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Radical and It's Applications in BCH-Algebras

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Abstract. In this paper, for any ideal I of BCH-algebra X, we introduce the concept of $\sqrt{I}$ and show that it is an ideal of X, when I is a closed ideal. Then we verify some useful properties of it and prove that it is the union of all $k-$nil ideals of I. Moreover, if I is a closed ideal of X, then $\sqrt{I}$ is a closed translation ideal and so we can construct a quotient BCH-algebra. We prove this quotient BCH-algebra is a P-semisimple BCI-algebra and so it is an abelian group. Finally, we use the concept of radical in order to construct the second and the third isomorphism theorems.

Keywords: Ideal, radical, Quotient BCH-algebra, Maximal, Translation.

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1. Introduction and Preliminaries

In 1966, Imai and Iséki [13, 14] introduced two classes of abstract algebras : BCK-algebras and BCI-algebras. It is well-known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. Since then many authors work on various aspects of these algebras such as hyper and fuzzy structure [1, 8, 9, 20], topological view [19]. In 1983, Hu and Li [10, 11] introduced a new class of algebras so-called BCH-*algebras*. They proved that the class of BCI-algebras is a proper subclass of BCH-algebras. They studied some properties of this algebra. In [6], Dudek and Jun introduced the notion of *k-nil*

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radical in BCH-algebras. They showed that when I is a translation ideal of X, then the k-nil radical of I is also a translation ideal of X. In this paper, we generalize this concept and define the notion of *radical* in BCH-algebras. We prove that in any BCH-algebra (BCI-algebra) the radical (the k-nil radical) of a closed ideal (of an ideal) is a translation ideal. Then we verify some properties of radical and use it to construct a BCH-algebra without any nilpotent elements.

Definition 1.1. [10, 11] A *BCH-algebra* is an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the following conditions:
(BCH1) $(x * y) * z = (x * z) * y$,
(BCH2) $x * x = 0$,
(BCH3) $x * y = 0$ and $y * x = 0$ imply $y = x$.

In any BCH-algebra X, the following hold: for any $x, y \in X$,
(BCH4) $x * 0 = x$,
(BCH5) $0 * (x * y) = (0 * x) * (0 * y)$,
(BCH6) $0 * (0 * (0 * x)) = 0 * x$.

The set $P = \{x \in X | 0 * (0 * x) = x\}$ is called *P-semisimple* part of X. A BCH-algebra X is said to be *P-semisimple* if $P = X$.
A BCH-algebra X is called *BCI-algebra* if $((x * y) * (x * z)) * (z * y) = 0$, for all $x, y, z \in X$. The set $B = \{x \in X | 0 * x = 0\}$ is called the BCK-part of X. Moreover, if X is a BCI-algebra and $B = \{0\}$, then X is a P-semisimple BCI-algebra. We will also use the following notation for simplicity: $x * y^n = (...(x * \overbrace{y) * ...) * y}^{n\ time}$, where $x, y \in X$ and $n \in \mathbb{N}$. A BCI-algebra X is called *nilpotent* if for any $x \in X$ there is $n \in \mathbb{N}$ such that $0 * x^n = 0$.

Definition 1.2. [10, 11] A non-empty subset I of a BCH-algebra X is called an *ideal* if $0 \in I$ and $y * x \in I$ and $x \in I$ imply $y \in I$, for all $x, y \in X$. An ideal I is called *proper*, if $I \neq X$ and it is called *closed*, if $x * y \in I$, for all $x, y \in I$. If S is a subset of X, then the least ideal of X containing S is called the *generated ideal of X by S* and is denoted by $\langle S\rangle$. If X is a BCH-algebra, I and J are ideals of X, then we use $I + J$ to denote the ideal of X generated by $I \cup J$.

Theorem 1.3. [21] *A BCI-algebra X is nilpotent if and only if all ideals of X are closed.*

Theorem 1.4. [21] *Let S be a nonempty subset of a BCI-algebra X and*

$$A = \{x \in X | (...((x * a_1) * a_2) * ...) * a_n = 0,\ \textit{for some}\ n \in \mathbb{N}\ \textit{and}\ a_1, ..., a_n \in S\ \}$$

Then $\langle S\rangle = A \cup \{0\}$. If S contains a nilpotent element, then $\langle S\rangle = A$.

Note 1.5. If I is a closed ideal of BCH-algebra X and $x \in I$, then $0 * x \in I$. Moreover, if J is an ideal of BCH-algebra X such that $0*x \in J$, for any $x \in J$, then by (BCH1), $(x * y) * x = 0 * y \in J$, for any $x, y \in J$. Since J is an ideal, then $x * y \in J$, for any $x, y \in J$. Therefore, J is a closed ideal of X.

Lemma 1.6. [6] *Let X be a BCH-algebra. Then the following hold:*
(i) $0 * (0 * x)^n = 0 * (0 * x^n)$*, for any* $n \in \mathbb{N}$*,*
(ii) $0 * (x * y)^n = (0 * x^n) * (0 * y^n)$*, for any* $n \in \mathbb{N}$*.*

Definition 1.7. [18] Let X and Y be two BCH-algebras. A map $f : X \to Y$ is called a *BCH-homomorphism* if $f(x * y) = f(x) * f(y)$, for all $x, y \in X$. Clearly, if f is a BCH-homomorphism, then $f(0) = 0$.

Lemma 1.8. *Let X be a BCH-algebra. Then the map $f_n : X \to X$, is defined by $f_n(x) = 0 * x^n$, is a BCH-homomorphism, for all $n \in \mathbb{N}$.*

Proof. See Lemma 1.6(ii). □

Lemma 1.9. [5] *Let X be a BCH-algebra and f_0 be the map is defined in Lemma 1.8, Then $f_0(X)$ is a BCI-algebra.*

Note 1.10. Let A be an ideal of a BCI-algebra X. Define a binary relation θ on X as follows: $(x, y) \in \theta$ if and only if $x*y, y*x \in A$, for all $x, y \in X$. Then, θ is a congruence relation and it is called the congruence relation induced by A. We usually denote A_x for $[x] = \{y \in X | (x, y) \in \theta\}$. Moreover A_0 is the greatest closed ideal of X contained in A. Set $X/A = \{A_x | x \in X\}$. Then $(X/A, *, A_0)$ is a BCI-algebra, where $A_x * A_y = A_{x*y}$, for all $x, y \in X$ (See [21]).

