OD-characterization of Almost Simple Groups Related to $D_4(4)$

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**Abstract.** Let $G$ be a finite group and $\pi_e(G)$ be the set of orders of all elements in $G$. The set $\pi_e(G)$ determines the prime graph (or Grunberg-Kegel graph) $\Gamma(G)$ whose vertex set is $\pi(G)$. The set of primes dividing the order of $G$, and two vertices $p$ and $q$ are adjacent if and only if $pq \in \pi_e(G)$. The degree $\text{deg}(p)$ of a vertex $p \in \pi(G)$, is the number of edges incident on $p$. Let $\pi(G) = \{p_1, p_2, ..., p_k\}$ with $p_1 < p_2 < ... < p_k$. We define $D(G) := (\text{deg}(p_1), \text{deg}(p_2), ..., \text{deg}(p_k))$, which is called the degree pattern of $G$. The group $G$ is called $k$-fold OD-characterizable if there exist exactly $k$ non-isomorphic groups $M$ satisfying conditions $|G| = |M|$ and $D(G) = D(M)$. Usually a 1-fold OD-characterizable group is simply called OD-characterizable. In this paper, we classify all finite groups with the same order and degree pattern as an almost simple groups related to $D_4(4)$.

**Keywords:** Degree pattern, $k$-fold OD-characterizable, Almost simple group.

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Let $G$ be a finite group, $\pi(G)$ the set of all prime divisors of $|G|$ and $\pi_e(G)$ be the set of orders of elements in $G$. The prime graph (or Grunberg-Kegel graph) $\Gamma(G)$ of $G$ is a simple graph with vertex set $\pi(G)$ in which two vertices $p$ and $q$ are joined by an edge (and we write $p \sim q$) if and only if $G$ contains an element of order $pq$ (i.e. $pq \in \pi_e(G)$).

The degree $\deg(p)$ of a vertex $p \in \pi(G)$ is the number of edges incident on $p$. If $\pi(G) = \{p_1, p_2, \ldots, p_k\}$ with $p_1 < p_2 < \ldots < p_k$, then we define $D(G) := (\deg(p_1), \deg(p_2), \ldots, \deg(p_k))$, which is called the degree pattern of $G$, and leads a following definition.

**Definition 1.1.** The finite group $G$ is called $k$-fold OD-characterizable if there exist exactly $k$ non-isomorphic groups $H$ satisfying conditions $|G| = |H|$ and $D(G) = D(H)$. In particular, a 1-fold OD-characterizable group is simply called OD-characterizable.

The interest in characterizing finite groups by their degree patterns started in [7] by M. R. Darafsheh and et. al., in which the authors proved that the following simple groups are uniquely determined by their order and degree patterns: All sporadic simple groups, the alternating groups $A_p$ with $p$ and $p - 2$ primes and some simple groups of Lie type. Also in a series of articles (see [4, 6, 8, 9, 14, 17]), it was shown that many finite simple groups are OD-characterizable.

Let $A$ and $B$ be two groups then a split extension is denoted by $A : B$. If $L$ is a finite simple group and $\text{Aut}(L) \cong L : A$, then if $B$ is a cyclic subgroup of $A$ of order $n$ we will write $L : n$ for the split extension $L : B$. Moreover if there are more than one subgroup of orders $n$ in $A$, then we will denote them by $L : n_1$, $L : n_2$, etc.

**Definition 1.2.** A group $G$ is said to be an almost simple group related to $S$ if and only if $S \leq G \leq \text{Aut}(S)$, for some non-abelian simple group $S$.

In many papers (see [2, 3, 10, 13, 15, 16]), it has been proved, up to now, that many finite almost simple groups are OD-characterizable or $k$-fold OD-characterizable for certain $k \geq 2$.

We denote the socle of $G$ by $\text{Soc}(G)$, which is the subgroup generated by the set of all minimal normal subgroups of $G$. For $p \in \pi(G)$, we denote by $G_p$ and $\text{Syl}_p(G)$ a Sylow $p$-subgroup of $G$ and the set of all Sylow $p$-subgroups of $G$ respectively, all further unexplained notation are standard and can be found in [11].

In this article our main aim is to show the recognizability of the almost simple groups related to $L := D(4)$ by degree pattern in the prime graph and
order of the group. In fact, we will prove the following theorem.

**Main Theorem** Let \( M \) be an almost simple group related to \( L := D_{4}(4) \). If \( G \) is a finite group such that \( D(G) = D(M) \) and \( |G| = |M| \), then the following assertions hold:

(a) If \( M = L \), then \( G \cong L \).
(b) If \( M = L : 2_{1} \), then \( G \cong L : 2_{1} \) or \( L : 2_{3} \).
(c) If \( M = L : 2_{2} \), then \( G \cong L : 2_{2} \) or \( Z_{2} \times L \).
(d) If \( M = L : 2_{3} \), then \( G \cong L : 2_{3} \) or \( L : 2_{1} \).
(e) If \( M = L : 3 \), then \( G \cong L : 3 \) or \( Z_{3} \times L \).
(f) If \( M = L : 2^{2} \), then \( G \cong L : 2^{2} \), \( Z_{2} \times (L : 2_{1}) \), \( Z_{2} \times (L : 2_{2}) \), \( Z_{2} \times (L : 2_{3}) \), \( Z_{4} \times L \) or \((Z_{2} \times Z_{2}) \times L \).
(g) If \( M = L : (D_{6})_{1} \), then \( G \cong L : (D_{6})_{1} \), \( L : 6 \), \( Z_{3} \times (L : 2_{1}) \), \( Z_{3} \times (L : 2_{3}) \) or \((Z_{3} \times L) \cdot Z_{2} \).
(h) If \( M = L : (D_{6})_{2} \), then \( G \cong L : (D_{6})_{2} \), \( Z_{2} \times (L : 3) \), \( Z_{3} \times (L : 2_{2}) \), \((Z_{3} \times L) \cdot Z_{2} \), \( Z_{2} \times L \) or \( D_{6} \times L \).
(i) If \( M = L : 6 \), then \( G \cong L : 6 \), \( L : (D_{6})_{1} \), \( Z_{3} \times (L : 2_{1}) \), \( Z_{3} \times (L : 2_{3}) \) or \((Z_{3} \times L) \cdot Z_{2} \).
(j) If \( M = L : D_{12} \), then \( G \cong L : D_{12} \), \( Z_{2} \times (L : (D_{6})_{1}) \), \( Z_{2} \times (L : (D_{6})_{2}) \), \( Z_{2} \times (L : 6) \), \( Z_{3} \times (L : 2^{2}) \), \( Z_{3} \times (L : 2_{1}) \), \( Z_{2} \), \( Z_{2} \times (L : 2_{2}) \), \( Z_{2} \times (L : 3) \), \( Z_{2} \times (L : 2_{3}) \), \( Z_{2} \times (L : 2_{3}) \), \( Z_{3} \times (L : 2_{1}) \), \( Z_{6} \times (L : 2_{2}) \), \( Z_{6} \times (L : 2_{3}) \), \( Z_{4} \times L \), \( Z_{2} \cdot D_{6} \times (L : 2_{1}) \), \( D_{6} \times (L : 2_{2}) \), \( D_{6} \times (L : 2_{3}) \), \( Z_{12} \times L \), \( (Z_{2} \times Z_{6}) \times L \), \( (Z_{2} \times L) \cdot D_{6} \), \( A_{4} \times L \), \( L \cdot A_{4} \), \( D_{12} \times L \) or \( T \times L \).

2. Preliminary Results

It is well-known that \( \text{Aut}(D_{4}(4)) \cong D_{4}(4) : D_{12} \) where \( D_{12} \) denotes the dihedral group of order 12. We remark that \( D_{12} \) has the following non-trivial proper subgroups up to conjugacy: three subgroups of order 2, one cyclic subgroup each of order 3 and 6, two subgroups isomorphic to \( D_{6} \cong S_{3} \) and one subgroup of order 4 isomorphic to the Klein’s four group denoted by \( 2^{2} \). The field and the duality automorphisms of \( D_{4}(4) \) are denoted by \( 2_{1} \) and \( 2_{2} \) respectively, and we set \( 2_{3} = 2_{1} \cdot 2_{2} \) (field*duality which is called the diagonal automorphism).

Therefore up to conjugacy we have the following almost simple groups related to \( D_{4}(4) \).

