OD-characterization of Almost Simple Groups Related to $D_4(4)$

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Abstract. Let $G$ be a finite group and $\pi_e(G)$ be the set of orders of all elements in $G$. The set $\pi_e(G)$ determines the prime graph (or Grunberg-Kegel graph) $\Gamma(G)$ whose vertex set is $\pi(G)$. The set of primes dividing the order of $G$, and two vertices $p$ and $q$ are adjacent if and only if $pq \in \pi_e(G)$. The degree $deg(p)$ of a vertex $p \in \pi(G)$, is the number of edges incident on $p$. Let $\pi(G) = \{p_1, p_2, ..., p_k\}$ with $p_1 < p_2 < ... < p_k$. We define $D(G) := (deg(p_1), deg(p_2), ..., deg(p_k))$, which is called the degree pattern of $G$. The group $G$ is called $k$-fold OD-characterizable if there exist exactly $k$ non-isomorphic groups $M$ satisfying conditions $|G| = |M|$ and $D(G) = D(M)$. Usually a 1-fold OD-characterizable group is simply called OD-characterizable. In this paper, we classify all finite groups with the same order and degree pattern as an almost simple groups related to $D_4(4)$.

Keywords: Degree pattern, $k$-fold OD-characterizable, Almost simple group.

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1. Introduction

Let $G$ be a finite group, $\pi(G)$ the set of all prime divisors of $|G|$ and $\pi_e(G)$ be the set of orders of elements in $G$. The prime graph (or Grunberg-Kegel graph) $\Gamma(G)$ of $G$ is a simple graph with vertex set $\pi(G)$ in which two vertices $p$ and $q$ are joined by an edge (and we write $p \sim q$) if and only if $G$ contains an element of order $pq$ (i.e. $pq \in \pi_e(G)$).

The degree $\text{deg}(p)$ of a vertex $p \in \pi(G)$ is the number of edges incident on $p$. If $\pi(G) = \{p_1, p_2, ..., p_k\}$ with $p_1 < p_2 < ... < p_k$, then we define $D(G) := (\text{deg}(p_1), \text{deg}(p_2), ..., \text{deg}(p_k))$, which is called the degree pattern of $G$, and leads a following definition.

**Definition 1.1.** The finite group $G$ is called $k$-fold OD-characterizable if there exist exactly $k$ non-isomorphic groups $H$ satisfying conditions $|G| = |H|$ and $D(G) = D(H)$. In particular, a 1-fold OD-characterizable group is simply called OD-characterizable.

The interest in characterizing finite groups by their degree patterns started in [7] by M. R. Darafsheh and et. all, in which the authors proved that the following simple groups are uniquely determined by their order and degree patterns: All sporadic simple groups, the alternating groups $A_p$ with $p$ and $p - 2$ primes and some simple groups of Lie type. Also in a series of articles (see [4, 6, 8, 9, 14, 17]), it was shown that many finite simple groups are OD-characterizable.

Let $A$ and $B$ be two groups then a split extension is denoted by $A : B$. If $L$ is a finite simple group and $\text{Aut}(L) \cong L : A$, then if $B$ is a cyclic subgroup of $A$ of order $n$ we will write $L : n$ for the split extension $L : B$. Moreover if there are more than one subgroup of orders $n$ in $A$, then we will denote them by $L : n_1$, $L : n_2$, etc.

**Definition 1.2.** A group $G$ is said to be an almost simple group related to $S := D_4(4)$ if and only if $S \leq G \leq \text{Aut}(S)$, for some non-abelian simple group $S$.

In many papers (see [2, 3, 10, 13, 15, 16]), it has been proved, up to now, that many finite almost simple groups are OD-characterizable or $k$-fold OD-characterizable for certain $k \geq 2$.

We denote the socle of $G$ by $\text{Soc}(G)$, which is the subgroup generated by the set of all minimal normal subgroups of $G$. For $p \in \pi(G)$, we denote by $G_p$ and $\text{Syl}_p(G)$ a Sylow $p$-subgroup of $G$ and the set of all Sylow $p$-subgroups of $G$ respectively, all further unexplained notation are standard and can be found in [11].

In this article our main aim is to show the recognizability of the almost simple groups related to $L := D_4(4)$ by degree pattern in the prime graph and
order of the group. In fact, we will prove the following Theorem.

**Main Theorem** Let $M$ be an almost simple group related to $L := D_4(4).$ If $G$ is a finite group such that $D(G) = D(M)$ and $|G| = |M|$, then the following assertions hold:

(a) If $M = L$, then $G \cong L$.
(b) If $M = L : 2$, then $G \cong L : 2$ or $L : 2$.
(c) If $M = L : 2^2$, then $G \cong L : 2^2$ or $Z_2 \times L$.
(d) If $M = L : 2^3$, then $G \cong L : 2^3$ or $L : 2^2$.
(e) If $M = L : 3$, then $G \cong L : 3$ or $Z_3 \times L$.
(f) If $M = L : 2^4$, then $G \cong L : 2^4$, $Z_2 \times (L : 2^1)$, $Z_2 \times (L : 2^2)$, $Z_2 \times (L : 2^3)$, $Z_4 \times L$ or $(Z_2 \times Z_2) \times L$.

**Lemma 2.1.** If $G$ is an almost simple group related to $L := D_4(4)$, then $G$ is isomorphic to one of the following groups: $L, L : 2^1, L : 2^2, L : 2^3, L : 3, L : 2^2, L : (D_6)_1, L : (D_6)_2, L : 6, L : D_{12}$.

**Lemma 2.2** ([5]). Let $G$ be a Frobenius group with kernel $K$ and complement $H$. Then:

(a) $K$ is a nilpotent group.
(b) $|K| \equiv 1 \pmod{|H|}$.
Let $p \geq 5$ be a prime. We denote by $\mathcal{S}_p$, the set of all simple groups with prime divisors at most $p$. Clearly, if $q \leq p$, then $\mathcal{S}_q \subseteq \mathcal{S}_p$. We list all the simple groups in class $\mathcal{S}_{17}$ with their order and the order of their outer automorphisms in TABLE 1, taken from [12].