Theorem 1.11. [18] *Let X, Y be two BCH-algebras and $f : X \to Y$ be a BCH-algebra homomorphism. Then $f(X) \cong X/Ker(f)$.*

Theorem 1.12. [21] *Let A be a closed ideal of a BCI-algebra X, $I(X, A)$ be the collection of all ideals of X containing A and $I(X/A)$ be the collection of all ideals of X/A. Then $\varphi : I(X, A) \to I(X/A)$, defined by $I \mapsto I/A$, is a bijection.*

Theorem 1.13. [4] *The category of BCH-algebras has arbitrary products. Let $\{X_j | j \in J\}$ be a family of BCH-algebras. Then $\prod_{j \in J} X_j = \{(x_j)_{j\in J} | x_j \in X_j, \forall j \in J\}$ is the product of this family.*

Definition 1.14. Let $(X, ., 0)$ be an abelian group. Then $(X, *, 0)$ is a P-semisimple BCI-algebra, where $x * y = x.y^{-1}$, for all $x, y \in X$. This BCI-algebra is called the adjoint BCI-algebra of $(X, ., 0)$ (See [21], Example 1.3.1.).

From now on, in this paper, we assume $X = (X, *, 0)$ be a BCH-algebra , unless otherwise stated.

2. Radical in BCH-algebras

In 1992, W. P. Huang [12] introduced the notion of nil ideal in BCI-algebras. In 1999, E. H. Roh and Y. B. Jun introduced nil ideals in BCH-algebras. They introduced nil subsets using nilpotent elements. Then W. A. Dudek and Y. B. Jun [6] introduced the notion of k-nil radicals in BCH-algebras. They showed that, if I is an ideal of X, then k-nil radical of I is an ideal, too. Moreover, k-nil radical of a translation ideal is again a translation ideal.

Lemma 2.1. *For any $x \in X$ and $n, m \in \mathbb{N}$, the following hold:*
(i) $0 * (0 * (0 * x^n)) = 0 * x^n$.
(ii) $0 * (0 * x^n)^m = 0 * (0 * x^{nm})$.

Proof. (i) Let $x \in X$ and $n \in \mathbb{N}$. Then

$$\begin{aligned}
0 * (0 * (0 * x^n)) &= 0 * (0 * (0 * x)^n), \quad \text{by Lemma 1.6(i)} \\
&= 0 * (0 * (0 * x))^n, \quad \text{by Lemma 1.6(i)} \\
&= (0 * (0 * (0 * x))) * (0 * (0 * x))^{n-1} \\
&= (0 * x) * (0 * (0 * x))^{n-1}, \quad \text{by (BCH6)} \\
&= ((0 * x) * (0 * (0 * x))) * (0 * (0 * x))^{n-2} \\
&= ((0 * (0 * (0 * x))) * x) * (0 * (0 * x))^{n-2}, \quad \text{by (BCH1)} \\
&= (0 * x^2) * (0 * (0 * x))^{n-2}, \quad \text{by (BCH6)} \\
&\ \vdots \\
&= (0 * x^{n-1}) * (0 * (0 * x)) \\
&= (0 * (0 * (0 * x))) * x^{n-1}, \quad \text{by (BCH1)} \\
&= (0 * x) * x^{n-1}, \quad \text{by (BCH6)} \\
&= 0 * x^n.
\end{aligned}$$

(ii) Suppose that $x \in X$ and $n, m \in \mathbb{N}$. Then

$$\begin{aligned}
0 * (0 * x^n)^m &= (...(0 * \overbrace{(0 * x^n)) * ...) * (0 * x^n)}^{m\ time} \\
&= (...((0 * (0 * x^n)) * \overbrace{(0 * x^n)) * ...) * (0 * x^n)}^{m-1\ time} \\
&= (...((0 * (0 * x)^n) * \overbrace{(0 * x^n)) * ...) * (0 * x^n)}^{m-1\ time}, \quad \text{by Lemma 1.6(i)} \\
&= ((...(0 * \overbrace{(0 * x^n)) * ...) * (0 * x^n)}^{m-1\ time})) * (0 * x)^n, \quad \text{by (BCH1)} \\
&= ((...(0 * \overbrace{(0 * x^n)) * ...) * (0 * x^n)}^{m-2\ time})) * (0 * x)^{2n} \\
&\ \vdots \\
&= 0 * (0 * x)^{mn}.
\end{aligned}$$

Now, by Lemma 1.6(i), we obtain $0 * (0 * x^n)^m = 0 * (0 * x^{nm})$. □

Definition 2.2. Let I be an ideal of X. The set

$$\{x \in X |\ 0 * x^n \in I \ \text{ and } \quad 0 * (0 * x^n) \in I, \text{ for some } n \in \mathbb{N}\}$$

is called the *radical* of I and is denoted by $\sqrt{I}$.