**Lemma 2.1.** If \( G \) is an almost simple group related to \( L := D_{4}(4) \), then \( G \) is isomorphic to one of the following groups:

- \( L, L : 2_{1}, L : 2_{2}, L : 2_{3}, L : 3, L : 2^{2}, L : (D_{6})_{1}, L : (D_{6})_{2}, L : 6, L : D_{12} \).
Let $p \geq 5$ be a prime. We denote by $\mathcal{S}_p$ the set of all simple groups with prime divisors at most $p$. Clearly, if $q \leq p$, then $\mathcal{S}_q \subseteq \mathcal{S}_p$. We list all the simple groups in class $\mathcal{S}_{17}$ with their order and the order of their outer automorphisms in TABLE 1, taken from [12].

**TABLE 1: Simple groups in $\mathcal{S}_p$, $p \leq 17$.**

| $S$  | $|S|$  | $|\text{Out}(S)|$ | $S$  | $|S|$  | $|\text{Out}(S)|$ |
|------|-------|-------------------|------|-------|-------------------|
| $A_5$ | $2^2 \cdot 3 \cdot 5$ | 2 | $G_2(3)$ | $2^6 \cdot 3^2 \cdot 7 \cdot 13$ | 2 |
| $A_6$ | $2^3 \cdot 3^2 \cdot 5$ | 4 | $3D_4(2)$ | $2^{12} \cdot 3^4 \cdot 7^2 \cdot 13$ | 3 |
| $S_4(3)$ | $2^6 \cdot 3^4 \cdot 5$ | 2 | $L_2(64)$ | $2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$ | 6 |
| $L_2(7)$ | $2^3 \cdot 3 \cdot 7$ | 2 | $U_3(5)$ | $2^7 \cdot 3^4 \cdot 5^6 \cdot 7 \cdot 13$ | 4 |
| $L_2(8)$ | $2^3 \cdot 3^2 \cdot 7$ | 3 | $L_2(9)$ | $2^7 \cdot 3^6 \cdot 5 \cdot 7 \cdot 13$ | 4 |
| $U_3(3)$ | $2^3 \cdot 3^3 \cdot 7$ | 2 | $S_6(3)$ | $2^9 \cdot 3^3 \cdot 5 \cdot 7 \cdot 13$ | 2 |
| $A_7$ | $2^3 \cdot 3^2 \cdot 5 \cdot 7$ | 2 | $O_7(3)$ | $2^9 \cdot 3^3 \cdot 5 \cdot 7 \cdot 13$ | 2 |
| $L_2(49)$ | $2^3 \cdot 3 \cdot 5^2 \cdot 7^2$ | 4 | $G_2(4)$ | $2^{12} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13$ | 2 |
| $U_3(5)$ | $2^3 \cdot 3^2 \cdot 5^3 \cdot 7$ | 6 | $S_4(8)$ | $2^{12} \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 13$ | 6 |
| $L_2(4)$ | $2^6 \cdot 3^2 \cdot 5 \cdot 7$ | 12 | $O_6^+(3)$ | $2^{12} \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$ | 24 |
| $A_8$ | $2^6 \cdot 3^2 \cdot 5 \cdot 7$ | 2 | $L_3(3)$ | $2^9 \cdot 3^{10} \cdot 5 \cdot 11 \cdot 13$ | 2 |
| $A_9$ | $2^6 \cdot 3^4 \cdot 5 \cdot 7$ | 2 | $A_{13}$ | $2^{10} \cdot 3^5 \cdot 5 \cdot 7 \cdot 11 \cdot 13$ | 2 |
| $J_2$ | $2^7 \cdot 3^3 \cdot 5^2 \cdot 7$ | 2 | $A_{14}$ | $2^{10} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11 \cdot 13$ | 2 |
| $A_{10}$ | $2^7 \cdot 3^4 \cdot 5^2 \cdot 7$ | 2 | $A_{15}$ | $2^{10} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$ | 2 |
| $U_4(3)$ | $2^7 \cdot 3^6 \cdot 5 \cdot 7$ | 8 | $L_6(3)$ | $2^{11} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 12^2$ | 4 |
| $S_4(7)$ | $2^6 \cdot 3^2 \cdot 5^2 \cdot 7^4$ | 2 | $S_{12}$ | $2^{13} \cdot 3^4 \cdot 5 \cdot 7 \cdot 11 \cdot 13$ | 2 |
| $S_6(2)$ | $2^9 \cdot 3^4 \cdot 5 \cdot 7$ | 1 | $A_{16}$ | $2^{14} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$ | 2 |
| $O_6^+(2)$ | $2^{12} \cdot 3^6 \cdot 5^2 \cdot 7$ | 6 | $F_{422}$ | $2^{17} \cdot 3^8 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$ | 2 |
| $L_2(11)$ | $2^2 \cdot 3 \cdot 5 \cdot 11$ | 2 | $L_2(17)$ | $2^4 \cdot 3^2 \cdot 17$ | 2 |
| $M_{11}$ | $2^3 \cdot 3^2 \cdot 5 \cdot 11$ | 1 | $L_2(16)$ | $2^3 \cdot 5 \cdot 17$ | 4 |
| $M_{12}$ | $2^6 \cdot 3^3 \cdot 5 \cdot 11$ | 2 | $S_4(4)$ | $2^8 \cdot 3^5 \cdot 5^2 \cdot 17$ | 4 |
| $U_3(2)$ | $2^{10} \cdot 3^5 \cdot 5 \cdot 11$ | 2 | $He$ | $2^{10} \cdot 3^5 \cdot 5 \cdot 7 \cdot 17$ | 2 |
| $M_{22}$ | $2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$ | 2 | $O_{6}(2)$ | $2^{12} \cdot 3^4 \cdot 5 \cdot 7 \cdot 17$ | 2 |
| $A_{11}$ | $2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$ | 2 | $L_4(4)$ | $2^{12} \cdot 3^4 \cdot 5 \cdot 7 \cdot 17$ | 4 |
| $M_{10}^L$ | $2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$ | 2 | $S_6(2)$ | $2^{16} \cdot 3^5 \cdot 5 \cdot 7 \cdot 17$ | 1 |
| $H_8$ | $2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$ | 2 | $U_4(4)$ | $2^{12} \cdot 3^4 \cdot 5 \cdot 7 \cdot 13 \cdot 17$ | 4 |
| $A_{12}$ | $2^9 \cdot 3^5 \cdot 5 \cdot 7 \cdot 11$ | 2 | $U_3(17)$ | $2^6 \cdot 3^4 \cdot 7 \cdot 13 \cdot 17^2$ | 6 |
| $U_6(2)$ | $2^{15} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11$ | 6 | $O_{16}^{2}(2)$ | $2^{20} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11 \cdot 17$ | 2 |
| $L_3(3)$ | $2^2 \cdot 3^3 \cdot 13$ | 2 | $L_2(13)^2$ | $2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 17$ | 4 |
| $L_2(25)$ | $2^3 \cdot 3 \cdot 5^2 \cdot 13$ | 4 | $S_4(13)$ | $2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13^4 \cdot 17$ | 2 |
| $U_3(4)$ | $2^6 \cdot 3 \cdot 5^2 \cdot 13$ | 4 | $L_3(16)$ | $2^{12} \cdot 3^2 \cdot 5 \cdot 7 \cdot 13 \cdot 17$ | 24 |
| $S_4(5)$ | $2^6 \cdot 3 \cdot 5^4 \cdot 13$ | 2 | $S_6(4)$ | $2^{18} \cdot 3^4 \cdot 5 \cdot 7 \cdot 13 \cdot 17$ | 2 |
| $L_4(3)$ | $2^7 \cdot 3^6 \cdot 5 \cdot 13$ | 4 | $O_{8}^{2}(4)$ | $2^{24} \cdot 3^3 \cdot 5^4 \cdot 7 \cdot 13 \cdot 17^2$ | 12 |
| $2F_4(2)'$ | $2^{11} \cdot 3^3 \cdot 5^2 \cdot 13$ | 2 | $F_4(2)$ | $2^{24} \cdot 3^6 \cdot 5 \cdot 7^2 \cdot 13 \cdot 17$ | 2 |
| $L_2(13)$ | $2^2 \cdot 3 \cdot 7 \cdot 13$ | 2 | $A_{17}$ | $2^{14} \cdot 3^6 \cdot 5 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17$ | 2 |
| $L_2(27)$ | $2^2 \cdot 3^3 \cdot 7 \cdot 13$ | 6 | $A_{18}$ | $2^{15} \cdot 3^8 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17$ | 2 |
Definition 2.3. A completely reducible group will be called a CR-group. The center of a CR-group is a direct product of the abelian factor in the decomposition. Hence, a CR-group is centerless, that is, has trivial center, if and only if it is a direct product of non-abelian simple groups. The following Lemma determines the structure of the automorphism group of a centerless CR-group.