**TABLE 1: Simple groups in $\mathcal{S}_p$, $p \leq 17$.**

| $S$  | $|S|$ | $|\text{Out}(S)|$ | $S$  | $|S|$ | $|\text{Out}(S)|$ |
|------|------|------------------|------|------|------------------|
| $A_5$ | $2^3 \cdot 3 \cdot 5$ | 2 | $G_2(3)$ | $2^6 \cdot 3^3 \cdot 7 \cdot 13$ | 2 |
| $A_6$ | $2^3 \cdot 3^2 \cdot 5$ | 4 | $3D_4(2)$ | $2^{12} \cdot 3^3 \cdot 7 \cdot 13$ | 3 |
| $S_4(3)$ | $2^6 \cdot 3^4 \cdot 5$ | 2 | $L_2(64)$ | $2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$ | 6 |
| $L_2(7)$ | $2^3 \cdot 3 \cdot 7$ | 2 | $U_4(5)$ | $2^7 \cdot 3^4 \cdot 5^7 \cdot 13$ | 4 |
| $L_2(8)$ | $2^3 \cdot 3^2 \cdot 7$ | 3 | $L_2(9)$ | $2^7 \cdot 3^6 \cdot 5 \cdot 7 \cdot 13$ | 4 |
| $U_3(3)$ | $2^3 \cdot 3^3 \cdot 7$ | 2 | $S_6(3)$ | $2^9 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$ | 2 |
| $A_7$ | $2^3 \cdot 3 \cdot 5 \cdot 7$ | 2 | $O_7(3)$ | $2^9 \cdot 3^3 \cdot 5 \cdot 7 \cdot 13$ | 2 |
| $L_2(49)$ | $2^3 \cdot 5^2 \cdot 7^2$ | 4 | $G_2(4)$ | $2^{12} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13$ | 2 |
| $U_3(5)$ | $2^3 \cdot 3^2 \cdot 5^3 \cdot 7$ | 6 | $S_4(8)$ | $2^{12} \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$ | 6 |
| $L_3(4)$ | $2^6 \cdot 3^2 \cdot 5 \cdot 7$ | 12 | $O_7^+(3)$ | $2^{12} \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$ | 24 |
| $A_8$ | $2^3 \cdot 3 \cdot 5 \cdot 7$ | 2 | $L_3(3)$ | $2^{10} \cdot 3^2 \cdot 5 \cdot 11 \cdot 13 \cdot 17$ | 2 |
| $A_9$ | $2^6 \cdot 3^3 \cdot 5 \cdot 7$ | 2 | $A_{12}$ | $2^9 \cdot 3^3 \cdot 5^7 \cdot 11 \cdot 13 \cdot 17$ | 2 |
| $J_2$ | $2^7 \cdot 3^3 \cdot 5 \cdot 7$ | 2 | $A_{14}$ | $2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17$ | 2 |
| $A_{10}$ | $2^7 \cdot 3^4 \cdot 5^2 \cdot 7$ | 2 | $A_{15}$ | $2^{10} \cdot 3^3 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17$ | 2 |
| $U_4(3)$ | $2^7 \cdot 3^3 \cdot 5 \cdot 7$ | 8 | $L_4(3)$ | $2^{11} \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17^3$ | 2 |
| $S_4(7)$ | $2^9 \cdot 3 \cdot 5 \cdot 7 \cdot 11$ | 12 | $S_4(8)$ | $2^{13} \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17$ | 2 |
| $S_6(2)$ | $2^9 \cdot 3 \cdot 5 \cdot 7 \cdot 11$ | 1 | $A_{16}$ | $2^{14} \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17$ | 2 |
| $O_7^+(2)$ | $2^{12} \cdot 3^2 \cdot 5^2 \cdot 7^2$ | 6 | $F_{422}$ | $2^{17} \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17$ | 2 |
| $L_2(11)$ | $2^2 \cdot 3 \cdot 5 \cdot 7$ | 2 | $L_2(17)$ | $2^{24} \cdot 3 \cdot 7 \cdot 11 \cdot 13 \cdot 17$ | 2 |
| $M_{11}$ | $2^3 \cdot 3 \cdot 5 \cdot 11$ | 1 | $L_2(16)$ | $2^4 \cdot 3 \cdot 5 \cdot 11$ | 2 |
| $M_{12}$ | $2^3 \cdot 3 \cdot 5 \cdot 11$ | 2 | $S_4(4)$ | $2^8 \cdot 3^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17$ | 4 |
| $U_4(2)$ | $2^{10} \cdot 3 \cdot 5 \cdot 11$ | 2 | $He$ | $2^{10} \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17$ | 2 |
| $M_{22}$ | $2^5 \cdot 3 \cdot 5 \cdot 7 \cdot 11$ | 2 | $O_7^+(2)$ | $2^{12} \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17$ | 2 |
| $A_{11}$ | $2^7 \cdot 3 \cdot 5 \cdot 7 \cdot 11$ | 2 | $L_4(4)$ | $2^{12} \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17$ | 2 |
| $M_{24}$ | $2^7 \cdot 3 \cdot 5 \cdot 7 \cdot 11$ | 2 | $S_4(2)$ | $2^{16} \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17$ | 1 |
| $H_8$ | $2^9 \cdot 3 \cdot 5 \cdot 7 \cdot 11$ | 2 | $U_4(4)$ | $2^{12} \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17$ | 4 |
| $A_{12}$ | $2^9 \cdot 3 \cdot 5 \cdot 7 \cdot 11$ | 2 | $U_4(3)$ | $2^{26} \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17^3$ | 6 |
| $U_6(2)$ | $2^{15} \cdot 3 \cdot 5 \cdot 7 \cdot 11$ | 6 | $O_7^+(2)$ | $2^{20} \cdot 3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17$ | 2 |
| $L_3(3)$ | $2^4 \cdot 3^4 \cdot 13$ | 2 | $L_2(137)$ | $2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 17$ | 4 |
| $L_2(25)$ | $2^3 \cdot 3 \cdot 5 \cdot 13$ | 4 | $S_4(13)$ | $2^6 \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 17$ | 2 |
| $U_3(4)$ | $2^6 \cdot 3 \cdot 5 \cdot 13$ | 4 | $L_3(16)$ | $2^{12} \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 17$ | 24 |
| $S_4(5)$ | $2^6 \cdot 3^2 \cdot 5 \cdot 13$ | 2 | $S_6(4)$ | $2^{18} \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 17$ | 2 |
| $L_4(3)$ | $2^7 \cdot 3 \cdot 5 \cdot 7 \cdot 11$ | 2 | $O_7^+(4)$ | $2^{24} \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 17$ | 12 |
| $2F_4(2)$ | $2^{11} \cdot 3 \cdot 5 \cdot 7 \cdot 11$ | 2 | $F_4(2)$ | $2^{24} \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 17$ | 2 |
| $L_2(13)$ | $2^2 \cdot 3 \cdot 7 \cdot 13$ | 2 | $A_{18}$ | $2^{14} \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17$ | 2 |
| $L_2(27)$ | $2^2 \cdot 3 \cdot 7 \cdot 13$ | 6 | $A_{17}$ | $2^{15} \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17$ | 2 |
Definition 2.3. A completely reducible group will be called a CR-group. The center of a CR-group is a direct product of the abelian factor in the decomposition. Hence, a CR-group is centerless, that is, has trivial center, if and only if it is a direct product of non-abelian simple groups. The following Lemma determines the structure of the automorphism group of a centerless CR-group.

Lemma 2.3 ([11]). Let \( R \) be a finite centerless CR-group and write \( R = R_1 \times R_2 \times ... \times R_k \), where \( R_i \) is a direct product of \( n_i \) isomorphic copies of a simple group \( H_i \), and \( H_i \) and \( H_j \) are not isomorphic if \( i \neq j \). Then \( \text{Aut}(R) = \text{Aut}(R_1) \times \text{Aut}(R_2) \times ... \times \text{Aut}(R_k) \) and \( \text{Aut}(R_i) \cong \text{Aut}(H_i) \wr S_{n_i} \), where in this wreath product \( \text{Aut}(H_i) \) appears in its right regular representation and the symmetric group \( S_{n_i} \) in its natural permutation representation. Moreover, these isomorphisms induce isomorphisms \( \text{Out}(R) \cong \text{Out}(R_1) \times \text{Out}(R_2) \times ... \times \text{Out}(R_k) \) and \( \text{Out}(R_i) \cong \text{Out}(H_i) \wr S_{n_i} \).

3. OD-Caracterization of Almost Simple Groups Related to \( D_4(4) \)

In this section, we study the problem of characterizing almost simple groups by order and degree pattern. Especially we will focus our attention on almost simple groups related to \( L = D_4(4) \), namely, we will prove the Main Theorem of Sec. 1. We break the proof into a number of separate propositions.

By assumption, we depict all possibilities for the prime graph associated with \( G \) by use of the variables for some vertices in each proposition. Also, we need to know the structure of \( \Gamma(M) \) to determine the possibilities for \( G \) in some proposition, therefore we depict the prime graph of all extension of \( L \) in pages 18 to 20. Note that the set of order elements in each of the following propositions is calculated using Magma.

Proposition 3.1. If \( M = L \), then \( G \cong L \).

Proof. By TABLE 1 \( |L| = 2^{24} \cdot 3^5 \cdot 5^4 \cdot 7 \cdot 13 \cdot 17^2 \). \( \pi_e(L) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 15, 17, 20, 21, 30, 34, 51, 63, 65, 85, 255\} \), so \( D(L) = (3, 4, 4, 1, 1, 3) \). Since \( |G| = |L| \) and \( D(G) = D(L) \), we conclude that the prime graph of \( G \) has following form:

![Figure 3.1](image-url)

where \( \{a, b\} = \{7, 13\} \).
We will show that \( G \) is isomorphic to \( L = D_4(4) \). We break up the proof into several steps.

**Step 1.** Let \( K \) be the maximal normal solvable subgroup of \( G \). Then \( K \) is a \( \{2, 3, 5\} \)-group. In particular, \( G \) is non-solvable.