Example 2.3. Let $(\mathbb{Z}, -, 0)$ be the adjoint BCI-algebra of the abelian group $(\mathbb{Z}, +, 0)$. Let $Y = \{0, b, c, d\}$. Define the binary operation "$*'$" on Y by the following table:

Table 1

$*'$	0	b	c	d
0	0	0	0	0
b	b	0	d	d
c	c	0	0	c
d	d	0	0	0

Then $(Y, *', 0)$ is a BCH-algebra (See [3], Example 2.3). Now, let $X = \mathbb{Z} \cup \{b, c, d\}$ and define the operation "$*$" on X, by

$$x * y = \begin{cases} x - y & \text{if } x, y \in \mathbb{Z}, \\ x *' y & \text{if } x, y \in Y, \\ -y & \text{if } x \in Y, y \in \mathbb{Z} - \{0\}, \\ x & \text{if } x \in \mathbb{Z}, y \in Y. \end{cases}$$

Clearly, $*$ is well-defined and $x * x = 0$, for any $x \in X$. Let $x * y = 0 = y * x$, for some $x, y \in X$. Since $(Y, *', 0)$ and $(\mathbb{Z}, -, 0)$ are BCH-algebras, then $x, y \in \mathbb{Z}$ or $x, y \in Y$, implies $x = y$. If $x \in Y$ and $y \in \mathbb{Z} - \{0\}$, then $0 = x * y = -y$ and so $y = 0$. On the other hand, $0 = x * y = x * 0 = x$. Hence $x = y$. By a similar way if $x \in \mathbb{Z} - \{0\}$ and $y \in Y$, then $0 = x * y = x$ and so $x = 0$. Hence "$*$" satisfies in $(BCH3)$. Moreover, if $x, y, z \in X$. Then clearly, $x, y, z \in Y$ or $x, y, z \in \mathbb{Z}$ implies $(x*y)*z = (x*z)*y$. If $x \in \mathbb{Z}$, then $(x*y)*z = x = (x*z)*y$. Now, let $x \in Y$. If $y = 0$ or $z = 0$, then clearly, $(x * y) * z = (x * z) * y$. Let $y, z \in \mathbb{Z} - \{0\}$, then $(x * y) * z = -y - z = -z - y = (x * z) * y$. If $y \in Y$, then $(x*y)*z = (x*'y)*z = -z = -z*y = (x*z)*y$. Finally, if $z \in Y$ and $y \in \mathbb{Z}$, then $(x*y)*z = (-y)*z = -y = (x*z)*y$. Therefore, $(X, *, 0)$ is a BCH-algebra. Let $I = \{0\}$. If $x \in \mathbb{Z}$, then $0 * x^n = 0$ implies $-nx = 0$ and so $x = 0$, for all $n \in \mathbb{N}$. Hence $\sqrt{I} = \{x \in X | 0 * x^n = 0, 0 * (0 * x^n) = 0 \text{ for some } n \in \mathbb{N}\} = \{0, b, c, d\}$.

Corollary 2.4. *If I is a closed ideal, then*

$$\sqrt{I} = \{x \in X |\ 0 * x^n \in I, \textit{ for some } n \in \mathbb{N}\}.$$

Definition 2.5. [6] Let I be a non-empty subset of X. Then the set $\sqrt[k]{I} = \{x \in X | 0 * x^k \in I\}$ is called the k-nil radical of I.

In Corollary 2.6 we will obtain the relation between $\sqrt{I}$ and $\sqrt[n]{I}$, for any $n \in \mathbb{N}$.

Corollary 2.6. *Let I be a closed ideal of X. Then,*

(i) $\sqrt{I} = \bigcup_{n\in\mathbb{N}} \sqrt[n]{I}$.

(ii) If $x, y \in \sqrt{I}$, then there exists $m \in \mathbb{N}$ such that, $x, y \in \sqrt[m]{I}$.

Proof. (i) Let $x \in X$. Then

$$\begin{aligned} x \in \sqrt{I} \quad &\Leftrightarrow \quad 0 * x^n, 0 * (0 * x^n) \in I, \quad \text{for some } n \in \mathbb{N} \\ &\Leftrightarrow \quad 0 * x^n \in I, \quad \text{since } I \text{ is a closed ideal} \\ &\Leftrightarrow \quad x \in \sqrt[n]{I}, \quad \text{for some } n \in \mathbb{N}\ . \end{aligned}$$

Therefore, $\sqrt{I} = \bigcup_{n\in\mathbb{N}} \sqrt[n]{I}$, for some $n \in \mathbb{N}$.

(ii) Let $x, y \in \sqrt{I}$. Then there are $s, t \in \mathbb{N}$ such that, $0 * x^s, 0 * (0 * x^s) \in I$ and $0 * y^t, 0 * (0 * y^t) \in I$. Since I is closed, we have $0 * (0 * x^s)^t \in I$ and $0*(0*y^t)^s \in I$. Now, Lemma 2.1(ii), implies $0*(0*x^{st}) \in I$ and $0*(0*y^{ts}) \in I$ and so $0 * (0 * (0 * x^{st})) \in I$ and $0 * (0 * (0 * y^{ts})) \in I$. Hence by Lemma 2.1(i), we have $0 * x^{st}, 0 * y^{ts} \in I$, so $x, y \in \sqrt[st]{I}$. □

Theorem 2.7. *Let I be a closed ideal of X. Then $\sqrt{I}$ is a closed ideal of X.*

Proof. Obviously, $0 \in \sqrt{I}$. Let $x, y \in X$ such that $x * y, y \in \sqrt{I}$. Since I is closed, so by Corollary 2.4, $0*(x*y)^n \in I$ and $0*y^m \in I$, for some $n, m \in \mathbb{N}$. By lemma 2.1(ii), we have $(0*(0*(x*y)^{2n})) = (0*(0*(x*y))^n)*(0*(x*y))^n \in I$. By a similar argument we get that $(0 * (0 * (x * y)^{mn})) \in I$. Since I is a closed ideal of X, $0*(0*(0*(x*y)^{mn}))) \in I$. Then, by Lemma 2.1(i), $0*(x*y)^{mn} \in I$. Likewise, we can we obtain $0 * y^{mn} \in I$. Since I is an ideal of X, by Lemma 1.6, $0*x^{mn} \in I$ and so $x \in \sqrt{I}$. Hence $\sqrt{I}$ is an ideal of X. Now, let $x, y \in \sqrt{I}$. By a similar way as the proof of the last part, we can obtain $0 * x^{mn} \in I$ and $0*y^{mn} \in I$, for some $m, n \in \mathbf{N}$. Hence,$0*(x*y)^{mn} = (0*x^{mn})*(0*y^{mn}) \in I$ and so $x * y \in \sqrt{I}$. Therefore, $\sqrt{I}$ is a closed ideal of X. □

Definition 2.8. An element x of X is called *nilpotent* if $0 * x^n = 0$, for some $n \in \mathbb{N}$. The set of all nilpotent elements of X is denoted by $N(X)$ or $\sqrt{0}$.