Lemma 2.3 ([11]). Let \( R \) be a finite centerless CR-group and write \( R = R_1 \times R_2 \times \ldots \times R_k \), where \( R_i \) is a direct product of \( n_i \) isomorphic copies of a simple group \( H_i \), and \( H_i \) and \( H_j \) are not isomorphic if \( i \neq j \). Then \( \text{Aut}(R) = \text{Aut}(R_1) \times \text{Aut}(R_2) \times \ldots \times \text{Aut}(R_k) \cong \text{Aut}(H_i) \wr S_{n_i} \), where in this wreath product \( \text{Aut}(H_i) \) appears in its right regular representation and the symmetric group \( S_{n_i} \) in its natural permutation representation. Moreover, these isomorphisms induce isomorphisms \( \text{Out}(R) \cong \text{Out}(R_1) \times \text{Out}(R_2) \times \ldots \times \text{Out}(R_k) \cong \text{Out}(H_i) \wr S_{n_i} \).

3. OD-Characterization of Almost Simple Groups Related to \( D_4(4) \)

In this section, we study the problem of characterizing almost simple groups by order and degree pattern. Especially we will focus our attention on almost simple groups related to \( L = D_4(4) \), namely, we will prove the Main Theorem of Sec. 1. We break the proof into a number of separate propositions.

By assumption, we depict all possibilities for the prime graph associated with \( G \) by use of the variables for some vertices in each proposition. Also, we need to know the structure of \( \Gamma(M) \) to determine the possibilities for \( G \) in some proposition, therefore we depict the prime graph of all extension of \( L \) in pages 18 to 20. Note that the set of order elements in each of the following propositions is calculated using Magma.

Proposition 3.1. If \( M = L \), then \( G \cong L \).

Proof. By TABLE 1 \( |L| = 2^{24} \cdot 3^5 \cdot 5^4 \cdot 7 \cdot 13 \cdot 17^2 \). \( \pi_e(L) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 15, 17, 20, 21, 30, 34, 51, 63, 65, 85, 255\} \), so \( D(L) = \{3, 4, 4, 1, 1, 3\} \). Since \( |G| = |L| \) and \( D(G) = D(L) \), we conclude that the prime graph of \( G \) has following form:

```
3
a

\[\begin{array}{c}
2 \\
\text{X} \\
17 \\
b
\end{array}\]
```

Figure 3.1

where \( \{a, b\} = \{7, 13\} \).
We will show that $G$ is isomorphic to $L = D_4(4)$. We break up the proof into a several steps.

**Step 1.** Let $K$ be the maximal normal solvable subgroup of $G$. Then $K$ is a $\{2,3,5\}$-group. In particular, $G$ is non-solvable.

First we show that $K$ is a $17'$-group. Assume the contrary and let $17 \in \pi(K)$. Then $13$ dose not divide the order of $K$. Otherwise, we may suppose that $T$ is a Hall $\{13,17\}$-subgroup of $K$. It is seen that $T$ is a nilpotent subgroup of order $13.17^i$ for $i = 1$ or $2$. Thus, $13.17 \in \pi_e(K) \subseteq \pi_e(G)$, a contradiction. Thus $\{17\} \subseteq \pi(K) \subseteq \pi(G) - \{13\}$. Let $K_{17} \in \text{Syl}_{17}(K)$. By Frattini argument, $G = KN_G(K_{17})$. Therefore, $N_G(K_{17})$ contains an element $x$ of order $13$. Since $G$ has no element of order $13.17$, $(x)$ should act fixed point freely on $K_{17}$, that is implying $\langle x \rangle K_{17}$ is a Frobenius group. By Lemma 2.2(b), $\vert \langle x \rangle \vert/|K_{17}| = 1$. It follows that $13|17^i - 1$ for $i = 1$ or $2$, which is a contradiction.

Next, we show that $K$ is a $p'$-group for $p \in \{a,b\}$. Let $p||K|$ and $K_p \in \text{Syl}_p(K)$. Now by Frattini argument, $G = KN_G(K_p)$, so $17$ must divide the order of $N_G(K_p)$. Therefore, the normalizer $N_G(K_p)$ contains an element of order $17$, say $x$. So $\langle x \rangle K_p$ is a cyclic subgroup of $G$ of order $17.p$, and so $p \sim 17$ in $\Gamma(G)$, which is a contradiction. Therefore $K$ is a $\{2,3,5\}$-group. In addition, since $K$ is a proper subgroup of $G$, it follows that $G$ is non-solvable.

**Step 2.** The quotient $G/K$ is an almost simple group. In fact, $S \leq G/K \subsetneq \text{Aut}(S)$, where $S$ is a finite non-abelian simple group isomorphic to $L := D_4(4)$.

Let $\overline{G} = G/K$. Then $S := \text{Soc}(\overline{G}) = P_1 \times P_2 \times \ldots \times P_m$, where $P_i$'s are finite non-abelian simple groups and $S \trianglelefteq \frac{G}{K} \subsetneq \text{Aut}(S)$. If we show that $m = 1$, the proof of Step 2 will be completed.

Suppose that $m \geq 2$. In this case, we claim that $13$ does not divide $|S|$. Assume the contrary and let $13 \mid |S|$, on the other hand, $\{2,3\} \subset \pi(P_i)$ for every $i$ (by TABLE 1), hence $2 \sim 13$ and $3 \sim 13$, which is a contradiction. Now, by step 1, we observe that $13 \in \pi(\overline{G}) \subseteq \pi(\text{Aut}(S))$. But $\text{Aut}(S) = \text{Aut}(S_1) \times \text{Aut}(S_2) \times \ldots \times \text{Aut}(S_t)$, where the groups $S_j$ are direct products of isomorphic $P_i$'s such that $S = S_1 \times S_2 \times \ldots \times S_t$. Therefore, for some $j$, $13$ divides the order of an automorphism group of a direct product $S_j$ of $t$ isomorphic simple groups $P_i$. Since $P_i \in \mathcal{S}_{17}$, it follows that $|\text{Out}(P_i)|$ is not divisible by $13$ (see TABLE 1). Now, by Lemma 2.3, we obtain $|\text{Aut}(S_j)| = |\text{Aut}(P_i)|^{t_i} t_i!$. Therefore, $t \geq 13$ and so $2^{26}$ must divide the order of $G$, which is a contradiction. Therefore $m = 1$ and $S = P_1$.

By TABLE 1 and Step 1, it is evident that $|S| = 2^a.3^b.5^7.13.17^2$, where $2 \leq a \leq 24, 1 \leq b \leq 5$ and $0 \leq \gamma \leq 4$. Now, using collected results contained in TABLE 1, we deduce that $S \cong D_4(4)$ and by Step 2, $L \trianglelefteq G/K \subsetneq \text{Aut}(L)$ is completed. As $|G| = |L|$, we deduce $K = 1$, so $G \cong L$ and the proof is completed.

□
Proposition 3.2. If $M = L : 2_1$, then $G \cong L : 2_1$ or $L : 2_3$.

Proof. As $|L : 2_1| = 2^{25}.3^5.5^4.7.13.17^2$ and $\pi_e(L : 2_1) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 24, 30, 34, 40, 42, 51, 60, 63, 65, 68, 85, 102, 126, 130, 170, 255\}$, then $D(L : 2_1) = (4, 4, 2, 1, 3)$. Since $|G| = |L : 2_1|$ and $D(G) = D(L : 2_1)$, we conclude that there exist several possibilities for $\Gamma(G)$:

$$\begin{array}{ccc}
  7 & a & 17 \\
  b & c & 13 \\
\end{array}$$

Figure 3.2

where $\{a, b, c\} = \{2, 3, 5\}$.

**Step 1.** Let $K$ be the maximal normal solvable subgroup of $G$. Then $K$ is a $\{2, 3, 5\}$-group. In particular, $G$ is non-solvable.

By a similar argument to that in Proposition 3.1, we can obtain this assertion.

**Step 2.** The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \leq \frac{G}{K} \leq \text{Aut}(S)$, where $S$ is a finite non-abelian simple group.

The proof is similar to Step 2 of Proposition 3.1.

By TABLE 1 and Step 1, it is evident that $|S| = 2^\alpha.3^\beta.5^\gamma.7.13.17^2$, where $2 \leq \alpha \leq 25$, $1 \leq \beta \leq 5$ and $0 \leq \gamma \leq 4$. Now, using collected results contained in TABLE 1, we conclude that $S \cong D_{4}(4)$ and by Step 2, $L \leq \frac{G}{K} \leq \text{Aut}(L)$. As $|G| = |L : 2_1| = 2|L|$, we deduce $|K| = 1$ or $2$.