First we show that \( K \) is a \( 17^\ast \)-group. Assume the contrary and let \( 17 \in \pi(K) \). Then 13 does not divide the order of \( K \). Otherwise, we may suppose that \( T \) is a Hall \( \{13, 17\} \)-subgroup of \( K \). It is seen that \( T \) is a nilpotent subgroup of order \( 13.17^i \) for \( i = 1 \) or \( 2 \). Thus, \( 13.17 \in \pi_c(K) \subseteq \pi_c(G) \), a contradiction. Thus \( \{17\} \subseteq \pi(K) \subseteq \pi(G) - \{13\} \). Let \( K_{17} \in \text{Syl}_{17}(K) \). By Frattini argument, \( G = KN_G(K_{17}) \). Therefore, \( N_G(K_{17}) \) contains an element \( x \) of order 13. Since \( G \) has no element of order 13, \( \langle x \rangle \) should act fixed point freely on \( K_{17} \), that is implying \( \langle x \rangle K_{17} \) is a Frobenius group. By Lemma 2.2(b), \( |\langle x \rangle||(|K_{17}| - 1) \).

It follows that \( 13|17^1 - 1 \) for \( i = 1 \) or \( 2 \), which is a contradiction.

Next, we show that \( K \) is a \( p' \)-group for \( p \in \{a, b\} \). Let \( p||K| \) and \( K_p \in \text{Syl}_p(K) \). Now by Frattini argument, \( G = KN_G(K_p) \), so 17 must divide the order of \( N_G(K_p) \). Therefore, the normalizer \( N_G(K_p) \) contains an element of order 17, say \( x \). So \( \langle x \rangle K_p \) is a cyclic subgroup of \( G \) of order 17, \( p \), and so \( p \sim 17 \) in \( \Gamma(G) \), which is a contradiction. Therefore \( K \) is a \( \{2, 3, 5\} \)-group. In addition, since \( K \) is a proper subgroup of \( G \), it follows that \( G \) is non-solvable.

**Step 2.** The quotient \( G/K \) is an almost simple group. In fact, \( S \leq G/K \leq \text{Aut}(S) \), where \( S \) is a finite non-abelian simple group isomorphic to \( L := D_4(4) \).

Let \( \overline{G} = G/K \). Then \( S := \text{Soc}(\overline{G}) = P_1 \times P_2 \times \ldots \times P_m \), where \( P_i \)'s are finite non-abelian simple groups and \( S \leq \overline{G} \leq \text{Aut}(S) \). If we show that \( m = 1 \), the proof of Step 2 will be completed.

Suppose that \( m \geq 2 \). In this case, we claim that 13 does not divide \( |S| \). Assume the contrary and let \( 13 \mid |S| \), on the other hand, \( \{2, 3\} \subseteq \pi(P_i) \) for every \( i \) (by TABLE 1), hence \( 2 \sim 13 \) and \( 3 \sim 13 \), which is a contradiction. Now, by step 1, we observe that \( 13 \in \pi(\overline{G}) \subseteq \pi(\text{Aut}(S)) \). But \( \text{Aut}(S) = \text{Aut}(S_1) \times \text{Aut}(S_2) \times \ldots \times \text{Aut}(S_r) \), where the groups \( S_j \) are direct products of isomorphic \( P_i \)'s such that \( S = S_1 \times S_2 \times \ldots \times S_r \). Therefore, for some \( j \), 13 divides the order of an automorphism group of a direct product \( S_j \) of \( t \) isomorphic simple groups \( P_i \). Since \( P_i \in \mathcal{S}_{17} \), it follows that \( |\text{Out}(P_i)| \) is not divisible by 13 (see TABLE 1). Now, by Lemma 2.3, we obtain \( |\text{Aut}(S_j)| = |\text{Aut}(P_i)|^t t! \). Therefore, \( t \geq 13 \) and so \( 2^{26} \) must divide the order of \( G \), which is a contradiction. Therefore \( m = 1 \) and \( S = P_1 \).

By TABLE 1 and Step 1, it is evident that \( |S| = 2^\alpha 3^\beta 5^\gamma 7.13.17^2 \), where \( 2 \leq \alpha \leq 24, 1 \leq \beta \leq 5 \) and \( 0 \leq \gamma \leq 4 \). Now, using collected results contained in TABLE 1, we deduce that \( S \cong D_4(4) \) and by Step 2, \( L \leq G/K \leq \text{Aut}(L) \) is completed. As \( |G| = |L| \), we deduce \( K = 1 \), so \( G \cong L \) and the proof is completed.
Proposition 3.2. If \( M = L : 2_1 \), then \( G \cong L : 2_1 \) or \( L : 2_3 \).

Proof. As \( |L : 2_1| = 2^{25}.3^5.5^4.7.13.17.19^2 \) and \( \pi_e(L : 2_1) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 24, 30, 34, 40, 42, 51, 60, 63, 65, 68, 85, 102, 126, 130, 170, 255\} \), then \( D(L : 2_1) = (4, 4, 2, 1, 3) \). Since \( |G| = |L : 2_1| \) and \( D(G) = D(L : 2_1) \), we conclude that there exist several possibilities for \( \Gamma(G) \):

\[
\begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
7 & a & 17
\end{array}
\]

where \( \{a, b, c\} = \{2, 3, 5\} \).

Step 1. Let \( K \) be the maximal normal solvable subgroup of \( G \). Then \( K \) is a \( \{2, 3, 5\} \)-group. In particular, \( G \) is non-solvable.

By a similar argument to that in Proposition 3.1, we can obtain this assertion.

Step 2. The quotient \( \frac{G}{K} \) is an almost simple group. In fact, \( S \leq \frac{G}{K} \leq \text{Aut}(S) \), where \( S \) is a finite non-abelian simple group.

The proof is similar to Step 2 of Proposition 3.1.

By TABLE 1 and Step 1, it is evident that \( |S| = 2^\alpha.3^\beta.5^\gamma.7.13.17^2 \), where \( 2 \leq \alpha \leq 25, 1 \leq \beta \leq 5 \) and \( 0 \leq \gamma \leq 4 \). Now, using collected results contained in TABLE 1, we conclude that \( S \cong D_4(4) \) and by Step 2, \( L \trianglelefteq \frac{G}{K} \trianglelefteq \text{Aut}(L) \). As \( |G| = |L : 2_1| = 2|L| \), we deduce \( |K| = 1 \) or \( 2 \).

If \( |K| = 1 \), then \( G \cong L : 2_1, L : 2_2 \) or \( L : 2_3 \). Obviously, \( G \cong L : 2_1 \) or \( L : 2_3 \) because \( \text{deg}(2) = 5 \) in \( \Gamma(L : 2_2) \) (see page 16).

If \( |K| = 2 \), then \( K \leq \text{Z}(G) \) and so \( \text{deg}(2) = 5 \), which is a contradiction. \( \square \)

Proposition 3.3. If \( M = L : 2_2 \), then \( G \cong L : 2_2 \) or \( \mathbb{Z}_2 \times L \).

Proof. As \( |L : 2_2| = 2^{25}.3^5.5^4.7.13.17^2 \) and \( \pi_e(L : 2_2) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 17, 18, 20, 21, 24, 26, 30, 34, 40, 42, 51, 60, 63, 65, 68, 85, 102, 126, 130, 170, 255\} \), then \( D(L : 2_2) = (5, 4, 4, 2, 2, 3) \). By assumption \( |G| = |L : 2_2| \) and \( D(G) = D(L : 2_2) \), so the prime graph of \( G \) has following form:

\[
\begin{array}{ccc}
3 & 5 & 2 \\
a & b & 17
\end{array}
\]

where \( \{a, b\} = \{7, 13\} \).
Step 1. Let \( K \) be the maximal normal solvable subgroup of \( G \). Then \( K \) is a \( \{2, 3, 5\} \)-group. In particular, \( G \) is non-solvable.

By similar arguments as in the proof of Step 1 in Proposition 3.1, we conclude that \( K \) is a \( \{2, 3, 5\} \)-group and \( G \) is non-solvable.

Step 2. The quotient \( \frac{G}{K} \) is an almost simple group. In fact, \( S \leq \frac{G}{K} \leq \text{Aut}(S) \), where \( S \) is a finite non-abelian simple group.

