

$$u_{n+1}(x) = T(u_n(x)) = f(x) + \lambda \int_a^b k(x,t)u_n(t)dt, \tag{3.3}$$

is T-stable in $L^2[a,b]$. First, we show that the nonlinear mapping T has a fixed point. For $m, n \in N$ we have

$$\begin{aligned} \|T(u_m(x)) - T(u_n(x))\| &= \|u_{m+1}(x) - u_{n+1}(x)\| \\ &= \left\| \lambda \int_a^b k(x,t)(u_m(t) - u_n(t))dt \right\| \\ &\le |\lambda| \left[\int_a^b \int_a^b k^2(x,t)dxdt \right]^{1/2} \|u_m(x) - u_n(x)\| . \end{aligned} \tag{3.4}$$

Therefore, if

$$|\lambda| < \left[\int_a^b \int_a^b k^2(x,t)dxdt \right]^{-1/2}, \tag{3.5}$$

then the nonlinear mapping T has a fixed point.
Second, we show that the nonlinear mapping T satisfies (1.1) and so (3.3) holds. Putting $L = 0$ and $\alpha = |\lambda| \left[\int_a^b \int_a^b k^2(x,t)dxdt \right]^{1/2}$ shows that (1.1) holds for the nonlinear mapping T. All of the conditions of Theorem 1 hold for the nonlinear mapping T and hence it is T-stable. As a result, we can state the following theorem.

Theorem 3.1. *Consider the iteration scheme*

$$\begin{aligned} u_0(x) &= f(x), \\ u_{n+1}(x) &= T(u_n(x)) = f(x) + \lambda \int_a^b k(x,t)u_n(t)dt, \end{aligned} \tag{3.6}$$

for $n = 0, 1, 2, ...$ to construct a sequence of successive iterations $\{u_n(x)\}$ to the solution of (3.2). In addition, let

$$|\lambda| < \left[\int_a^b \int_a^b k^2(x,t)dxdt\right]^{-1/2}, \tag{3.7}$$

$L = 0$ and $\alpha = |\lambda| \left[\int_a^b \int_a^b k^2(x,t)dxdt\right]^{1/2}$. Then the nonlinear mapping T, in the norm of $L^2(a,b)$, is T-stable.

In view of Eq.(2.2), we define following convex homotopy $H(u,p)$ for (3.1)

$$(1-p)[u(x) - f(x)] + p[u(x) - f(x) - \lambda \int_a^b k(x,t)u(t)dt] = 0, \tag{3.8}$$

or

$$u(x) = f(x) + p\lambda \int_a^b k(x,t)u(t)dt. \tag{3.9}$$

Substituting Eq.(2.4) into Eq.(3.9), and equating the terms with identical powers of p, we have

$$\begin{aligned} p^0 &: u_0 = f(x), \\ p^1 &: u_1 = \lambda \int_a^b k(x,t)(u_0)dt, \\ p^2 &: u_2 = \lambda \int_a^b k(x,t)(u_1)dt, \\ p^3 &: u_3 = \lambda \int_a^b k(x,t)(u_2)dt, \\ &\vdots \end{aligned}$$

Therefore, we obtain iteration formula for Eq.(3.1) as follow:

$$\begin{aligned} u_0(x) &= f(x), \\ u_m(x) &= \lambda \int_a^b k(x,t)u_{m-1}(t)dt, \ m > 0, \end{aligned} \tag{3.10}$$

according to Eq.(3.10), we define partial sum as follow:

$$\begin{aligned} s_0(x) &= f(x), \\ s_n(x) &= \sum_{i=0}^{n} u_i(x), \end{aligned} \tag{3.11}$$

in view of Eqs.(3.10) and (3.11) we have

$$\begin{aligned} s_0(x) &= f(x), \\ s_{n+1}(x) &= f(x) + \lambda \int_a^b k(x,t)s_n(t)dt. \end{aligned}$$

Definition 3.2. *Let $u_1, u_2, u_3, \ldots$ be a sequence of functions. The series $\sum_{n=1}^{\infty} u_n$ is said to converge to u if the sequence $\{s_n\}$ of partial sums defined by*

$$s_n(x) = \sum_{k=1}^{\infty} u_k(x), \tag{3.12}$$

converges to u [12].