If $|K| = 1$, then $G \cong L : 2_1$, $L : 2_2$, or $L : 2_3$. Obviously, $G \cong L : 2_1$ or $L : 2_3$ because $\text{deg}(2) = 5$ in $\Gamma(L : 2_3)$ (see page 16).

If $|K| = 2$, then $K \leq Z(G)$ and so $\text{deg}(2) = 5$, which is a contradiction. $\Box$

Proposition 3.3. If $M = L : 2_2$, then $G \cong L : 2_2$ or $\mathbb{Z}_2 \times L$.

Proof. As $|L : 2_2| = 2^{25}.3^5.5^4.7.13.17^2$ and $\pi_e(L : 2_2) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 17, 18, 20, 21, 24, 26, 30, 34, 40, 42, 51, 60, 63, 65, 68, 85, 102, 126, 130, 170, 255\}$, then $D(L : 2_2) = (5, 4, 4, 2, 2, 3)$. By assumption $|G| = |L : 2_2|$ and $D(G) = D(L : 2_2)$, so the prime graph of $G$ has following form:

$$\begin{array}{ccc}
  3 & 5 & 2 \\
  a & b & 17 \\
\end{array}$$

Figure 3.3

where $\{a, b\} = \{7, 13\}$.

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**Step 1.** Let $K$ be the maximal normal solvable subgroup of $G$. Then $K$ is a $\{2, 3, 5\}$-group. In particular, $G$ is non-solvable. By similar arguments as in the proof of Step 1 in Proposition 3.1, we conclude that $K$ is a $\{2, 3, 5\}$-group and $G$ is non-solvable.

**Step 2.** The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \leq \frac{G}{K} \leq \text{Aut}(S)$, where $S$ is a finite non-abelian simple group.

Let $\overline{G} = \frac{G}{K}$. Then $S := \text{Soc}(\overline{G})$, $S = P_1 \times P_2 \times \ldots \times P_m$, where $P_i$'s are finite non-abelian simple groups and $S \leq \frac{G}{K} \leq \text{Aut}(S)$. We are going to prove that $m = 1$ and $S = P_1$. Suppose that $m \geq 2$. We claim $a$ does not divide $|S|$. Assume the contrary and let $a \mid |S|$, we conclude that a just divide the order of one of the simple groups $P_1$'s. Without loss of generality, we assume that $a \mid |P_1|$. Then the rest of the $P_i$'s must be $\{2, 3\}$-group (because only 2 and 3 are adjacent to a in $\Gamma(G)$), this is a contradiction because $P_i$'s are finite non-abelian simple groups. Now, by Step 1, we observe that $a \in \pi(\overline{G}) \subseteq \pi(\text{Aut}(S))$. But $\text{Aut}(S) = \text{Aut}(S_1) \times \text{Aut}(S_2) \times \ldots \times \text{Aut}(S_t)$, where the groups $S_j$ are direct products of isomorphic $P_i$'s such that $S = S_1 \times S_2 \times \ldots \times S_r$. Therefore, for some $j$, $a$ divides the order of an automorphism group of a direct product $S_j$ of $t$ isomorphic simple groups $P_i$. Since $P_i \in \mathfrak{S}_{17}$, it follows that $|\text{Out}(P_i)|$ is not divisible by $a$ (see TABLE 1), so $a$ does not divide the order of $\text{Aut}(P_i)$. Now, by Lemma 2.3, we obtain $|\text{Aut}(S_j)| = |\text{Aut}(P_i)|!^t$. Therefore, $t \geq a$ and so $3^a$ must divide the order of $G$, which is a contradiction. Therefore $m = 1$ and $S = P_1$.

By TABLE 1 and Step 1, it is evident that $|S| = 2^a 3^b 5^c 7^d 13^e 17^f$, where $2 \leq a \leq 25$, $1 \leq b \leq 5$ and $0 \leq c \leq 4$. Now, using collected results contained in TABLE 1, we conclude that $S \cong D_4(4)$ and by Step 2, $L \cong \frac{G}{K} \leq \text{Aut}(L)$. As $|G| = |L : Z| = 2 |L|$, we deduce $|K| = 1$ or 2.

If $|K| = 1$, then $G \cong L : 21$, $L : 22$ or $L : 23$ because $|G| = 2 |L|$. It is obvious that $G \cong L : 22$, because $deg(13) = 1$ in $\Gamma(L : 21)$ and $\Gamma(L : 23)$ (see page 17).

If $|K| = 2$, then $G/K \cong L$ and $K \leq Z(G)$. It follows that $G$ is a central extension of $K$ by $L$. If $G$ is a non-split extension of $K$ by $L$, then $|K|$ must divide the Schur multiplier of $L$, which is 1. But this is a contradiction, so we obtain that $G$ split over $|K|$. Hence $G \cong Z_2 \times L$. □

**Proposition 3.4.** If $M = L : 2_3$, then $G \cong L : 2_3$ or $L : 2_1$.

**Proof.** As $|L : 2_3| = 2^{25} 3^3 5^4 7^2 13 17^2$ and $\pi_r(L : 2_3) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 24, 30, 34, 51, 63, 65, 85, 255\}$, then $D(L : 2_3) = (4, 4, 4, 2, 1, 3)$. Since $|G| = |L : 2_3|$ and $D(G) = D(L : 2_3)$, we conclude that $\Gamma(G)$ has the following form similarly to Proposition 3.2:
OD-characterization of Almost Simple Groups Related to $D_4(4)$

Figure 3.4

where $\{a, b, c\} = \{2, 3, 5\}$.

**Step 1.** Let $K$ be the maximal normal solvable subgroup of $G$. Then $K$ is a $\{2, 3, 5\}$-group. In particular, $G$ is non-solvable.

We can prove this by the similar way to that in Proposition 3.2.

**Step 2.** The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \leq \frac{G}{K} \preceq \text{Aut}(S)$, where $S$ is a finite non-abelian simple group.

By using a similar argument, as in the proof of Proposition 3.2, we can verify that $\frac{G}{K}$ is an almost simple group.

By TABLE 1 and Step 1, it is evident that $|S| = 2^\alpha 3^\beta 5^\gamma 7^\delta$, where $2 \leq \alpha \leq 25$, $1 \leq \beta \leq 5$ and $0 \leq \gamma \leq 4$. Now, using collected results contained in TABLE 1, we conclude that $S \cong D_4(4)$ and by Step 2, $L \leq \frac{G}{K} \preceq \text{Aut}(L)$. As $|G| = |L : 2_3| = 2|L|$, we deduce $|K| = 1$ or 2.

If $|K| = 1$, then $G \cong L : 2_1$, $L : 2_2$ or $L : 2_3$ because $|G| = 2|L|$. Obviously, $G \cong L : 2_3$ or $L : 2_1$, because $\deg(2) = 5$ in $\Gamma(L : 2_2)$ (see page 16).

If $|K| = 2$, then $K \leq Z(G)$ and so $\deg(2) = 5$, which is a contradiction. □

**Proposition 3.5.** If $M = L : 3$, then $G \cong L : 3$ or $\mathbb{Z}_3 \times L$.

*Proof.* As $|L : 3| = 2^{24} 3^{36} 5^4 7^1 13^1 17^2$ and $\pi_e(L : 3) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 15, 17, 18, 20, 21, 24, 30, 34, 39, 45, 51, 63, 65, 85, 255\}$, then $D(L : 3) = (3, 5, 4, 1, 2, 3)$. Since $|G| = |L : 3|$ and $D(G) = D(L : 3)$, we conclude that $\Gamma(G)$ has the following form (like $\Gamma(L : 3)$):

![Figure 3.5](image-url)

**Step 1.** Let $K$ be the maximal normal solvable subgroup of $G$. Then $K$ is a $\{2, 3\}$-group. In particular, $G$ is non-solvable.

First, we show that $K$ is a $p'$-group for $p = 7, 13$ and 17. Since the proof is quite similar to the proof of Step 1 in Proposition 3.1, so we avoid here full explanation of all details.
Next we consider $K$ is a $5'$-group. Assume the contrary, $5 \in \pi_e(K)$. Let $K_5 \in \text{Syl}_5(K)$. By Frattini argument, $G = KN_G(K_5)$. Therefore, $N_G(K_5)$ has an element $x$ of order 7. Since $G$ has no element of order 5, $\langle x \rangle$ should act fixed point freely on $K_5$, implying $\langle x \rangle K_5$ is a Frobenius group. By Lemma 2.2(b), $\langle x \rangle | |(|K_5| - 1)$, which is impossible. Therefore $K$ is a $\{2, 3\}$-group.