Proposition 2.9. *$\sqrt{0}$ is a closed ideal of X.*

Proof. Since $I = \{0\}$ is a closed ideal of X, then by Theorem 2.7, $\sqrt{0}$ is a closed ideal of X. □

Example 2.10. (i) Let $(G, ., e)$ be the cyclic group of order three, $X = (\mathbb{Z}, *, 0)$ and $Y = (G, *, e)$ be the adjoint BCI-algebras of the abelian groups $(\mathbb{Z}, +, 0)$, and $(G, ., 0)$ respectively. Then by Theorem 1.13, we have $X \times Y$ is a BCH-algebra. In $X \times Y$, we have $(0, e) * (x, e)^n \neq (0, e)$, for all $x \in \mathbb{Z}\backslash\{0\}$. Hence $(x, e) \notin \sqrt{(0, e)}$, for all $x \in \mathbb{Z}\backslash\{0\}$. Also, $(0, e) * (0, y)^3 = (0, e)$, for all $y \in G$. Therefore, $\sqrt{(0, e)}$ is a proper ideal of $X \times Y$.
(ii) Let $X = (\mathbb{R}, *, 0)$ be the adjoint BCI-algebra of abelian group $(\mathbb{R}, +, 0)$.

That is $x * y = x + (-y)$, for all $x, y \in \mathbb{R}$. Let $a \in \mathbb{Q}$, where $\mathbb{Q}$ is the set of all rational numbers and $\langle\{a, -a\}\rangle$ be the ideal generated by $\{a, -a\}$. Then $\langle\{a, -a\}\rangle = \{x \in X \mid x * a^n = 0, \text{ for some } n \in \mathbb{N}\} = \{..., -2a, -a, 0, a, 2a, 3a, ...\}$ and so

$$\begin{aligned}\sqrt{\langle\{a,-a\}\rangle} &= \{x \in \mathbb{R} \mid 0 * x^n \in \langle\{a,-a\}\rangle, \exists n \in \mathbb{N}\} \\ &= \{x \in \mathbb{R} \mid -(nx) = \pm ma, \exists n, m \in \mathbb{N}\} \\ &= \{\pm \frac{m}{n} a \mid n, m \in \mathbb{N}\} \subseteq \mathbb{Q}.\end{aligned}$$

Therefore, $\sqrt{\langle\{a,-a\}\rangle}$ is a proper ideal of $(\mathbb{R}, *, 0)$.

In Proposition 2.11, we want to verify relation between $\sqrt{I}$ and the set of all nilpotent elements of X/I, for any ideal I of X.

Proposition 2.11. *Let X be a BCI-algebra, I be an ideal of X and I_x is an equivalence class of X containing x with respect to the congruence relation which is defined in Note 1.10, for any $x \in X$. Then $\sqrt{I} = \{x \in X \mid I_x \in N(X/I)\}$.*

Proof. Let $x \in X$. Then

$$\begin{aligned} I_x \in N(X/I) &\Leftrightarrow I_0 * I_x^n = I_0, \quad \text{for some } n \in \mathbb{N} \\ &\Leftrightarrow I_{0*x^n} = I_0 \\ &\Leftrightarrow 0 * x^n, 0 * (0 * x^n) \in I \\ &\Leftrightarrow x \in \sqrt{I}. \end{aligned}$$

Hence $\sqrt{I} = \{x \in X \mid I_x \in N(X/I)\}$. □

Theorem 2.12. *Let X be a BCI-algebra and I be an ideal of X. Then $\sqrt{I}$ is a closed ideal of X.*

Proof. Let $y, x * y \in \sqrt{I}$. Then by Proposition 2.11, $I_{x*y}, I_y \in N(X/I)$. By Proposition 2.9, we obtain $I_x \in N(X/I)$. Now, Proposition 2.11, implies $x \in \sqrt{I}$. Hence $\sqrt{I}$ is an ideal of X.
Let $x, y \in \sqrt{I}$. Then Proposition 2.11 implies that $I_x, I_y \in N(X/I)$. Now, by proposition 2.11, we have $I_{x*y} = I_x * I_y \in N(X/I)$. Therefore, $x * y \in \sqrt{I}$. □

In the next proposition, we try to obtain some useful properties of radical in BCH-algebras.

Proposition 2.13. *Let I and J be two ideals of X. Then the following assertions hold:*
(i) If I is a closed ideal of X, then $I \subseteq \sqrt{I}$.
(ii) If $I \subseteq J$, then $\sqrt{I} \subseteq \sqrt{J}$.
(iii) If I and J are closed, then $\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$.

(iv) If I is a closed or X is a BCI-algebra, then $\sqrt{\sqrt{I}} = \sqrt{I}$.
(v) Let Y be a BCH-algebra, and $f : X \to Y$ be a BCH-homomorphism. If I is an ideal of X and J is an ideal of Y, then $\sqrt{f^{-1}(J)} = f^{-1}(\sqrt{J})$ and $f(\sqrt{I}) \subseteq \sqrt{f(I)}$. Moreover, if f is onto and $kerf \subseteq I$, then $f(\sqrt{I}) = \sqrt{f(I)}$.