Let \( \overline{G} = \frac{G}{K} \). Then \( S := \text{Soc}(\overline{G}) \), \( S = P_1 \times P_2 \times \ldots \times P_m \), where \( P_i \)'s are finite non-abelian simple groups and \( S \leq \frac{G}{K} \subseteq \text{Aut}(S) \). We are going to prove that \( m = 1 \) and \( S = P_1 \). Suppose that \( m \geq 2 \). We claim \( a \) does not divide \( |S| \).

Assume the contrary and let \( a \mid |S| \), we conclude that \( a \) just divide the order of one of the simple groups \( P_i \)'s. Without loss of generality, we assume that \( a \| P_1 \). Then the rest of the \( P_i \)'s must be \( \{2, 3\} \)-group (because only 2 and 3 are adjacent to a in \( \Gamma(G) \)), this is a contradiction because \( P_i \)'s are finite non-abelian simple groups. Now, by Step 1, we observe that \( a \in \pi(\overline{G}) \subseteq \pi(\text{Aut}(S)) \). But \( \text{Aut}(S) = \text{Aut}(S_1) \times \text{Aut}(S_2) \times \ldots \times \text{Aut}(S_r) \), where the groups \( S_j \) are direct products of isomorphic \( P_i \)'s such that \( S = S_1 \times S_2 \times \ldots \times S_r \). Therefore, for some \( j \), \( a \) divides the order of an automorphism group of a direct product \( S_j \) of \( t \) isomorphic simple groups \( P_i \). Since \( P_i \in \mathcal{S}_7 \), it follows that \( |\text{Out}(P_i)| \) is not divisible by \( a \) (see TABLE 1), so a does not divide the order of \( \text{Aut}(P_i) \). Now, by Lemma 2.3, we obtain \( |\text{Aut}(S_j)| = |\text{Aut}(P_i)|^t \cdot t! \). Therefore, \( t \geq a \) and so \( 3^a \) must divide the order of \( G \), which is a contradiction. Therefore \( m = 1 \) and \( S = P_1 \).

By TABLE 1 and Step 1, it is evident that \( |S| = 2^\alpha \cdot 3^\beta \cdot 5^\gamma \cdot 7 \cdot 13 \cdot 17^2 \), where \( 2 \leq \alpha \leq 25 \), \( 1 \leq \beta \leq 5 \) and \( 0 \leq \gamma \leq 4 \). Now, using collected results contained in TABLE 1, we conclude that \( S \cong D_4(4) \) and by Step 2, \( L \leq \frac{G}{K} \subseteq \text{Aut}(L) \). As \( |G| = |L : 2| \cdot |2| \), we deduce \( |K| = 1 \) or 2.

If \( |K| = 1 \), then \( G \cong L : 2L : 2 \) or \( L : 23 \) because \( |G| = 2|L| \). It is obvious that \( G \cong L : 2 \), because \( \text{deg}(13) = 1 \) in \( \Gamma(L : 21) \) and \( \Gamma(L : 23) \) (see page 17).

If \( |K| = 2 \), then \( G/K \cong L \) and \( K \leq Z(G) \). It follows that \( G \) is a central extension of \( K \) by \( L \). If \( G \) is a non-split extension of \( K \) by \( L \), then \( |K| \) must divide the Schur multiplier of \( L \), which is 1. But this is a contradiction, so we obtain that \( G \) split over \( |K| \). Hence \( G \cong \mathbb{Z}_2 \times L \). \( \square \)

Proposition 3.4. If \( M = L : 23 \), then \( G \cong L : 23 \) or \( L : 21 \).

Proof. As \( |L : 23| = 2^{25} \cdot 3^2 \cdot 5^4 \cdot 7 \cdot 13 \cdot 17^2 \) and \( \pi_r(L : 23) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 24, 30, 34, 51, 63, 65, 85, 255\} \), then \( D(L : 23) = (4, 4, 2, 1, 3) \). Since \( |G| = |L : 23| \) and \( D(G) = D(L : 23) \), we conclude that \( \Gamma(G) \) has the following form similarly to Proposition 3.2:
Step 1. Let $K$ be the maximal normal solvable subgroup of $G$. Then $K$ is a $\{2, 3, 5\}$-group. In particular, $G$ is non-solvable.

We can prove this by the similar way to that in Proposition 3.2.

Step 2. The quotient $G/K$ is an almost simple group. In fact, $S \leq G/K \preceq \text{Aut}(S)$, where $S$ is a finite non-abelian simple group.

By using a similar argument, as in the proof of Proposition 3.2, we can verify that $G/K$ is an almost simple group.

By TABLE 1 and Step 1, it is evident that $|S| = 2^a \cdot 3^b \cdot 5^c \cdot 7^d \cdot 13^e \cdot 17^f$, where $2 \leq a \leq 25, 1 \leq b \leq 5$ and $0 \leq c \leq 4$. Now, using collected results contained in TABLE 1, we conclude that $S \cong D_4(4)$ and by Step 2, $L \preceq G/K \preceq \text{Aut}(L)$.

If $|K| = 1$, then $G \cong L : 21, L : 22$ or $L : 23$ because $|G| = 2|L|$. Obviously, $G \cong L : 23$ or $L : 21$, because $\deg(2) = 5$ in $\Gamma(L : 21)$ (see page 16).

If $|K| = 2$, then $K \leq Z(G)$ and so $\deg(2) = 5$, which is a contradiction. □

Proposition 3.5. If $M = L : 3$, then $G \cong L : 3$ or $\mathbb{Z}_3 \times L$.

Proof. As $|L : 3| = 2^{24} \cdot 3^6 \cdot 5^4 \cdot 7 \cdot 13 \cdot 17^2$ and $\pi_e(L : 3) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 15, 17, 18, 20, 21, 24, 30, 34, 39, 45, 51, 63, 65, 85, 255\}$, then $D(L : 3) = (3, 5, 4, 1, 2, 3)$. Since $|G| = |L : 3|$ and $D(G) = D(L : 3)$, we conclude that $\Gamma(G)$ has the following form (like $\Gamma(L : 3)$):

Step 1. Let $K$ be the maximal normal solvable subgroup of $G$. Then $K$ is a $\{2, 3\}$-group. In particular, $G$ is non-solvable.

First, we show that $K$ is a $p'$-group for $p = 7, 13$ and 17. Since the proof is quite similar to the proof of Step 1 in Proposition 3.1, so we avoid here full explanation of all details.
Next we consider $K$ is a $5'$-group. Assume the contrary, $5 \in \pi_e(K)$. Let $K_5 \in \text{Syl}_5(K)$. By Frattini argument, $G = KN_G(K_5)$. Therefore, $N_G(K_5)$ has an element $x$ of order 7. Since $G$ has no element of order 5, $\langle x \rangle$ should act fixed point freely on $K_5$, implying $\langle x \rangle K_5$ is a Frobenius group. By Lemma 2.2(b), $|\langle x \rangle|/(|K_5| - 1)$, which is impossible. Therefore $K$ is a $\{2, 3\}$-group. In addition since $K$ is a proper subgroup of $G$, then $G$ is non-solvable and the proof of this step is completed.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \leq \frac{G}{K} \lesssim \text{Aut}(S)$, where $S$ is a finite non-abelian simple group.

In a similar way as in the proof of Step 2 in Proposition 3.1, we conclude that $\frac{G}{K}$ is an almost simple group.

By TABLE 1 and Step 1, it is evident that $|S| = 2^a.3^2.5^4.7.13.17^2$, where $2 \leq \alpha \leq 24$ and $1 \leq \beta \leq 6$. Now, using collected results contained in TABLE 1, we conclude that $S \cong D_4(4)$ and by Step 2, $L \leq \frac{G}{K} \lesssim \text{Aut}(L)$. As $|G| = |L : 3| = 3|L|$, we deduce $|K| = 1$ or 3.

If $|K| = 1$, then $G \cong L : 3$.