In addition since $K$ is a proper subgroup of $G$, then $G$ is non-solvable and the proof of this step is completed.

**Step 2.** The quotient $G/K$ is an almost simple group. In fact, $S \leq G/K \lesssim \text{Aut}(S)$, where $S$ is a finite non-abelian simple group.

In a similar way as in the proof of Step 2 in Proposition 3.1, we conclude that $G/K$ is an almost simple group.

By TABLE 1 and Step 1, it is evident that $|S| = 2^a \cdot 3^b \cdot 5^c \cdot 7^{13} \cdot 17^2$, where $2 \leq a \leq 24$ and $1 \leq b \leq 6$. Now, using collected results contained in TABLE 1, we conclude that $S \cong D_4(4)$ and by Step 2, $L \leq G/K \lesssim \text{Aut}(L)$. As $|G| = |L : 3| = 3|L|$, we deduce $|K| = 1$ or 3.

If $|K| = 1$, then $G \cong L : 3$.

If $|K| = 3$, then $G/K \cong L$. In this case we have $G/C_G(K) \lesssim \text{Aut}(K) \cong \mathbb{Z}_2$. Thus $|G/C_G(K)| = 1$ or 2. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is, $G$ is a central extension of $K$ by $L$. If $G$ is a non-split extension of $K$ by $L$, then $|K|$ must divide the Schur multiplier of $L$, which is 1. But this is a contradiction, so we obtain that $G$ split over $K$. Hence $G \cong \mathbb{Z}_3 \times L$. If $|G/C_G(K)| = 2$, then $K < C_G(K)$ and $1 \neq C_G(K)/K \leq G/K \cong L$, which is a contradiction since $L$ is simple. \hfill \Box

**Proposition 3.6.** If $M = L : 2^2$, then $G \cong L : 2^2$, $\mathbb{Z}_2 \times (L : 2_1)$, $\mathbb{Z}_2 \times (L : 2_2)$, $\mathbb{Z}_2 \times (L : 2_3)$, $\mathbb{Z}_2 \times L$ or $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times L$.

**Proof.** As $|L : 2^2| = 2^{26} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 13 \cdot 17^2$ and $\pi_e(L : 2^2) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 24, 26, 30, 34, 42, 51, 60, 63, 65, 68, 85, 102, 126, 130, 170, 255\}$, then $D(L : 2^2) = (5, 4, 4, 2, 2, 3)$. Since $|G| = |L : 2^2|$ and $D(G) = D(L : 2^2)$, so the prime graph of $G$ has following form similarly to Proposition 3.3:

![Figure 3.6](image_url)
Step 1. Let $K$ be the maximal normal solvable subgroup of $G$. Then $K$ is a $\{2, 3, 5\}$-group. In particular, $G$ is non-solvable.

According to Step 1 in Proposition 3.3, we have $K$ is a $\{2, 3, 5\}$-group and $G$ is non-solvable.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \leq \frac{G}{K} \leq \text{Aut}(S)$, where $S$ is a finite non-abelian simple group.

We can prove this by the similar argument in Step 2 in Proposition 3.3.

By TABLE 1 and Step 1, it is evident that $|S| = 2^\alpha 3^\beta 5^\gamma 7.13.17^2$, where $2 \leq \alpha \leq 26$, $1 \leq \beta \leq 5$ and $0 \leq \gamma \leq 4$. Now, using collected results contained in TABLE 1, we conclude that $S \cong D_4(4)$ and by Step 2, $L \leq \frac{G}{K} \leq \text{Aut}(L)$. As $|G| = |L: 2^2| = 4|L|$, we deduce $|K| = 1, 2$ or 4.

If $|K| = 1$, then $G \cong L : 2^2$.

If $|K| = 2$, then $K \leq Z(G)$. In this case $G$ is a central extension of $Z_2$ by $L : 2_1$, $L : 2_2$ or $L : 2_3$. If $G$ splits over $K$ then $G \cong Z_2 \times (L : 2_1), Z_2 \times (L : 2_2)$ or $Z_2 \times (L : 2_3)$, otherwise we get a contradiction because $|K|$ must divide the Schur multiplier of $L : 2_1, L : 2_2$ and $L : 2_3$, which is impossible.

If $|K| = 4$, then $G/K \cong L$. In this case we have $G/C_G(K) \cong \text{Aut}(K) \cong Z_2$ or $S_3$. Thus $|G/C_G(K)| = 1, 2, 3$ or 6. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is, $G$ is a central extension of $K$ by $L$. If $G$ is a non-split extension of $K$ by $L$, then $|K|$ must divide the Schur multiplier of $L$, which is 1, but this is a contradiction. Therefore $G$ splits over $K$. Hence $G \cong K \times L$. So we have $G \cong Z_2 \times L$ or $(Z_2 \times Z_2) \times L$ because $K \cong Z_4$ or $Z_2 \times Z_2$. If $|G/C_G(K)| = 2, 3$ or 6, then $K < C_G(K)$ and $1 \neq C_G(K)/K \leq G/K \cong L$. Which is a contradiction, since $L$ is simple.

\[ \square \]

Proposition 3.7. If $M = L : (D_6)_1$, then $G \cong L : (D_6)_1, L : 6, Z_3 \times (L : 2_1)$, $Z_3 \times (L : 2_3)$ or $(Z_3 \times L).Z_2$.

Proof. As $|L : (D_6)_1| = 2^{25} 3^6 5^4 7.13.17^2$ and $\pi_c(L : (D_6)_1) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 24, 30, 34, 39, 42, 45, 51, 60, 63, 65, 85, 255\}$, then $D(L : (D_6)_1) = \{4, 5, 4, 2, 2, 3\}$. Since $|G| = |L : (D_6)_1|$ and $D(G) = D(L : (D_6)_1)$, we conclude that there exist several possibilities for $\Gamma(G)$:

\[ \begin{figure}[h]
\centering
\begin{tikzpicture}
\node at (0,0) [circle,fill,inner sep=2pt] (a) {a};
\node at (1,1) [circle,fill,inner sep=2pt] (b) {b};
\node at (2,0) [circle,fill,inner sep=2pt] (c) {c};
\node at (0,1) [circle,fill,inner sep=2pt] (d) {d};
\node at (1,-1) [circle,fill,inner sep=2pt] (e) {e};
\path[dashed, line width=1pt] (a) edge (b);
\path[dashed, line width=1pt] (a) edge (c);
\path[dashed, line width=1pt] (a) edge (d);
\path[dashed, line width=1pt] (a) edge (e);
\path[dashed, line width=1pt] (b) edge (c);
\path[dashed, line width=1pt] (b) edge (d);
\path[dashed, line width=1pt] (b) edge (e);
\end{tikzpicture}
\caption{Figure 3.7}
\end{figure} \]
where \( \{a, b\} = \{7, 13\} \).

**Step 1.** Let \( K \) be the maximal normal solvable subgroup of \( G \). Then \( K \) is a \( \{2, 3, 5\} \)-group. In particular, \( G \) is non-solvable.

By the similar argument to that in Step 1 in Proposition 3.1, we can obtain this assertion.

**Step 2.** The quotient \( \frac{G}{K} \) is an almost simple group. In fact, \( S \leq \frac{G}{K} \simeq \text{Aut}(S) \), where \( S \) is a finite non-abelian simple group.

The proof is similar to Step 2 in Proposition 3.3.

By TABLE 1 and Step 1, it is evident that \( |S| = 2^a \cdot 3^b \cdot 5^c \cdot 7 \cdot 11 \cdot 17 \cdot 19 \), where \( 2 \leq a \leq 25, 1 \leq b \leq 6 \) and \( 0 \leq c \leq 4 \). Now, using collected results contained in TABLE 1, we conclude that \( S \simeq D_4(4) \) and by Step 2, \( L \leq \frac{G}{K} \simeq \text{Aut}(L) \). As \( |G| = |L : D_6|_1 = 6|L| \), we deduce \( |K| = 1, 2, 3 \) or 6.

If \( |K| = 1 \), then \( G \simeq L : (D_6)_1 \), \( L : (D_6)_2 \) or \( L : 6 \) because \( |G| = 6|L| \).

Obviously, \( G \simeq L : (D_6)_1 \) or \( L : 6 \) because \( \text{deg}(2) = 5 \) in \( \Gamma(L : (D_6)_2) \).