Proof. (i) Let $x \in I$. Since I is closed, then $0 * x \in I$ and so $x \in \sqrt{I}$.
(ii) Straightforward.
(iii) Since I and J are closed ideals, $I \cap J$ is also a closed ideal. Now, let $x \in \sqrt{I \cap J}$. Then by Corollary 2.6(i), there is an $n \in \mathbb{N}$ such that $0 * x^n, 0 * (0 * x^n) \in I \cap J$ and so $x \in \sqrt{I} \cap \sqrt{J}$. Hence, we have $\sqrt{I \cap J} \subseteq \sqrt{I} \cap \sqrt{J}$. Let $x \in \sqrt{I} \cap \sqrt{J}$. Then, there exist $m, n \in \mathbb{N}$ such that $0 * x^n \in \sqrt{I}$ and $0 * x^m \in \sqrt{J}$. By using the proof of Theorem 2.7, we have $0 * x^{mn} \in I \cap J$. Since $I \cap J$ is a closed ideal of X, then $x \in \sqrt{I \cap J}$. Hence $\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$.
(iv) Let I be a closed ideal of X. Then by (i), $I \subseteq \sqrt{I}$ and so (ii), implies $\sqrt{I} \subseteq \sqrt{\sqrt{I}}$. Now, let $x \in \sqrt{\sqrt{I}}$. Then there exists $n \in \mathbb{N}$ such that $0*x^n \in \sqrt{I}$. Likewise, there is $m \in \mathbb{N}$ such that $0*(0*x^n)^m \in I$. By Lemma 2.1(ii), we have $0 * (0 * x^{mn}) \in I$. Since I is closed we obtain $0 * x^{mn} = 0 * (0 * (0 * x^{mn})) \in I$. Hence $x \in \sqrt{I}$, whence $\sqrt{I} \subseteq \sqrt{\sqrt{I}}$. Now, let X be a BCI-algebra and I be an ideal of X. Then by (i), (ii), and Theorem 2.12, we have $\sqrt{I} \subseteq \sqrt{\sqrt{I}}$. Let $J = \sqrt{I}$ and $x \in \sqrt{J}$. Then there exists $n \in \mathbb{N}$ such that, $0*x^n, 0*(0*x^n) \in J$. Thus, $0 * (0 * x^n)^m, 0 * (0 * (0 * x^n)^m) \in I$, for some $m \in \mathbb{N}$. Also, by Lemma 2.1(ii), we have $0 * (0 * x^n)^m = 0 * (0 * x^{nm})$ and $0 * (0 * (0 * x^n)^m) = 0 * x^{mn}$. Hence $0 * x^{mn}, 0 * (0 * x^{mn}) \in I$ and so $x \in \sqrt{I} = J$. Therefore, $\sqrt{J} = J$.
(v) Let $x \in X$. Then

$$\begin{aligned} x \in \sqrt{f^{-1}(J)} &\Leftrightarrow 0 * x^n, 0 * (0 * x^n) \in f^{-1}(J), \quad \text{for some } n \in \mathbb{N} \\ &\Leftrightarrow f(0) * f(x)^n, f(0) * (f(0) * f(x)^n) \in J, \quad \text{for some } n \in \mathbb{N} \\ &\Leftrightarrow 0 * f(x)^n, 0 * (0 * f(x)^n) \in J, \quad \text{for some } n \in \mathbb{N} \\ &\Leftrightarrow f(x) \in \sqrt{J} \Leftrightarrow x \in f^{-1}(\sqrt{J}). \end{aligned}$$

Hence $f^{-1}(\sqrt{J}) = \sqrt{f^{-1}(J)}$.
Let $b \in f(\sqrt{I})$. Then there exists $a \in \sqrt{I}$ such that $f(a) = b$ and so $0 * a^n \in I$ and $0 * (0 * a^n) \in I$, for some $n \in \mathbb{N}$. Since f is a homomorphism, we have $0 * f(a)^n, 0 * (0 * f(a)^n) \in f(I)$. Hence, $b = f(a) \in \sqrt{f(I)}$, whence $f(\sqrt{I}) \subseteq \sqrt{f(I)}$. Now, let f be an onto homomorphism such that $kerf \subseteq I$ and $y \in \sqrt{f(I)}$. Then there exists $m \in \mathbb{N}$ such that $0 * y^m \in f(I)$ and $0 * (0 * y^m) \in f(I)$. Since f is onto, then $y = f(x)$, for some $x \in X$ and so $f(0 * x^m) = 0 * f(x)^m = 0 * y^m \in f(I)$. Hence there is $b \in I$, such that $f(0 * x^m) = f(b)$ and so $f((0 * x^m) * b) = f(0 * x^m) * f(b) = 0$. It follows that $(0 * x^m) * b \in kerf \subseteq I$. Since $b \in I$, then $0 * x^m \in I$. By a similar way we have $0 * (0 * x^m) \in I$ and so $x \in \sqrt{I}$. Therefore, $y = f(x) \in f(\sqrt{I})$, so $\sqrt{f(I)} \subseteq f(\sqrt{I})$. □

Proposition 2.14. *Let I and J be two closed ideals of BCI-algebra X. Then* $\sqrt{I+J} = \sqrt{\sqrt{I}+\sqrt{J}}$.

Proof. Since I, J are closed ideals, we have $I \subseteq \sqrt{I}$ and $J \subseteq \sqrt{J}$ and so $I + J \subseteq \sqrt{I} + \sqrt{J}$. Hence by Proposition 2.13(ii), $\sqrt{I+J} \subseteq \sqrt{\sqrt{I}+\sqrt{J}}$. Let $u \in \sqrt{\sqrt{I}+\sqrt{J}}$. Then $0 * u^n \in \sqrt{I} + \sqrt{J}$, for some $n \in \mathbb{N}$. By Theorem 1.4, there are $m \in \mathbb{N}$ and $a_1, ..., a_m \in \sqrt{I}$ such that

$$(...((0 * u^n) * a_1) * ...) * a_m \in \sqrt{J}, \quad (1)$$

By Corollary 2.6(ii), we can find $s \in \mathbb{N}$ such that $0 * a_i^s \in I$, for all $i \in \{1, 2, ..., m\}$. On the other hand, (1) implies there is $t \in \mathbb{N}$ such that $0 * ((...((0 * u^n) * a_1) * ...) * a_m)^t \in J$. Since I and J are closed ideals of X, likewise the proof of Theorem 2.7, we have $0 * a_i^{ts} \in I$, for all $i \in \{1, 2, ..., m\}$ and $0 * ((...((0 * u^n) * a_1) * ...) * a_m)^{ts} \in J$ and so by Lemma 1.6(ii),

$$(...((0 * (0 * u^n)^{ts}) * (0 * a_1^{ts})) * ...) * (0 * a_m^{ts}) \in J, \quad (2) \quad .$$

Since I is an ideal of X and $0 * a_i^{ts} \in I$, for all $i \in \{1, 2, ..., m\}$, then $0 * (0 * u^n)^{st} \in I + J$. Hence $0 * u^n \in \sqrt{I+J}$ and so $u \in \sqrt{\sqrt{I+J}}$. Hence by Proposition 2.13(iv), $u \in \sqrt{I+J}$. Summing up the above statements, we get $\sqrt{I+J} = \sqrt{\sqrt{I}+\sqrt{J}}$. □

The following example shows that if, I and J are not closed then, Proposition 2.14 may not be true.