If $|K| = 3$, then $G/K \cong L$. In this case we have $G/C_G(K) \lesssim \text{Aut}(K) \cong \mathbb{Z}_2$. Thus $|G/C_G(K)| = 1$ or 2. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is, $G$ is a central extension of $K$ by $L$. If $G$ is a non-split extension of $K$ by $L$, then $|K|$ must divide the Schur multiplier of $L$, which is 1. But this is a contradiction, so we obtain that $G$ split over $K$. Hence $G \cong \mathbb{Z}_2 \times L$. If $|G/C_G(K)| = 2$, then $K < C_G(K)$ and $1 \neq C_G(K)/K \leq G/K \cong L$, which is a contradiction since $L$ is simple.

\textbf{Proposition 3.6.} If $M = L : 2^2$, then $G \cong L : 2^2$, $\mathbb{Z}_2 \times (L : 2_1)$, $\mathbb{Z}_2 \times (L : 2_2)$, $\mathbb{Z}_2 \times (L : 2_3)$, $\mathbb{Z}_3 \times L$ or $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times L$.

\textbf{Proof.} As $|L : 2^2| = 2^{26}.3^5.5^4.7.13.17^2$ and $\pi_e(L : 2^2) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 24, 26, 30, 34, 42, 51, 60, 63, 65, 68, 85, 102, 126, 130, 170, 255\}$, then $D(L : 2^2) = \langle 5, 4, 4, 2, 2, 3 \rangle$. Since $|G| = |L : 2^2|$ and $D(G) = D(L : 2^2)$, so the prime graph of $G$ has following form similarly to Proposition 3.3:

![Figure 3.6](image-url)

where $\{a, b\} = \{7, 13\}$. 
**Step 1.** Let $K$ be the maximal normal solvable subgroup of $G$. Then $K$ is a $\{2, 3, 5\}$-group. In particular, $G$ is non-solvable.

According to Step 1 in Proposition 3.3, we have $K$ is a $\{2, 3, 5\}$-group and $G$ is non-solvable.

**Step 2.** The quotient $G/K$ is an almost simple group. In fact, $S \leq G/K \leq \text{Aut}(S)$, where $S$ is a finite non-abelian simple group.

We can prove this by the similar argument in Step 2 in Proposition 3.3.

By TABLE 1 and Step 1, it is evident that $|S| = 2^\alpha 3^\beta 5^\gamma 7 13 17^2$, where $2 \leq \alpha \leq 26$, $1 \leq \beta \leq 5$ and $0 \leq \gamma \leq 4$. Now, using collected results contained in TABLE 1, we conclude that $S \cong D_4(4)$ and by Step 2, $L \leq G/K \leq \text{Aut}(L)$. As $|G| = |L| 2^2$, we deduce $|K| = 1, 2$ or 4.

If $|K| = 1$, then $G \cong L 2^2$. If $|K| = 2$, then $K \leq Z(G)$. In this case $G$ is a central extension of $\mathbb{Z}_2$ by $L : 2_1, L : 2_2$ or $L : 2_3$. If $G$ splits over $K$ then $G \cong \mathbb{Z}_2 \times (L : 2_1), \mathbb{Z}_2 \times (L : 2_2)$ or $\mathbb{Z}_2 \times (L : 2_3)$, otherwise we get a contradiction because $|K|$ must divide the Schur multiplier of $L : 2_1, L : 2_2$ and $L : 2_3$, which is impossible.

If $|K| = 4$, then $G/K \cong L$. In this case we have $G/C_G(K) \leq \text{Aut}(K) \cong \mathbb{Z}_2$ or $S_3$. Thus $|G/C_G(K)| = 1, 2, 3$ or 6. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is, $G$ is a central extension of $K$ by $L$. If $G$ is a non-split extension of $K$ by $L$, then $|K|$ must divide the Schur multiplier of $L$, which is 1, but this is a contradiction. Therefore $G$ splits over $K$. Hence $G \cong K \times L$. So we have $G \cong \mathbb{Z}_4 \times L$ or $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times L$ because $K \cong \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. If $|G/C_G(K)| = 2, 3$ or 6, then $K < C_G(K)$ and $1 \neq C_G(K)/K \leq G/K \cong L$. Which is a contradiction, since $L$ is simple.

\[\square\]

**Proposition 3.7.** If $M = L : (D_6)_1$, then $G \cong L : (D_6)_1$, $L : 6, \mathbb{Z}_3 \times (L : 2_1)$, $\mathbb{Z}_3 \times (L : 2_2)$ or $(\mathbb{Z}_3 \times L) \mathbb{Z}_2$.

\[\text{Proof.}\] As $|L : (D_6)_1| = 2^{25} 3^6 5^4 7 13 17^2$ and $\pi_e(L : (D_6)_1) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 24, 30, 34, 39, 42, 45, 51, 60, 63, 65, 85, 255\}$, then $D(L : (D_6)_1) = \{4, 5, 4, 2, 2, 3\}$. Since $|G| = |L : (D_6)_1|$ and $D(G) = D(L : (D_6)_1)$, we conclude that there exist several possibilities for $\Gamma(G)$:

![Figure 3.7](image-url)
where \( \{a, b\} = \{7, 13\} \).

**Step 1.** Let \( K \) be the maximal normal solvable subgroup of \( G \). Then \( K \) is a \( \{2, 3, 5\} \)-group. In particular, \( G \) is non-solvable.

By the similar argument to that in Step 1 in Proposition 3.1, we can obtain this assertion.

**Step 2.** The quotient \( \frac{G}{K} \) is an almost simple group. In fact, \( S \leq \frac{G}{K} \leq \text{Aut}(S) \), where \( S \) is a finite non-abelian simple group.

The proof is similar to Step 2 in Proposition 3.3.

By TABLE 1 and Step 1, it is evident that \( |S| = 2^a.3^\beta.5^\gamma.7.13.17^2 \), where \( 2 \leq a \leq 25, 1 \leq \beta \leq 6 \) and \( 0 \leq \gamma \leq 4 \). Now, using collected results contained in TABLE 1, we conclude that \( S \cong D_4(4) \) and by Step 2, \( L \leq \frac{G}{K} \leq \text{Aut}(L) \). As \( |G| = |L : D_6| = 6|L| \), we deduce \( |K| = 1, 2, 3 \) or 6.

If \( |K| = 1 \), then \( G \cong L : (D_6)_1, L : (D_6)_2 \) or \( L : 6 \) because \( |G| = 6|L| \).

Obviously, \( G \cong L : (D_6)_1 \) or \( L : 6 \) because \( \text{deg}(2) = 5 \) in \( \Gamma(L : (D_6)_2) \).

If \( |K| = 2 \), then \( K \leq \text{Z}(G) \) and so \( \text{deg}(2) = 5 \), which is a contradiction (see page 18).

If \( |K| = 3 \), then \( G/K \cong L : 2_1, L : 2_2 \) or \( L : 2_3 \). But \( G/C_G(K) \leq \text{Aut}(K) \cong \mathbb{Z}_2 \). Thus \( |G/C_G(K)| = 1 \) or 2. If \( |G/C_G(K)| = 1 \), then \( K \leq \text{Z}(G) \), that is, \( G \) is a central extension of \( K \) by \( L : 2_1, L : 2_2 \) or \( L : 2_3 \). If \( G \) splits over \( K \), then \( G \cong \mathbb{Z}_3 \times (L : 2_1) \) or \( \mathbb{Z}_3 \times (L : 2_2) \) because in \( \Gamma(\mathbb{Z}_3 \times (L : 2_2)) \) the degree of 2 is 5. Otherwise we get a contradiction because \( |K| \) must divide the Schure multiplier of \( L : 2_1, L : 2_2 \) and \( L : 2_3 \), which is impossible. If \( |G/C_G(K)| = 2 \), then \( K < C_G(K) \) and \( 1 \neq C_G(K)/K \leq G/K \cong L : 2_1, L : 2_2 \) or \( L : 3_2 \), we obtain \( C_G(K)/K \cong L \). Since \( K \leq \text{Z}(C_G(K)) \), \( C_G(K) \) is a central extension of \( K \) by \( L \). If \( C_G(K) \) splits over \( K \), then \( C_G(K) \cong \mathbb{Z}_3 \times L \), otherwise we get a contradiction because \( |K| \) must divide the Schure multiplier of \( L \), which is impossible. Therefore, \( G \cong (\mathbb{Z}_3 \times L).\mathbb{Z}_2 \).