If \( |K| = 2 \), then \( K \leq Z(G) \) and so \( \text{deg}(2) = 5 \), which is a contradiction (see page 18).

If \( |K| = 3 \), then \( G/K \simeq L : 2_1, L : 2_2 \) or \( L : 2_3 \). But \( G/C_G(K) \simeq \text{Aut}(K) \simeq Z_2 \). Thus \( |G/C_G(K)| = 1 \) or 2. If \( |G/C_G(K)| = 1 \), then \( K \leq Z(G) \), that is, \( G \) is a central extension of \( K \) by \( L : 2_1, L : 2_2 \) or \( L : 2_3 \). If \( G \) splits over \( K \), then \( G \simeq Z_3 \times (L : 2_1) \) or \( Z_3 \times (L : 2_3) \) because in \( \Gamma(Z_3 \times (L : 2_2)) \) the degree of 2 is 5. Otherwise we get a contradiction because \( |K| \) must divide the Schur multiplier of \( L : 2_1, L : 2_2 \) and \( L : 2_3 \), which is impossible. If \( |G/C_G(K)| = 2 \), then \( K < C_G(K) \) and \( 1 \neq C_G(K)/K \leq G/K \leq L : 2_1, L : 2_2 \) or \( L : 2_3 \), we obtain \( C_G(K)/K \simeq L \). Since \( K \leq Z(C_G(K)) \), \( C_G(K) \) is a central extension of \( K \) by \( L \). If \( C_G(K) \) splits over \( K \), then \( C_G(K) \simeq Z_3 \times L \), otherwise we get a contradiction because \( |K| \) must divide the Schur multiplier of \( L \), which is impossible. Therefore, \( G \simeq (Z_3 \times L).Z_2 \).

If \( |K| = 6 \), then \( G/K \simeq L \) and \( K \simeq Z_6 \) or \( D_6 \).

If \( K \approx Z_6 \), then \( G/C_G(K) \simeq Z_2 \) and so \( |G/C_G(K)| = 1 \) or 2. If \( |G/C_G(K)| = 1 \), then \( K \leq Z(G) \). It follows that \( \text{deg}(2) = 5 \), a contradiction. If \( |G/C_G(K)| = 2 \), then \( K < C_G(K) \) and \( 1 \neq C_G(K)/K \leq G/K \leq L \), which is a contradiction because \( L \) is simple.

If \( K \approx D_6 \), then \( K \cap C_G(K) = 1 \) and \( G/C_G(K) \simeq D_6 \). Thus \( C_G(K) \neq 1 \).

Hence, \( 1 \neq C_G(K) \simeq C_G(K)/K \leq G/K \leq L \). It follows that \( L \simeq G/K \simeq C_G(K) \) because \( L \) is simple. Therefore, \( G \simeq D_6 \times L \), which implies that \( \text{deg}(2) = 5 \), a contradiction.

\[ \square \]

**Proposition 3.8.** If \( M = L : (D_6)_2 \), then \( G \simeq L : (D_6)_2, Z_2 \times (L : 3), Z_3 \times (L : 2_2), (Z_3 \times L).Z_2, Z_6 \times L \) or \( S_3 \times L \).
Proof. As \(|L : (D_6)_2| = 2^{25}.3^6.5^4.7.13.17^2\) and \(\pi_e (L : (D_6)_2) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 17, 18, 20, 21, 24, 26, 30, 34, 39, 40, 42, 45, 51, 60, 63, 65, 68, 85, 102, 126, 130, 170, 255\}\), then \(D(L : (D_6)_2) = (5, 5, 4, 2, 3, 3)\). Since \(|G| = |L : (D_6)_2|\) and \(D(G) = D(L : (D_6)_2)\), we conclude that \(\Gamma(G)\) has the following form (like \(\Gamma(L : (D_6)_2)\)):

\[
\begin{align*}
&\Gamma(G) = \Gamma(L : (D_6)_2) \\
&\text{Figure 3.8}
\end{align*}
\]

**Step 1.** Let \(K\) be the maximal normal solvable subgroup of \(G\). Then \(K\) is a \(\{2, 3\}\)-group. In particular, \(G\) is non-solvable.

The proof is similar to Step 1 in Proposition 3.5.

**Step 2.** The quotient \(\frac{G}{K}\) is an almost simple group. In fact, \(S \leq \frac{G}{K} \leq \text{Aut}(S)\), where \(S\) is a finite non-abelian simple group.

Let \(\mathcal{G} = \frac{G}{K}\). Then \(S := \text{Soc}(\mathcal{G})\), \(S = P_1 \times P_2 \times \ldots \times P_m\), where \(P_i\)'s are finite non-abelian simple groups and \(S \leq \frac{G}{K} \leq \text{Aut}(S)\). We are going to prove that \(m = 1\) and \(S = P_1\). Suppose that \(m \geq 2\). By the same argument in Step 2 of Proposition 3.3 and considering 7 instead of \(a\), we get a contradiction. Therefore \(m = 1\) and \(S = P_1\).

By TABLE 1 and Step 1, it is evident that \(|S| = 2^a.3^b.5^4.7.13.17^2\), where \(2 \leq a \leq 25\) and \(1 \leq b \leq 6\). Now, using collected results contained in TABLE 1, we conclude that \(S \cong D_4(4)\) and by Step 2, \(L \leq \frac{G}{K} \leq \text{Aut}(L)\). As \(|G| = |L : (D_6)_2| = 6|L|\), we deduce \(|K| = 1, 2, 3\) or \(6\).

If \(|K| = 1\), then \(G \cong L : (D_6)_1, L : (D_6)_2\) or \(L : 6\) because \(|G| = 6|L|\).

Obviously \(G \cong L : (D_6)_2\) because in \(\Gamma(L : (D_6)_1)\) and \(\Gamma(L : 6)\), we have \(\text{deg}(13) = 2\) (see page 17).

If \(|K| = 2\), then \(K \leq Z(G)\) and \(G/K \cong L : 3\). Hence \(G\) is a central extension of \(K\) by \(L : 3\). If \(G\) splits over \(K\), then \(G \cong \mathbb{Z}_2 \times (L : 3)\). Otherwise we get a contradiction because \(|K|\) must divide the Schur multiplier of \(L : 3\), which is impossible.

If \(|K| = 3\), then \(G/K \cong L : 2_1, L : 2_2\) or \(L : 2_3\). But \(G/C_G(K) \nsubseteq \text{Aut}(K) \cong \mathbb{Z}_2\). Thus \(|G/C_G(K)| = 1\) or \(2\). If \(|G/C_G(K)| = 1\), then \(K \leq Z(G)\), that is, \(G\) is a central extension of \(K\) by \(L : 2_1, L : 2_2\) or \(L : 2_3\). If \(G\) splits over \(K\), then only \(G \cong \mathbb{Z}_3 \times (L : 2_2)\) because \(2 \not\approx 13\) in \(\Gamma(\mathbb{Z}_3 \times (L : 2_1))\) and \(\Gamma(\mathbb{Z}_3 \times (L : 2_3))\). Otherwise we get a contradiction because \(|K|\) must divide the Schur multiplier of \(L : 2_1, L : 2_2\) and \(L : 2_3\), which is impossible. If
If $|K| = 6$, then $G/K \cong L$ and $K \cong Z_6$ or $D_6$. If $K \cong Z_6$, then $G/C_G(K) \cong C_2 \times Z_2$ and so $|G/C_G(K)| = 1$ or $2$. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$ and $G/K \cong L$. Therefore $G$ is a central extension of $K$ by $L$. If $G$ is a non-split extension of $K$ by $L$, then $|K|$ must divide the Schur multiplier of $L$, which is 1. But this is a contradiction. So we obtain that $G$ splits over $K$. Hence $G \cong Z_6 \times L$. If $|G/C_G(K)| = 2$, then $K < C_G(K)$ and $1 \neq C_G(K)/K \cong \mathbb{Z}/6\mathbb{Z}$, which is a contradiction because $L$ is simple. If $K \cong D_6$, then $K \cap C_G(K) = 1$ and $G/C_G(K) \cong D_6$. Thus $C(G)/K \not\cong 1$. Hence, $1 \neq C_G(K)/K \cong C_G(K)/K \cong G/K \cong L$. It follows that $L \cong G/K \cong C_G(K)$ because $L$ is simple. Therefore $G \cong D_6 \times L$.

**Proposition 3.9.** If $M = L : 6$, then $G \cong L : 6$, $L : (D_6)_1$, $Z_3 \times (L : 2)_1$, $Z_3 \times (L : 2)_3$ or $(Z_3 \times L, Z_2)$.