Example 2.15. Let $X = (\mathbb{Z}, -, 0)$ be the BCI-algebra in Example 2.10(i). Assume that $I = \{0, 3, 6, 9, ...\}$ and $J = \{0, -3, -6, -9, ...\}$. Then clearly, I and J are ideals of X. Since $9, 6 \in I$ and $6 * 9 = -3 \notin I$, I is not closed. By a similar way, we can deduced that J is not closed. Moreover,

$$\begin{aligned} \sqrt{I} &= \{x \in X \mid 0 * x^n, 0 * (0 * x^n) \in I, \quad \text{for some } n \in \mathbb{N} \} \\ &= \{x \in X \mid nx, -nx \in I, \quad \text{for some } n \in \mathbb{N} \} \\ &= \{0\}. \end{aligned}$$

Similarly, we can obtain $\sqrt{J} = \{0\}$. Therefore, $\sqrt{\sqrt{I}+\sqrt{J}} = \sqrt{\{0\}} = \{0\}$. Also we have

$$\begin{aligned} I + J = \langle\{3, -3\}\rangle &= \{x \in \mathbb{Z} \mid x * a^n = 0, \quad \text{for some } n \in \mathbb{N}, a \in \{3, -3\} \} \\ &= \{..., -6, -3, 0, 3, 6, ...\}. \end{aligned}$$

Hence $\sqrt{I+J} = \{x \in \mathbb{Z} \mid 0 * x^n, 0 * (0 * x^n) \in I + J, \quad \text{for some } n \in \mathbb{N} \} = \mathbb{Z}$. Therefore, $\sqrt{I+J} \neq \sqrt{\sqrt{I}+\sqrt{J}}$.

Proposition 2.16. *Let M be a maximal ideal of a BCI-algebra X such that M is closed. Then* $\sqrt{M} = X$.

Proof. Since M is a closed ideal of X, then by Theorem 1.12, $\{M/M, X/M\}$ is the set of all ideals of X/M. Hence all ideal of X/M are closed (M/M is a zero ideal of X/M and so it is closed). Thus, by Theorem 1.3, X/M is nilpotent and so

$$\forall x \in X, \exists n \in N \ \text{ such that } \quad M_0 * M_x^n = M_0 \Rightarrow M_{0*x^n} = M_0.$$

Hence for all $x \in X$, $0 * x^n \in M$ and so $\sqrt{M} = X$. □

In the next example, we will show that if the ideal M is not closed, then Proposition 2.16 may not be true, in general.

Example 2.17. Let X be the BCI-algebra in Example 2.15, and let $M = \mathbb{N} \cup \{0\}$. Clearly, M is not closed (Since $2 * 3 = 2 - 3 = -1$) and M is a maximal ideal of X (See [21], Example 5.3.2). Let $x \in X$. Then

$$\begin{aligned} x \in \sqrt{M} \quad &\Leftrightarrow \quad 0 * x^n, 0 * (0 * x^n) \in M \\ &\Leftrightarrow \quad 0 - nx \in M \ \text{ and } \ 0 - (0 - nx) \in M \\ &\Rightarrow \quad nx, -nx \in M \\ &\Leftrightarrow \quad x = 0. \end{aligned}$$

Therefore, $\sqrt{M} = \{0\}$.

By Note 1.10, if I is an ideal of BCI-algebra X, then the relation $\theta = \{(x, y) \in X \times X | x * y, y * x \in I\}$, is a congruence relation of X, but it is not true for BCH-algebra in general case.

Example 2.18. Let $X = \{0, a, b, c, d, e, f, g, h, i, j, k\}$. Define the binary operation "*" on X by the following table:

Table 2

*	0	a	b	c	d	e	f	g	h	i	j	k
0	0	0	0	0	0	0	0	0	h	h	h	h
a	a	0	a	0	a	0	a	0	h	h	h	h
b	b	b	0	0	f	f	f	f	i	h	k	k
c	c	b	a	0	g	f	g	f	i	h	k	k
d	d	d	0	0	0	0	d	d	j	h	h	j
e	e	e	a	0	a	0	e	d	j	h	h	j
f	f	f	0	0	0	0	0	0	k	h	h	h
g	g	f	a	0	a	0	a	0	k	h	h	h
h	h	h	h	h	h	h	h	h	0	0	0	0
i	i	i	h	h	k	k	k	k	b	0	f	f
j	j	j	k	k	k	k	j	j	d	0	0	d
k	k	k	h	h	h	h	h	h	f	0	0	0

Then $(X,*,0)$ is a BCH-algebra (See [2] Example 7). Let $I=\{0,b,d,f\}$. Clearly, I is an ideal of X. Let $\theta=\{(x,y)\in X\times X|x*y,y*x\in I\}$. Then $c*a=b$ and $a*c=0$ and so $(a,c)\in\theta$. Moreover, $e*c=0$ and $c*e=f$ and so $(e,c)\in\theta$. But, $(c*c,e*a)=(0,e)\notin\theta$. It follows that θ is not a congruence relation on X.

Definition 2.19. [18] A *translation* ideal of X is an ideal U of X such that:

$$\forall x,y,z\in X, x*y\in U, y*x\in U\Rightarrow (x*z)*(y*z)\in U, (z*x)*(z*y)\in U.$$

Remark 2.20. Let U be a translation ideal of BCH-algebra X. Then the relation θ, was defined in Note 1.10, is a congruence relation on X. By U_x we denote the equivalence class containing x and by X/U we denote the set of all equivalence classes with respect to this congruence relation. Then $(X/U,*,U_0)$ is a BCH-algebra, where $U_x*U_y=U_{x*y}$, for all $x,y\in X$. Moreover, $kerf$ is a translation ideal for any BCH-homomorphism f (See [18]).