We now recall a theorem from [4].

Theorem 3.3. *[4] Consider the iteration scheme*

$$s_0(x) = f(x),$$
$$s_{n+1}(x) = f(x) + \lambda \int_a^b k(x,t) s_n(t) dt,$$

for $n = 0, 1, 2, \ldots$ to construct a sequence of successive iterations $s_n(x)$ to the solution of Eq. (3.1). In addition, let

$$\int_a^b \int_a^b k^2(x,t) dx dt = B^2 < \infty, \tag{3.13}$$

and assume that $f(x) \in L^2(a,b)$. Then, if $|\lambda| < \frac{1}{B}$, the above iteration converges in the norm of $L^2(a,b)$ to the solution of Eq. (3.1).

Corollary Consider the iteration scheme

$$\begin{aligned} s_0(x) &= f(x), \\ s_{n+1}(x) &= T(s_n(x)) = f(x) + \lambda \int_a^b k(x,t) s_n(t) dt, \end{aligned}$$

for $n = 0, 1, 2, \ldots$ If

$$|\lambda| < \left[\int_a^b \int_a^b k^2(x,t) dx dt \right]^{-1/2}, \tag{3.14}$$

$L = 0$ and $\alpha = |\lambda| \left[\int_a^b \int_a^b k^2(x,t) dx dt \right]^{1/2}$, then stability of the nonlinear mapping T is a coefficient condition for the above iteration to converge solution of (3.1) in the norm of $L^2(a,b)$.

4. Test Examples

In this section, we present some test examples to show that the stability of the HPM for solving integral equations. In fact the stability interval is a subset of converges interval.

Example 4.1. *[11] Consider the integral equation*

$$u(x) = \sqrt{x} + \lambda \int_0^1 x t u(t) dt. \tag{4.1}$$

Its iteration formula reads

$$s_{n+1}(x) = \sqrt{x} + \lambda \int_0^1 (xt) s_n(t) dt, \tag{4.2}$$

and

$$u_0(x) = \sqrt{x}.$$

In view of Eq.(3.9), we obtain

$$u(x) = \sqrt{x} + p\lambda \int_0^1 xtu(t)dt. \tag{4.3}$$

Substituting Eq. (2.4) into Eq. (4.3), we have the following results:

$$\begin{aligned}
&p^0 : u_0(x) = \sqrt{x},\\
&p^1 : u_1(x) = \lambda \int_0^1 xt\sqrt{t}dt = \frac{2\lambda x}{5},\\
&p^2 : u_2(x) = \lambda \int_0^1 xt\frac{2\lambda t}{5}dt = \frac{2\lambda^2}{15}x,\\
&p^3 : u_3(x) = \lambda \int_0^1 xt\frac{2\lambda^2}{15}tdt = \frac{2\lambda^3}{45}x,\\
&\quad\vdots
\end{aligned}$$

Continuing in this way, we obtain

$$s_n(x) = \sqrt{x} + [\frac{2}{5.3^0}\lambda + \frac{2}{5.3^1}\lambda^2 + \frac{2}{5.3^2}\lambda^3 + \cdots] = \sqrt{x} + [\frac{6}{5}\sum_{i=1}^{n}(\frac{\lambda}{3})^i x]. \tag{4.4}$$

The above sequence is convergent if $|\lambda| < 3$.
On the other hand, we have

$$\left[\int_a^b \int_a^b k^2(x,t)dxdt\right]^{\frac{1}{2}} = \left[\int_0^1 \int_0^1 (xt)^2 dxdt\right]^{\frac{1}{2}} = \frac{1}{3}. \tag{4.5}$$