**Proof.** As $|L : 6| = 2^{25}.3^9.5^4.7^5.13.17^2$ and $\pi_c(L : 6) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 22, 24, 30, 34, 36, 39, 42, 45, 48, 51, 63, 65, 85, 255\}$, then $D(L : 6) = \{4, 5, 4, 2, 2, 3\}$. Since $|G| = |L : 6|$ and $D(G) = D(L : 6)$, there exist several possibilities for $\Gamma(G)$ similarly to Proposition 3.7:

![Figure 3.9](image_url)

where $\{a, b\} = \{7, 13\}$.

**Step 1.** Let $K$ be the maximal normal solvable subgroup of $G$. Then $K$ is a $\{2, 3, 5\}$-group. In particular, $G$ is non-solvable. The proof is similar to that in Proposition 3.3.

**Step 2.** The quotient $G/K$ is an almost simple group. In fact, $S \leq G/K \leq Aut(S)$, where $S$ is a finite non-abelian simple group.

Again we refer to Step 2 of proposition 3.3 to get the proof.

By TABLE 1 and Step 1, it is evident that $|S| = 2^\alpha.3^\beta.5^\gamma.7^{13.17^2}$, where $2 \leq \alpha \leq 25$, $1 \leq \beta \leq 6$ and $0 \leq \gamma \leq 4$. Now, using collected results contained
in TABLE 1, we conclude that $S \cong D_4(4)$ and by Step 2, $L \leq G \cong \text{Aut}(L)$. As $|G| = |L : 6| = 6|L|$, we deduce $|K| = 1, 2, 3$ or 6.

If $|K| = 1$, then $G \cong L : 6$, $L : (D_6)_1$ or $L : (D_6)_2$ because $|G| = 6|L|$. Obviously, $G \cong L : 6$ or $L : (D_6)_1$ because $\text{deg}(2) = 5$ in $\Gamma(\frac{L}{D_6})_2$ (see page 18).

If $|K| = 2$, then $K \leq Z(G)$ and so $\text{deg}(2) = 5$, which is a contradiction.

If $|K| = 3$, then $G/K \cong L : 2_1, L : 2_2$ or $L : 2_3$. But $G/C_G(K) \leq \text{Aut}(K) \cong Z_2$. Thus $|G/C_G(K)| = 1$ or 2. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is, $G$ is a central extension of $K$ by $L : 2_1, L : 2_2$ or $L : 2_3$. If $G$ splits over $K$, then $G \cong Z_3 \times (L : 2_1)$ or $Z_3 \times (L : 2_3)$ because in $\Gamma(Z_3 \times (L : 2))$ the degree of 2 is 5. Otherwise we get a contradiction because $|K|$ must divide the Schur multiplier of $L : 2_1, L : 2_2$ and $L : 2_3$, which is impossible. If $|G/C_G(K)| = 2$, then $K < C_G(K)$ and $1 \neq C_G(K)/K \cong G/K \cong L : 2_1, L : 2_2$ or $L : 2_3$, we obtain $C_G(K)/K \cong L$. Since $K \leq Z(C_G(K))$, $C_G(K)$ is a central extension of $K$ by $L$. If $C_G(K)$ splits over $K$, then $C_G(K) \cong Z_3 \times L$, otherwise we get a contradiction because $|K|$ must divide the Schur multiplier of $L$, which is impossible. Therefore, $G \cong (Z_3 \times L).Z_2$.

If $|K| = 6$, then $G/K \cong L$ and $K \cong Z_6$ or $D_6$. If $K \cong Z_6$, then $G/C_G(K) \cong Z_2$ and so $|G/C_G(K)| = 1$ or 2. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$. It follows that $\text{deg}(2) = 5$, a contradiction. If $|G/C_G(K)| = 2$, then $K < C_G(K)$ and $1 \neq C_G(K)/K \cong G/K \cong L$, which is a contradiction because $L$ is simple. If $K \cong D_6$, then $K \cap C_G(K) = 1$ and $G/C_G(K) \cong D_6$. Thus $C_G(K) \neq 1$. Hence, $1 \neq C_G(K) \cong C_G(K)/K \cong G/K \cong L$. It follows that $L \cong G/K \cong C_G(K)$ because $L$ is simple. Therefore, $G \cong D_6 \times L$, which implies that $\text{deg}(2) = 5$, a contradiction.

\begin{proposition}
If $M = L : D_{12}$, then $G \cong L : D_{12}$, $Z_2 \times (L : (D_6)_1), Z_2 \times (L : (D_6)_2), Z_2 \times (L : 6), Z_2 \times (L : 2^2), (Z_3 \times (L : 2_1)).Z_2, (Z_3 \times (L : 2_3)).Z_2, Z_4 \times (L : 3), (Z_2 \times Z_2) \times (L : 3), (Z_4 \times L).Z_3, ((Z_2 \times Z_2) \times L).Z_3, Z_6 \times (L : 2_1), Z_6 \times (L : 2_2), Z_6 \times (L : 2_3), (Z_6 \times L).Z_2, S_3 \times (L : 2_1), S_3 \times (L : 2_3), S_3 \times (L : 2_5), Z_{12} \times L, (Z_2 \times Z_6) \times L, D_{12} \times L, (Z_2 \times L).D_6, (Z_4 \times L, L.A_4 \times T \times L).
\end{proposition}

\begin{proof}
As $|L : D_{12}| = 2^{26}.3^9.5^4.7.13.17^2$ and $\pi_c(L : (D_{12})) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 24, 26, 30, 34, 39, 40, 42, 45, 51, 60, 63, 65, 68, 85, 102, 126, 130, 170, 255\}$, then $D(L : D_{12}) = (5, 5, 4, 2, 3, 3)$. Since $|G| = |L : D_{12}|$ and $D(G) = D(L : D_{12})$, we conclude that $\Gamma(G)$ has the following form (like $\Gamma(L : D_{12})$):
Step1. Let $K$ be the maximal normal solvable subgroup of $G$. Then $K$ is a $\{2, 3\}$-group. In particular, $G$ is non-solvable.

The proof is similar to Step 1 in Proposition 3.5.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \leq \frac{G}{K} \lessdot \text{Aut}(S)$, where $S$ is a finite non-abelian simple group.

To get the proof, follow the way in the proof of Step 2 in Proposition 3.5.

By TABLE 1 and Step 1, it is evident that $|S| = 2^\alpha \cdot 3^\beta \cdot 5 \cdot 7 \cdot 13 \cdot 17^2$, where $2 \leq \alpha \leq 26$ and $1 \leq \beta \leq 6$. Now, using collected results contained in TABLE 1, we conclude that $S \cong D_4(4)$ and by Step 2, $L \leq \frac{G}{K} \lessdot \text{Aut}(L)$. As $|G| = |L : D_{12}| = 12|L|$, we deduce $|K| = 1, 2, 3, 4, 6$ or 12.

If $|K| = 1$, then $G \cong L : D_{12}$.

If $|K| = 2$, then $G/K \cong L : (D_6)_1, L : (D_6)_2$ or $L : 6$ and $K \leq Z(G)$. It follows that $G$ is a central extension of $K$ by $L : (D_6)_1, L : (D_6)_2$ or $L : 6$. If $G$ splits over $K$, then $G \cong \mathbb{Z}_2 \times (L : (D_6)_1), \mathbb{Z}_2 \times (L : (D_6)_2)$ or $\mathbb{Z}_2 \times (L : 6)$. Otherwise $G \cong \mathbb{Z}_2.(L : (D_6)_1)$ or $\mathbb{Z}_2.(L : (D_6)_2)$.

If $|K| = 3$, then $G/K \cong L : 2^2$. But $G/C_G(K) \lessdot \text{Aut}(K) \cong \mathbb{Z}_2$. Thus $|G/C_G(K)| = 1$ or 2. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is, $G$ is a central extension of $K$ by $L : 2^2$. If $G$ splits over $K$, then $G \cong \mathbb{Z}_3 \times (L : 2^2)$, otherwise we get a contradiction because $|K|$ must divide the Schur multiplier of $L : 2^2$, which is impossible. If $|G/C_G(K)| = 2$, then $K \neq C_G(K)$ and $1 \neq C_G(K)/K \leq G/K \cong L : 2^2$, and we obtain $C_G(K)/K \cong L : 2^1, L : 2^2$ or $L : 2^3$. Since $K \leq Z(C_G(K))$, $C_G(K)$ is a central extension of $K$ by $L : 2^1, L : 2^2$ or $L : 2^3$. Thus $C_G(K) \cong \mathbb{Z}_3 \times (L : 2^1), \mathbb{Z}_3 \times (L : 2^2)$ or $\mathbb{Z}_3 \times (L : 2^3)$, otherwise we get a contradiction because 3 must divide the Schur multiplier of $L : 2^1, L : 2^2$ or $L : 2^3$, which is impossible. Therefore, $G \cong (\mathbb{Z}_3 \times (L : 2^1)).\mathbb{Z}_2, (\mathbb{Z}_3 \times (L : 2^2)).\mathbb{Z}_2$ or $(\mathbb{Z}_3 \times (L : 2^3)).\mathbb{Z}_2$.