Dudek and Jun in [6], prove that if U is a translation ideal of X, then so is $\sqrt[n]{U}$, for any $n\in\mathbb{N}$. In Theorem 2.21, we will show that if I is a closed ideal of X, then $\sqrt[n]{I}$ is a translation ideal of X, for any $n\in\mathbb{N}$.

Theorem 2.21. *Let I be a closed ideal of X. Then,*
(i) $\sqrt[n]{I}$ is a translation ideal of X, for all $n\in\mathbb{N}$.
(ii) $\sqrt{I}$ is a translation ideal of X.

Proof. (i) Let $x,y,z\in I$, such that $x*y,y*x\in\sqrt[n]{I}$. Then $0*(x*y)^n\in I$ and $0*(y*x)^n\in I$. By Lemma 1.9, we have

$$([(0*(0*x^n))*(0*(0*z^n))]*[(0*(0*y^n))*(0*(0*z^n))])*[(0*(0*x^n))*(0*(0*y^n))]=0.$$

Since I is a closed ideal, then $0*(0*(x*y)^n)\in I$ and so by Lemma 1.6(ii), $(0*(0*x^n))*(0*(0*y^n))\in I$. Hence $[(0*(0*x^n))*(0*(0*z^n))]*[(0*(0*y^n))*(0*(0*z^n))]\in I$. Now, since I is closed, then $0*([(0*(0*x^n))*(0*(0*z^n))]*[(0*(0*y^n))*(0*(0*z^n))])\in I$, so Lemma 2.1(i) and 1.6, imply that

$$0*((x*z)*(y*z))^n=[(0*x^n)*(0*z^n)]*[(0*y^n)*(0*z^n)]\in I.$$

Hence, $(x*z)*(y*z)\in\sqrt[n]{I}$. By a similar way, $(z*x)*(z*y)\in\sqrt[n]{I}$. Thus, $\sqrt[n]{I}$ is a translation ideal of X.
(ii) Let $x,y,z\in X$ such that $x*y,y*x\in\sqrt{I}$. By Corollary 2.6(ii), there is $n\in\mathbb{N}$ such that $x*y,y*x\in\sqrt[n]{I}$ and so by (i), $(x*z)*(y*z),(z*x)*(z*y)\in\sqrt[n]{I}$. Hence, by Corollary 2.6(i), $(x*z)*(y*z),(z*x)*(z*y)\in\sqrt{I}$. Therefore, $\sqrt{I}$ is a translation ideal of X. □

Corollary 2.22. *Let I be a closed ideal of X. Then $(X/\sqrt{I},*,(\sqrt{I})_0)$ is a BCH-algebra.*

Proof. By Theorem 2.21, $\sqrt{I}$ is a translation ideal of X, so by Remark 2.20, $(X/\sqrt{I},*,0)$ is a BCH-algebra. □

In Corollary 2.22, we proved that if I is a closed ideal of X, then $X/\sqrt{I}$ is a BCH-algebra. In the next proposition we show that it has no non zero nilpotent elements.

Proposition 2.23. *Let J be a closed ideal of X and $I = \sqrt{J}$. Then BCH-algebra $(X/I, *, I_0)$ does not have any non zero nilpotent elements.*

Proof. Let $I_x \in X/\sqrt{0}$. Then

$$\begin{aligned} I_x \text{ is nilpotent} \Leftrightarrow I_0 * I_x^n = I_0, \text{ for some } n \in \mathbb{N} \quad &\Leftrightarrow \quad I_{0*x^n} = I_0 \\ &\Leftrightarrow \quad 0 * x^n, 0 * (0 * x^n) \in I. \end{aligned}$$

Therefore, $0 * (0 * x^n)^m \in J$ and $0 * (0 * (0 * x^n))^t \in J$, for some $n, t \in \mathbb{N}$. By $0 * (0 * x^n)^m \in J$ and Lemma 2.1(ii), one has $0 * (0 * x^{mn}) \in J$. Also by Lemma2.1 and 1.6(i), the following hold:

$$0 * (0 * x)^{mn} = 0 * (0 * x^{mn}) \in J. \quad (1)$$

By $0 * (0 * (0 * x^n))^t \in J$ and Lemma 1.6(i), we have

$$0 * (0 * (0 * x^{nt}) = 0 * (0 * (0 * x^n)^t) = 0 * (0 * (0 * x^n))^t \in J.$$

and so Lemma 2.1(i), implies that

$$0 * (0 * (0 * x^{nt}) = 0 * x^{nt} \in J \quad (2) \quad .$$

It follows from (1),(2) and Corollary 2.4 that $0 * x, x \in \sqrt{J} = I$. Thus $I_0 = I_x$. Therefore, I_0 is the only nilpotent element of X/I. □

Proposition 2.24. *For any $x, y, z \in X$, we have $((x * y) * (x * z)) * (z * y) \in N(X)$. Moreover, $\{((x * y) * (x * z)) * (z * y) | \ x, y, z \in X\} \subseteq \sqrt{I}$, for all ideal I of X.*

Proof. Let $x, y, z \in X$. Then Lemma 1.6(i), implies

$$0*(((x*y)*(x*z))*(z*y)) = (((0*x)*(0*y))*((0*x)*(0*z)))*((0*z)*(0*y)).$$

By Lemma 1.9, $f_0(X)$ is a BCI-algebra, thus

$$(((0 * x) * (0 * y)) * ((0 * x) * (0 * z))) * ((0 * z) * (0 * y)) = 0$$

Therefore, $0*(((x*y)*(x*z))*(z*y)) = 0$. That is $(((x*y)*(x*z))*(z*y)) \in N(X)$. Now, let I be an ideal of X. Then by Proposition 2.13(ii), $N(X) \subseteq \sqrt{I}$ and so $\{((x * y) * (x * z)) * (z * y) | x, y, z \in X\} \subseteq \sqrt{I}$. It completes the proof of this proposition. □