Then, if $|\lambda| < 3$ for mapping

$$s_{n+1}(x) = T(s_n(x)) = \sqrt{x} + \lambda \int_0^1 xts_n(t)dt,$$

we have

$$\begin{aligned}
\|T(s_m(x)) - T(s_n(x))\| &= \|s_{m+1}(x) - s_{n+1}(x)\|\\
&= \left\|\lambda \int_0^1 xt(s_m(x) - s_n(x))dt\right\| \qquad (4.6)\\
&\le |\lambda| \left[\int_0^1 \int_0^1 (xt)^2 dxdt\right]^{1/2} \|s_m(x) - s_n(x)\|\\
&\le \frac{|\lambda|}{3}\|s_m(x) - s_n(x)\|,
\end{aligned}$$

which implies that T has a fixed point. Also, putting $L = 0$ and $\alpha = \frac{|\lambda|}{3}$ shows that (1.1) holds for the nonlinear mapping T. All of the conditions of Theorem 1.1 hold for the nonlinear mapping T and hence it is T-stable.

Example 4.2. *[11]Consider the integral equation*

$$u(x) = x + \lambda \int_0^1 (1 - 3xt)u(t)dt, \tag{4.7}$$

whose iteration formula reads

$$s_{n+1}(x) = x + \lambda \int_0^1 (1 - 3xt)s_n(t)dt, \tag{4.8}$$

and

$$u_0(x) = x.$$

In view of Eq. (3.9), we obtain

$$u(x) = x + p\lambda \int_0^1 (1 - 3xt)u(t)dt. \tag{4.9}$$

Substituting Eq.(2.4) into Eq. (4.9), we obtain the following results:

$$\begin{aligned}
&p^0 : u_0(x) = x,\\
&p^1 : u_1(x) = \lambda \int_0^1 (1 - 3xt)tdt = \frac{\lambda}{2} - \lambda x,\\
&p^2 : u_2(x) = \lambda \int_0^1 (1 - 3xt)(\frac{\lambda}{2} - \lambda t)dt = \frac{\lambda^2}{4}x,\\
&p^3 : u_3(x) = \lambda \int_0^1 (1 - 3xt)\frac{\lambda^2}{4}tdt = \frac{\lambda^3}{8} - \frac{\lambda^3}{4}x,\\
&\vdots
\end{aligned}$$

Continuing in this way, we obtain

$$s_n(x) = (1-\lambda)x+\frac{\lambda^2}{4}(1-\lambda)x+\frac{\lambda}{2}+\frac{\lambda^3}{8}+... = \sum_{i=0}^{n}[(\frac{\lambda^2}{4})^i\lambda(\frac{1}{2}-x)+(\frac{\lambda^2}{4})^i]+(1+(-1)^n)\frac{\lambda^{2n+2}}{2^{2n+3}}x. \tag{4.10}$$

The above sequence is convergent if $|\frac{\lambda}{2}| < 1$, that is, $-2 < \lambda < 2$.
On the other hand, we have

$$\left[\int_a^b \int_a^b k^2(x,t)dxdt\right]^{\frac{1}{2}} = \left[\int_0^1 \int_0^1 (1 - 3xt)^2 dxdt\right]^{\frac{1}{2}} = \frac{1}{\sqrt{2}}. \tag{4.11}$$

Then if $|\lambda| \leq \sqrt{2}$, for mapping

$$s_{n+1}(x) = T(s_n(x)) = x + \lambda \int_0^1 (1 - 3xt)s_n(t)dt,$$

we have

$$\begin{aligned}\|T(s_m(x))-T(s_n(x))\| &= \|s_{m+1}(x)-s_{n+1}(x)\| \\ &= \left\|\lambda\int_0^1(1-3xt)(s_m(x)-s_n(x))dt\right\| \qquad (4.12)\\ &\le |\lambda|\left[\int_0^1\int_0^1(1-3xt)^2dxdt\right]^{1/2}\|s_m(x)-s_n(x)\| \\ &\le \frac{|\lambda|}{\sqrt{2}}\|s_m(x)-s_n(x)\|,\end{aligned}$$

which implies that T has a fixed point. Also, putting $L=0$ and $\alpha=\frac{|\lambda|}{\sqrt{2}}$ shows that (1.1) holds for the nonlinear mapping T. All of the conditions of Theorem 1.1 hold for the nonlinear mapping T and hence it is T-stable.