If $|K| = 4$, then $G/K \cong L : 3$ and $K \cong \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. In this case we have $G/C_G(K) \lessdot \text{Aut}(K) \cong \mathbb{Z}_2$ or $S_3$. Thus $|G/C_G(K)| = 1, 2, 3$ or 6. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is, $G$ is a central extension of $K$ by $L : 3$. If $G$ split over $K$ by $L : 3$, then $G \cong \mathbb{Z}_4 \times (L : 3)$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times (L : 3)$. Otherwise we get a contradiction because $|K|$ must divide the Schur multiplier of $L : 3$, which is impossible. If $|G/C_G(K)| \neq 1$, since $|G/C_G(K)| = 2, 3$ or 6, it follows that $K \leq C_G(K)$. As $L$ is simple, we conclude that $1 \neq C_G(K)/K$ must
be an extension of $L$. Hence $|G/C_G(K)| = 3$ and therefore $C_G(K)/K \cong L$. Now, since $K \leq Z(C_G(K))$, we conclude that $C_G(K)$ is a central extension of $K$ by $L$. Thus $C_G(K) \cong \mathbb{Z}_4 \times L$, or $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times L$, otherwise $|K|$ must divide the Schur multiplier of $L$, which is $1$ and it is impossible. Therefore, $G \cong (\mathbb{Z}_4 \times L).\mathbb{Z}_3$ or $((\mathbb{Z}_2 \times \mathbb{Z}_2) \times L).\mathbb{Z}_3$.

If $|K| = 6$, then $G/K \cong L : 2_1$, $L : 2_2$ or $L : 2_3$ and $K \cong \mathbb{Z}_6$ or $D_6$. If $K \cong \mathbb{Z}_6$, then $G/C_G(K) \cong \mathbb{Z}_2$ and so $|G/C_G(K)| = 1$ or $2$. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is $G$ is a central extension of $\mathbb{Z}_6$ by $L : 2_1$, $L : 2_2$ or $L : 2_3$. If $G$ splits over $K$, we obtain $G \cong \mathbb{Z}_6 \times (L : 2_1)$, $\mathbb{Z}_6 \times (L : 2_2)$ or $\mathbb{Z}_6 \times (L : 2_3)$, otherwise we get a contradiction because $|K|$ must divide the Schur multiplier of $L$, which is $1$ and it is impossible. If $|G/C_G(K)| = 2$, then $K < C_G(K)$ and $1 \neq C_G(K)/K \leq G/K \cong L : 2_1$, $L : 2_2$ or $L : 2_3$, and we obtain $C_G(K)/K \cong L$. Since $K \leq Z(C_G(K))$, $C_G(K)$ is a central extension of $K$ by $L$. Thus $C_G(K) \cong \mathbb{Z}_6 \times L$, otherwise we get a contradiction because $|K|$ must divide the Schur multiplier of $L$. Therefore $G \cong (\mathbb{Z}_6 \times L).\mathbb{Z}_2$. If $K \cong D_6$, then $G/C_G(K) \cong D_6$ and so $|G/C_G(K)| = 1, 2, 3$ or $6$. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is a contradiction. If $|G/C_G(K)| = 2$, then we have $|KC_G(K)| = 6 |G|/2 = 3 |G|$ because $K \cap C_G(K) = 1$, which is a contradiction. If $|G/C_G(K)| = 3$, then we have $|KC_G(K)| = 6 |G|/3 = 2 |G|$ because $K \cap C_G(K) = 1$, which is a contradiction. If $|G/C_G(K)| = 6$, then $G/C_G(K) \cong D_6$ and $C_G(K) \neq 1$. Hence, $1 \neq C_G(K) \cong C_G(K)/K \leq G/K \cong L : 2_1, L : 2_2$ or $L : 2_3$. It follows that $C_G(K) \cong L : 2_1, L : 2_2$ or $L : 2_3$ because $L$ is simple. Therefore, $G \cong D_6 \times (L : 2_1)$, $D_6 \times (L : 2_2)$ or $D_6 \times (L : 2_3)$.

Before processing the last case, we recall the following facts.

There exist five non-isomorphic groups of order $12$. Two of them are abelian and three are non-abelian. The non-abelian groups are: alternating group $A_4$, dihedral group $D_{12}$ and the dicyclic group $T$ with generators $a$ and $b$, subject to the relations $a^6 = 1$, $a^3 = b^2$ and $b^{-1}ab = a^{-1}$.

If $|K| = 12$, then $G/K \cong L$ and $K \cong \mathbb{Z}_{12}, \mathbb{Z}_2 \times \mathbb{Z}_6, D_{12}, A_4$ or $T$. But $C_G(K)/K \leq G/K \cong L$. If $C_G(K)/K \cong L$, then $C_G(K) \leq K$ and hence $|L| = |G/K|||G/C_G(K)||\text{Aut}(K)|$. Thus $|L||\text{Aut}(K)|$, a contradiction. Therefore, $C_G(K)/K \neq 1$ and since $L$ is simple group, we conclude that $G = C_G(K)K$ and hence, $G/C_G(K) \cong K/Z(K)$. Now, we should consider the following cases:

If $K \cong \mathbb{Z}_{12}$ or $\mathbb{Z}_2 \times \mathbb{Z}_6$, then $G/C_G(K) = 1$. Therefore $K \leq Z(G)$, that is $G$ is a central extension of $\mathbb{Z}_{12}$ or $\mathbb{Z}_2 \times \mathbb{Z}_6$ by $L$. If $G$ splits over $K$, we obtain $G \cong \mathbb{Z}_{12} \times L$ or $(\mathbb{Z}_2 \times \mathbb{Z}_6) \times L$, otherwise we get a contradiction because $|K|$ must divide the Schur multiplier of $L$, which is $1$ and it is impossible.
If $K \cong D_{12}$, then $G = K.L$ and $G/C_G(K) \cong D_6$. Since $C_G(K)/Z(K) \cong G/K \cong L$ and $Z(K) \leq Z(C_G(K))$, we conclude that $C_G(K)$ is a central extension of $Z(K) \cong \mathbb{Z}_2$ by $L$. If $C_G(K)$ is a non-split extension, then 2 must divide the Schur multiplier of $L$, which is 1 and it is impossible. Thus $C_G(K) \cong \mathbb{Z}_2 \times L$ and hence, $G$ is a split extension of $K$ by $L$. Now, since $\text{Hom}(L, \text{Aut}(D_{12}))$ is trivial, we have $G \cong D_{12} \times L$.

If $K \cong A_4$, then $G/C_G(K) \cong A_4$. As $G = C_G(K)K$, it follows that $C_G(K) \cong L$. Therefore $G \cong L \times A_4$ or $L.A_4$.

If $K \cong T$, then By the similar way in case $K \cong D_{12}$, we can conclude that $G$ is a split extension of $K$ by $L$. Also, since $\text{Hom}(L, \text{Aut}(T))$ is trivial, we have $G \cong T \times L$. □

According to what we said before the proof, here we depict $\Gamma(M)$ by $|M|$ and $\pi_e(M)$, where $M$ is an almost simple group related to $L = D_4(4)$. 

\[ \begin{array}{c}
\text{3} \\
\text{5} \\
\text{7} \quad \text{2} \quad \text{13} \quad \text{17} \\
\end{array} \]

\begin{center}
$\Gamma(L)$
\end{center}

\[ \begin{array}{c}
\text{3} \\
\text{5} \\
\text{7} \quad \text{2} \quad \text{17} \quad \text{13} \\
\end{array} \]

\begin{center}
$\Gamma(L : 2_1)$
\end{center}

\[ \begin{array}{c}
\text{3} \\
\text{5} \\
\text{7} \quad \text{13} \quad \text{17} \\
\end{array} \]

\begin{center}
$\Gamma(L : 2_2)$
\end{center}
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OD-characterization of Almost Simple Groups Related to $D_4(4)$

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