If \( |K| = 6 \), then \( G/K \cong L \) and \( K \cong \mathbb{Z}_6 \) or \( D_6 \).

If \( K \cong \mathbb{Z}_6 \), then \( G/C_G(K) \leq \mathbb{Z}_2 \) and so \( |G/C_G(K)| = 1 \) or 2. If \( |G/C_G(K)| = 1 \), then \( K \leq \text{Z}(G) \). It follows that \( \text{deg}(2) = 5 \), a contradiction. If \( |G/C_G(K)| = 2 \), then \( K < C_G(K) \) and \( 1 \neq C_G(K)/K \leq G/K \cong L \), which is a contradiction because \( L \) is simple.

If \( K \cong D_6 \), then \( K \cap C_G(K) = 1 \) and \( G/C_G(K) \leq D_6 \). Thus \( C_G(K) \neq 1 \).

Hence, \( 1 \neq C_G(K) \cong C_G(K)/K \leq G/K \cong L \). It follows that \( L \cong G/K \cong C_G(K) \) because \( L \) is simple. Therefore, \( G \cong D_6 \times L \), which implies that \( \text{deg}(2) = 5 \), a contradiction.

\( \square \)

**Proposition 3.8.** If \( M = L : (D_6)_2 \), then \( G \cong L : (D_6)_2, \mathbb{Z}_2 \times (L : 3), \mathbb{Z}_3 \times (L : 2_2), (\mathbb{Z}_3 \times L).\mathbb{Z}_2, \mathbb{Z}_6 \times L \) or \( S_3 \times L \).
Proof. As $|L : (D_6)_2| = 2^{25} \cdot 3^6 \cdot 5^4 \cdot 7 \cdot 13 \cdot 17^2$ and $\pi_e(L : (D_6)_2) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 17, 18, 20, 21, 24, 26, 30, 34, 39, 40, 42, 45, 51, 60, 63, 65, 68, 85, 102, 126, 130, 170, 255\}$, then $D(L : (D_6)_2) = (5, 5, 4, 2, 3, 3)$. Since $|G| = |L : (D_6)_2|$ and $D(G) = D(L : (D_6)_2)$, we conclude that $\Gamma(G)$ has the following form (like $\Gamma(L : (D_6)_2)$):

$$\text{Figure 3.8}$$

**Step 1.** Let $K$ be the maximal normal solvable subgroup of $G$. Then $K$ is a $\{2, 3\}$-group. In particular, $G$ is non-solvable.

The proof is similar to Step 1 in Proposition 3.5.

**Step 2.** The quotient $G/K$ is an almost simple group. In fact, $S \leq G/K \lesssim \text{Aut}(S)$, where $S$ is a finite non-abelian simple group.

Let $S = \text{Soc}(G)$. Then $S := \text{Soc}(G)$, $S = P_1 \times P_2 \times \cdots \times P_m$, where $P_i$'s are finite non-abelian simple groups and $S \leq G/K \lesssim \text{Aut}(S)$. We are going to prove that $m = 1$ and $S = P_1$. Suppose that $m \geq 2$. By the same argument in Step 2 of Proposition 3.3 and considering 7 instead of a, we get a contradiction. Therefore $m = 1$ and $S = P_1$.

By TABLE 1 and Step 1, it is evident that $|S| = 2^a \cdot 3^b \cdot 5^4 \cdot 7 \cdot 13 \cdot 17^2$, where $2 \leq a \leq 25$ and $1 \leq b \leq 6$. Now, using collected results contained in TABLE 1, we conclude that $S \cong D_4(4)$ and by Step 2, $L \leq G/K \lesssim \text{Aut}(L)$. As $|G| = |L : (D_6)_2| = 6|L|$, we deduce $|K| = 1, 2, 3$ or 6.

If $|K| = 1$, then $G \cong L : (D_6)_1, L : (D_6)_2$ or $L : 6$ because $|G| = 6|L|$. Obviously $G \cong L : (D_6)_2$ because in $\Gamma(L : (D_6)_1)$ and $\Gamma(L : 6)$, we have $\deg(13) = 2$ (see page 17).

If $|K| = 2$, then $K \leq Z(G)$ and $G/K \cong L : 3$. Hence $G$ is a central extension of $K$ by $L : 3$. If $G$ splits over $K$, then $G \cong Z_2 \times (L : 3)$. Otherwise we get a contradiction because $|K|$ must divide the Schur multiplier of $L : 3$, which is impossible.

If $|K| = 3$, then $G/K \cong L : 2_1, L : 2_2$ or $L : 2_3$. But $G/C_G(K) \cong \text{Aut}(K) \cong Z_2$. Thus $|G/C_G(K)| = 1$ or 2. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is, $G$ is a central extension of $K$ by $L : 2_1, L : 2_2$ or $L : 2_3$. If $G$ splits over $K$, then only $G \cong Z_3 \times (L : 2_2)$ because $2 \sim 13$ in $\Gamma(Z_3 \times (L : 2_1))$ and $\Gamma(Z_3 \times (L : 2_3))$. Otherwise we get a contradiction because $|K|$ must divide the Schur multiplier of $L : 2_1, L : 2_2$ and $L : 2_3$, which is impossible. If
\( |G/C_G(K)| = 2 \), then \( K < C_G(K) \) and \( 1 \neq C_G(K)/K \trianglelefteq G/K \cong L : 2_1, L : 2_2 \)
or \( L : 2_3 \); we obtain \( C_G(K)/K \cong L \). Since \( K \leq Z(C_G(K)) \), \( C_G(K) \) is a central extension of \( K \) by \( L \). If \( C_G(K) \) splits over \( K \), then \( C_G(K) \cong \mathbb{Z}_3 \times L \), otherwise we get a contradiction because \( |K| \) must divide the Schur multiplier of \( L \), which is impossible. Therefore, \( G \cong (\mathbb{Z}_3 \times L).\mathbb{Z}_2 \).

If \( |K| = 6 \), then \( G/K \cong L \) and \( K \cong \mathbb{Z}_6 \) or \( D_6 \). If \( K \cong \mathbb{Z}_6 \), then \( G/C_G(K) \cong Z_2 \) and so \( |G/C_G(K)| = 1 \) or \( 2 \). If \( |G/C_G(K)| = 1 \), then \( K \leq Z(G) \) and \( G/K \cong L \). Therefore \( G \) is a central extension of \( K \) by \( L \). If \( G \) is a non-split extension of \( K \) by \( L \), then \( |K| \) must divide the Schur multiplier of \( L \), which is 1. But this is a contradiction. So we obtain that \( G \) splits over \( K \). Hence \( G \cong \mathbb{Z}_6 \times L \). If \( |G/C_G(K)| = 2 \), then \( K < C_G(K) \) and \( 1 \neq C_G(K)/K \trianglelefteq G/K \cong L \), which is a contradiction because \( L \) is simple. If \( K \cong D_6 \), then \( K \cap C_G(K) = 1 \) and \( G/C_G(K) \cong D_6 \). Thus \( C_G(K) \neq 1 \). Hence, \( 1 \neq C_G(K) \cong C_G(K)/K \trianglelefteq G/K \cong L \). It follows that \( L \cong G/K \cong C_G(K) \) because \( L \) is simple. Therefore \( G \cong D_6 \times L \). \( \square \)

**Proposition 3.9.** If \( M = L : 6 \), then \( G \cong L : 6, L : (D_6)_1, \mathbb{Z}_3 \times (L : 2_1), \mathbb{Z}_3 \times (L : 2_3) \) or \((\mathbb{Z}_3 \times L)\mathbb{Z}_2 \).

**Proof.** As \( |L : 6| = 2^{25}.3^6.5^4.7.13.17^2 \) and \( \pi_e(L : 6) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 24, 30, 34, 36, 39, 42, 45, 48, 51, 63, 65, 85, 255 \} \), then \( D(L : 6) = \{4, 5, 4, 2, 2, 3\} \). Since \( |G| = |L : 6| \) and \( D(G) = D(L : 6) \), there exist several possibilities for \( \Gamma(G) \) similarly to Proposition 3.7:

\[
\begin{array}{c}
3 \\
5 \\
\text{a}
\end{array}
\begin{array}{c}
2 \\
\text{b}
\end{array}
\begin{array}{c}
17
\end{array}
\]

Figure 3.9

where \( \{a, b\} = \{7, 13\} \).