Example 4.3. *Consider the integral equation*

$$u(x)=sinax+\lambda\frac{a}{2}\int_0^{\frac{\pi}{2a}}cos(ax)u(t)dt, \qquad (4.13)$$

with its iteration formula

$$s_{n+1}(x)=sinax+\lambda\frac{a}{2}\int_0^{\frac{\pi}{2a}}cos(ax)s_n(t)dt, \qquad (4.14)$$

and

$$u_0(x)=\sin ax.$$

In view of Eq. (3.9), we obtain

$$u(x)=sinax+p\lambda\frac{a}{2}\int_0^{\frac{\pi}{2a}}cos(ax)u_n(t)dt, \qquad (4.15)$$

Substituting Eq.(2.4) into Eq. (4.15), we have

$$\begin{aligned}&p^0: u_0(x)=sinax,\\ &p^1: u_1(x)=\lambda\frac{a}{2}\int_0^{\frac{\pi}{2a}}cos(ax)sin(at)dt=\frac{\lambda}{2}cos(ax),\\ &p^2: u_2(x)=\lambda\frac{a}{2}\int_0^{\frac{\pi}{2a}}cos(ax)\frac{\lambda}{2}cos(at)dt=\frac{\lambda^2}{4}cos(ax),\\ &p^3: u_3(x)=\lambda\frac{a}{2}\int_0^{\frac{\pi}{2a}}cos(ax)\frac{\lambda^2}{4}cos(at)dt=\frac{\lambda^3}{8}cos(ax),\\ &\vdots\end{aligned}$$

Continuing in this way, we obtain

$$\begin{aligned}s_n(x) &= sinax+\frac{\lambda}{2}cos(ax)+\frac{\lambda^2}{4}cos(ax)+\frac{\lambda^3}{8}cos(ax)+\cdots \qquad (4.16)\\ &= sinax+cos(ax)\sum_{i=1}^{\infty}(\frac{\lambda}{2})^i.\end{aligned}$$

The above sequence is convergent if $|\frac{\lambda}{2}| < 1$, that is, $-2 < \lambda < 2$.
On the other hand, we have

$$\left[\int_a^b \int_a^b k^2(x,t)dxdt\right]^{\frac{1}{2}} = \left[\int_0^{\frac{\pi}{2a}} \int_0^{\frac{\pi}{2a}} (\frac{a}{2}cos(ax))^2 dxdt\right]^{\frac{1}{2}} = \sqrt{\frac{\pi^2}{32}}. \quad (4.17)$$

Then if $|\lambda| < \frac{1}{\sqrt{\frac{\pi^2}{32}}} \cong 1.800$, for mapping

$$s_{n+1}(x) = T(s_n(x)) = x + \lambda\frac{a}{2}\int_0^{\frac{\pi}{2a}} cos(ax)s_n(t)dt,$$

we have

$$\begin{aligned} \|T(s_m(x)) - T(s_n(x))\| &= \|s_{m+1}(x) - s_{n+1}(x)\| \\ &= \left\|\lambda\int_0^{\frac{\pi}{2a}} \frac{a}{2}cos(ax)(s_m(x) - s_n(x))dt\right\| \quad (4.18) \\ &\leq |\lambda|\left[\int_0^{\frac{\pi}{2a}}\int_0^{\frac{\pi}{2a}} (\frac{a}{2}cos(ax))^2 dxdt\right]^{1/2} \|s_m(x) - s_n(x)\| \\ &\leq |\lambda|\sqrt{\frac{\pi^2}{32}}\,\|s_m(x) - s_n(x)\|, \end{aligned}$$

which implies that T has a fixed point. Also, putting $L = 0$ and $\alpha = |\lambda|\sqrt{\frac{\pi^2}{32}}$ shows that (1.1) holds for the nonlinear mapping T. All of the conditions of Theorem 1.1 hold for the nonlinear mapping T and hence it is T-stable.

5. Conclusion

In this work, we considered T-stability definition according to Qing and Rhoades [10] and we showed that the HPM was T-stable for solving integral equations. The sufficient condition for convergence of the method was presented and the examination of this condition for the integral equations and integro-differential equation was illustrated by presenting examples.

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