Corollary 2.25. *Let I be a closed ideal of X. Then $(X/J, *, J_0)$ is a P-semisimple BCI-algebra, where $J = \sqrt{I}$.*

Proof. By Corollary 2.22, $(X/J, *, J_0)$ is a BCH- algebra. Let $J_x, J_y, J_z \in X/J$. Then

$$((J_x * J_y) * (J_x * J_z)) * (J_z * J_y) = J_{((x*y)*(x*z))*(z*y)}.$$

By Proposition 2.24, $((x * y) * (x * z)) * (z * y) \in J$. Since J is a closed ideal of X, we obtain $J_{((x*y)*(x*z))*(z*y)} = J_0$. Hence $((J_x * J_y) * (J_x * J_z)) * (J_z * J_y) = J_0$. It follows that $(X/J, *, J_0)$ is a BCI-algebra. Now, by Proposition 2.23, $(X/J, *, J_0)$ does not have any nilpotent element and so BCK-part of X/J is the set $\{I_0\}$. Therefore, $(X/J, *, J_0)$ is a P-semisimple BCI-algebra. □

Remark 2.26. We know that each abelian group induces a P-semisimple BCI-algebra and the opposite process is still true (See [21]). Hence Corollary 2.25, implies for any closed ideal I of BCH-algebra X we can find an abelian group. It is $(X/J, .)$, where $J = \sqrt{I}$ and $J_x.J_y = J_{x*(0*y)}$, for all $x, y \in X$.

Theorem 2.27. *Let I and J be two ideals of X, such that $I \subseteq J$ and let I be a translation ideal of X. Then J/I is an ideal of X, where $J/I = \{I_x | x \in J\}$. Moreover, $I_x \in J/I$ if and only if $x \in J$ (See [18]).*

Theorem 2.28. *Let H be a subalgebra of X and K be a closed ideal of X. Then $\frac{H\sqrt{K}}{\sqrt{K}} \cong \frac{H}{H\cap\sqrt{K}}$, where $H\sqrt{K} = \bigcup\{(\sqrt{K})_h | h \in H\}$.*

Proof. Let $I = \sqrt{K}$. By Theorem 2.7, I is a closed ideal of X and so $I_0 = \{x \in X | x * 0, 0 * x \in I\} = I$. Hence $I \subseteq H\sqrt{K}$. If $x, y \in H\sqrt{K}$, then there are $a, b \in H$ such that $x \in I_a$ and $y \in I_b$ and so $I_x = I_a$ and $I_y = I_b$. Hence $x * y \in I_{x*y} = I_{a*b}$ and $a * b \in H$. It follows that $x * y \in H\sqrt{K}$, so $H\sqrt{K}$ is a subalgebra of X containing $\sqrt{K}$. Thus by Corollary 2.22, $\frac{H\sqrt{K}}{\sqrt{K}}$ is a BCH-algebra. Since $\sqrt{K}$ is a translation ideal of X, then $H\cap\sqrt{K}$ is a translation ideal of H and so by Remark 2.20, $\frac{H}{H\cap\sqrt{K}}$ is a BCH-algebra. Define $\varphi : H \to HI$ by $\varphi(h) = I_h$, for all $h \in H$. It is easily seen that, φ is a homomorphism. Let $I_x \in \frac{HI}{I}$. Then there exists $h \in H$ such that $x \in I_h$. Therefore, $I_x = I_h$ and so $I_x = \varphi(x) = \varphi(h)$. Thus φ is epimorphism. Moreover,

$$x \in ker\varphi \Leftrightarrow I_x = \varphi(x) = I_0 \Leftrightarrow x * 0 \in I \Leftrightarrow x \in H \cap I.$$

Therefore, $Ker(\varphi) = H \cap I$. Now, by Theorem 1.11, we have $\frac{H\sqrt{K}}{\sqrt{K}} \cong \frac{H}{H\cap\sqrt{K}}$. □

Theorem 2.29. *Let K and A be two closed ideals of X and $A \subseteq K$. Suppose that $\sqrt{K}/\sqrt{A} = \{(\sqrt{A})_x | x \in \sqrt{K}\}$. Then $\frac{X}{\sqrt{K}} \cong \frac{X/\sqrt{A}}{\sqrt{K}/\sqrt{A}}$.*

Proof. By Corollary 2.22, $\frac{X}{\sqrt{K}}$ and $\frac{X}{\sqrt{A}}$ are BCH-algebras. Now, let $f : \frac{X}{\sqrt{A}} \to \frac{X}{\sqrt{K}}$ be defined by $(\sqrt{A})_x \mapsto (\sqrt{K})_x$. If $(\sqrt{A})_x = (\sqrt{A})_y$, for $x, y \in X$, then $x * y, y * x \in \sqrt{A}$. Since $A \subseteq K$, by Proposition 2.13(ii), we have $x * y, y * x \in \sqrt{K}$. Hence $(\sqrt{K})_x = (\sqrt{K})_y$. Thus f is well defined. Clearly, f is an epimorphic. Now, let $(\sqrt{A})_x \in Ker(f)$. Then $(\sqrt{K})_x = (\sqrt{K})_0$ and so $x \in \sqrt{K}$. Hence $(\sqrt{A})_x \in \sqrt{K}/\sqrt{A}$. On the other hand, if $(\sqrt{A})_x \in \sqrt{K}/\sqrt{A}$, then $x \in \sqrt{K}$. Since $\sqrt{K}$ is closed, we have $(\sqrt{K})_x = (\sqrt{K})_0$. Hence $(\sqrt{A})_x \in Ker(f)$. Therefore, $Ker(f) = \sqrt{K}/\sqrt{A}$. Now, by Theorem 1.11, we have $\frac{X}{\sqrt{K}} \cong \frac{X/\sqrt{A}}{\sqrt{K}/\sqrt{A}}$. □

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