**Step 1.** Let \( K \) be the maximal normal solvable subgroup of \( G \). Then \( K \) is a \( \{2, 3, 5\} \)-group. In particular, \( G \) is non-solvable.

The proof is similar to that in Proposition 3.3.

**Step 2.** The quotient \( \frac{G}{K} \) is an almost simple group. In fact, \( S \leq \frac{G}{K} \trianglelefteq \text{Aut}(S) \), where \( S \) is a finite non-abelian simple group.

Again we refer to Step 2 of proposition 3.3 to get the proof.

By TABLE 1 and Step 1, it is evident that \( |S| = 2^\alpha.3^\beta.5^\gamma.7.13.17^2 \), where \( 2 \leq \alpha \leq 25, 1 \leq \beta \leq 6 \) and \( 0 \leq \gamma \leq 4 \). Now, using collected results contained
in TABLE 1, we conclude that \( S \cong D_4(4) \) and by Step 2, \( L \leq \frac{G}{K} \leq \text{Aut}(L) \). As \(|G| = |L : 6| = 6|L|\), we deduce \(|K| = 1, 2, 3 \) or 6.

If \(|K| = 1\), then \( G \cong L : 6, L : (D_6)_1 \) or \( L : (D_6)_2 \) because \(|G| = 6|L|\). Obviously, \( G \cong L : 6 \) or \( L : (D_6)_1 \) because \( \text{deg}(2) = 5 \) in \( \Gamma(L : (D_6)_2) \) (see page 18).

If \(|K| = 2\), then \( K \leq Z(G) \) and so \( \text{deg}(2) = 5\), which is a contradiction.

If \(|K| = 3\), then \( G/K \cong L : 2_1, L : 2_2 \) or \( L : 2_3 \). But \( G/C_G(K) \leq \text{Aut}(K) \cong \mathbb{Z}_2 \). Thus \( |G/C_G(K)| = 1 \) or 2. If \( |G/C_G(K)| = 1\), then \( K \leq Z(G)\), that is, \( G \) is a central extension of \( K \) by \( L : 2_1, L : 2_2 \) or \( L : 2_3 \). If \( G \) splits over \( K \), then \( G \cong \mathbb{Z}_3 \times (L : 2_1) \) or \( \mathbb{Z}_3 \times (L : 2_3) \) because in \( \Gamma(\mathbb{Z}_3 \times (L : 2_1)) \) the degree of 2 is 5. Otherwise we get a contradiction because \(|K|\) must divide the Schur multiplier of \( L : 2_1, L : 2_2 \) and \( L : 2_3 \), which is impossible. If \( |G/C_G(K)| = 2\), then \( K < C_G(K) \) and \( 1 \neq C_G(K)/K \leq G/K \cong L : 2_1, L : 2_2 \) or \( L : 2_3 \), we obtain \( C_G(K)/K \cong L \). Since \( K \leq Z(C_G(K)) \), \( C_G(K) \) is a central extension of \( K \) by \( L \). If \( C_G(K) \) splits over \( K \), then \( C_G(K) \cong \mathbb{Z}_3 \times L \), otherwise we get a contradiction because \(|K|\) must divide the Schur multiplier of \( L \), which is impossible. Therefore, \( G \cong \mathbb{Z}_3 \times L \times \mathbb{Z}_2 \).

If \(|K| = 6\), then \( G/K \cong L \) and \( K \cong \mathbb{Z}_6 \) or \( D_6 \). If \( K \cong \mathbb{Z}_6 \), then \( G/C_G(K) \cong \mathbb{Z}_2 \) and so \( |G/C_G(K)| = 1 \) or 2. If \( |G/C_G(K)| = 1\), then \( K \leq Z(G)\). It follows that \( \text{deg}(2) = 5\), a contradiction. If \( |G/C_G(K)| = 2\), then \( K < C_G(K) \) and \( 1 \neq C_G(K)/K \leq G/K \cong L \), which is a contradiction because \( L \) is simple. If \( K \cong D_6 \), then \( K \cap C_G(K) = 1 \) and \( G/C_G(K) \cong D_6 \). Thus \( C_G(K) \neq 1 \). Hence, \( 1 \neq C_G(K) \cong G/K \cong L \). It follows that \( L \cong G/K \cong C_G(K) \) because \( L \) is simple. Therefore, \( G \cong D_6 \times L \), which implies that \( \text{deg}(2) = 5\), a contradiction.

\begin{proposition}
If \( M = L : D_{12}, \) then \( G \cong L : D_{12}, \mathbb{Z}_2 \times (L : (D_6)_1), \mathbb{Z}_2 \times (L : (D_6)_2), \mathbb{Z}_2 \times (L : 6), \mathbb{Z}_3 \times (L : 2^2), \mathbb{Z}_3 \times (L : 2), (\mathbb{Z}_3 \times (L : 2_1)), \mathbb{Z}_2, (\mathbb{Z}_3 \times (L : 2_3)), \mathbb{Z}_2, \mathbb{Z}_4 \times (L : 3), (\mathbb{Z}_2 \times \mathbb{Z}_2) \times (L : 3), (\mathbb{Z}_4 \times L), \mathbb{Z}_3, (\mathbb{Z}_2 \times \mathbb{Z}_2) \times L, \mathbb{Z}_3, \mathbb{Z}_6 \times (L : 2_1), \mathbb{Z}_6 \times (L : 2_2), \mathbb{Z}_6 \times (L : 2_3), \mathbb{Z}_6 \times L, S_3 \times (L : 2_1), S_3 \times (L : 2_2), S_3 \times (L : 2_3), \mathbb{Z}_{12} \times L, (\mathbb{Z}_2 \times \mathbb{Z}_6) \times L, D_{12} \times L, (\mathbb{Z}_2 \times L), D_6, A_4 \times L, L.A_4 \) or \( T \times L \).
\end{proposition}

\textit{Proof.} As \( |L : D_{12}| = 2^{26}.3^6.5^4.7.13.17^2 \) and \( \pi_\infty(L : (D_{12})) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 24, 26, 30, 34, 39, 40, 42, 45, 51, 60, 63, 65, 68, 85, 102, 126, 130, 170, 255\} \), then \( D(L : D_{12}) = (5, 5, 4, 2, 3, 3) \). Since \( |G| = |L : D_{12}| \) and \( D(G) = D(L : D_{12}) \), we conclude that \( \Gamma(G) \) has the following form (like \( \Gamma(L : D_{12}) \)):
Step 1. Let $K$ be the maximal normal solvable subgroup of $G$. Then $K$ is a $\{2,3\}$-group. In particular, $G$ is non-solvable.

The proof is similar to Step 1 in Proposition 3.5.

Step 2. The quotient $G/K$ is an almost simple group. In fact, $S \leq G/K \leq \text{Aut}(S)$, where $S$ is a finite non-abelian simple group.

To get the proof, follow the way in the proof of Step 2 in Proposition 3.5.

By Table 1 and Step 1, it is evident that $|S| = 2^a \cdot 3^b \cdot 5^c \cdot 7^d \cdot 13^e \cdot 17^f$, where $2 \leq a \leq 26$ and $1 \leq b \leq 6$. Now, using collected results contained in Table 1, we conclude that $S \cong D_4(4)$ and by Step 2, $L \leq G/K \leq \text{Aut}(L)$. As $|G| = |L : D_{12}| = 12|L|$, we deduce $|K| = 1, 2, 3, 4, 6$ or 12.

If $|K| = 1$, then $G \cong L : D_{12}$.

If $|K| = 2$, then $G/K \cong (D_6)_1, (D_6)_2$ or $L : 6$ and $K \leq Z(G)$. It follows that $K$ is a central extension of $K$ by $L : (D_6)_1, (D_6)_2$ or $L : 6$. If $G$ splits over $K$, then $G \cong \mathbb{Z}_2 \times (L : (D_6)_1), \mathbb{Z}_2 \times (L : (D_6)_2)$ or $\mathbb{Z}_2 \times (L : 6)$.

Otherwise $G \cong \mathbb{Z}_2 : (L : (D_6)_1)$ or $\mathbb{Z}_2 : (L : (D_6)_2)$.

If $|K| = 3$, then $G/K \cong L : 2^3$. But $G/C_G(K) \leq \text{Aut}(L) \cong \mathbb{Z}_2$. Thus $|G/C_G(K)| = 1$ or 2. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is, $G$ is a central extension of $K$ by $L : 2^3$. If $G$ splits over $K$, then $G \cong \mathbb{Z}_3 \times (L : 2^3)$. Otherwise we get a contradiction because $|K|$ must divide the Schur multiplier of $L : 2^3$, which is impossible. If $|G/C_G(K)| = 2$, then $K \neq C_G(K)$ and $1 \neq C_G(K)/K \cong G/K \cong L : 2^3$, and we obtain $C_G(K)/K \cong L : 2_1, L : 2_2$ or $L : 2_3$. Since $K \leq Z(C_G(K))$, $C_G(K)$ is a central extension of $K$ by $L : 2_1, L : 2_2$ or $L : 2_3$. Thus $C_G(K) \cong \mathbb{Z}_3 \times (L : 2_1), \mathbb{Z}_3 \times (L : 2_2)$ or $\mathbb{Z}_3 \times (L : 2_3)$, otherwise we get a contradiction because 3 must divide the Schur multiplier of $L : 2_1, L : 2_2$ or $L : 2_3$, which is impossible. Therefore, $G \cong (\mathbb{Z}_3 \times (L : 2_1)), \mathbb{Z}_2, (\mathbb{Z}_3 \times (L : 2_2)), \mathbb{Z}_2$ or $(\mathbb{Z}_3 \times (L : 2_3)) \times \mathbb{Z}_2$.

If $|K| = 4$, then $G/K \cong L : 3$ and $K \cong \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. In this case we have $G/C_G(K) \leq \text{Aut}(K) \cong \mathbb{Z}_2$ or $S_3$. Thus $|G/C_G(K)| = 1, 2, 3$ or 6. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is, $G$ is a central extension of $K$ by $L : 3$. If $G$ splits over $K$ by $L : 3$, then $G \cong \mathbb{Z}_4 \times (L : 3)$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times (L : 3)$. Otherwise we get a contradiction because $|K|$ must divide the Schur multiplier of $L : 3$, which is impossible. If $|G/C_G(K)| \neq 1$, since $|G/C_G(K)| = 2, 3$ or 6, it follows that $K < C_G(K)$. As $L$ is simple, we conclude that $1 \neq C_G(K)/K$ must
be an extension of $L$. Hence $|G/C_G(K)| = 3$ and therefore $C_G(K)/K \cong L$. Now, since $K \leq Z(C_G(K))$, we conclude that $C_G(K)$ is a central extension of $K$ by $L$. Thus $C_G(K) \cong \mathbb{Z}_4 \times L$, or $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times L$, otherwise $|K|$ must divide the Schur multiplier of $L$, which is 1 and it is impossible. Therefore, $G \cong (\mathbb{Z}_4 \times L).\mathbb{Z}_3 \text{ or } ((\mathbb{Z}_2 \times \mathbb{Z}_2) \times L).\mathbb{Z}_3$.

If $|K| = 6$, then $G/K \cong L : 2$, $L : 2_2$ or $L : 2_3$ and $K \cong \mathbb{Z}_6$ or $D_6$. If $K \cong \mathbb{Z}_6$, then $G/C_G(K) \leq \mathbb{Z}_2$ and so $|G/C_G(K)| = 1$ or 2. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is $G$ is a central extension of $\mathbb{Z}_6$ by $L : 2_1$, $L : 2_2$ or $L : 2_3$. If $G$ splits over $K$, we obtain $G \cong \mathbb{Z}_6 \times (L : 2_1)$, $\mathbb{Z}_6 \times (L : 2_2)$ or $\mathbb{Z}_6 \times (L : 2_3)$, otherwise we get a contradiction because $|K|$ must divide the Schur multiplier of $L$, which is 1 and it is impossible. If $|G/C_G(K)| = 2$, then $K < C_G(K)$ and $1 \neq C_G(K)/K \unlhd G/K \cong L : 2_1$, $L : 2_2$ or $L : 2_3$, and we obtain $C_G(K)/K \cong L$. Since $K \leq Z(C_G(K))$, $C_G(K)$ is a central extension of $K$ by $L$. Thus $C_G(K) \cong \mathbb{Z}_6 \times L$, otherwise we get a contradiction because $|K|$ must divide the Schur multiplier of $L$. Therefore $G \cong (\mathbb{Z}_6 \times L).\mathbb{Z}_2$. If $K \cong D_6$, then $G/C_G(K) \leq D_6$ and so $|G/C_G(K)| = 1, 2, 3$ or 6. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is a contradiction. If $|G/C_G(K)| = 2$, then we have $|KC_G(K)| = 6, |G|/2 = 3|G|$ because $K \cap C_G(K) = 1$, which is a contradiction. If $|G/C_G(K)| = 3$, then we have $|KC_G(K)| = 6, |G|/3 = 2|G|$ because $K \cap C_G(K) = 1$, which is a contradiction. If $|G/C_G(K)| = 6$, then $G/C_G(K) \cong D_6$ and $C_G(K) \neq 1$. Hence, $1 \neq C_G(K) \cong C_G(K)/K \unlhd G/K \cong L : 2_1, L : 2_2$ or $L : 2_3$. It follows that $C_G(K) \cong L : 2_1, L : 2_2$ or $L : 2_3$ because $L$ is simple. Therefore, $G \cong D_6 \times (L : 2_1), D_6 \times (L : 2_2)$ or $D_6 \times (L : 2_3)$.

Before processing the last case, we recall the following facts.

There exist five non-isomorphic groups of order 12. Two of them are abelian and three are non-abelian. The non-abelian groups are: alternating group $A_4$, dihedral group $D_{12}$ and the dicyclic group $T$ with generators $a$ and $b$, subject to the relations $a^6 = 1$, $a^3 = b^2$ and $b^{-1}ab = a^{-1}$.

If $|K| = 12$, then $G/K \cong L$ and $K \cong \mathbb{Z}_{12}$, $\mathbb{Z}_2 \times \mathbb{Z}_6$, $D_{12}$, $A_4$ or $T$. But $C_G(K)/K \unlhd G/K \cong L$. If $C_G(K)/K \cong L$, then $C_G(K) \leq K$ and hence $|L| = |G/K||G/C_G(K)||\text{Aut}(K)|$. Thus $|L||\text{Aut}(K)|$, a contradiction. Therefore, $C_G(K)/K \not\unlhd L$ and since $L$ is simple group, we conclude that $G = C_G(K)K$ and hence, $G/C_G(K) \cong K/Z(K)$. Now, we should consider the following cases:

If $K \cong \mathbb{Z}_{12}$ or $\mathbb{Z}_2 \times \mathbb{Z}_6$, then $G/C_G(K) = 1$. Therefore $K \leq Z(G)$, that is $G$ is a central extension of $\mathbb{Z}_{12}$ or $\mathbb{Z}_2 \times \mathbb{Z}_6$ by $L$. If $G$ splits over $K$, we obtain $G \cong \mathbb{Z}_{12} \times L$ or $(\mathbb{Z}_2 \times \mathbb{Z}_6) \times L$, otherwise we get a contradiction because $|K|$ must divide the Schur multiplier of $L$, which is 1 and it is impossible.
If \( K \cong D_{12} \), then \( G = K.L \) and \( G/C_G(K) \cong D_6 \). Since \( C_G(K)/Z(K) \cong G/K \cong L \) and \( Z(K) \leq Z(C_G(K)) \), we conclude that \( C_G(K) \) is a central extension of \( Z(K) \cong \mathbb{Z}_2 \) by \( L \). If \( C_G(K) \) is a non-split extension, then 2 must divide the Schur multiplier of \( L \), which is 1 and it is impossible. Thus \( C_G(K) \cong \mathbb{Z}_2 \times L \) and hence, \( G \) is a split extension of \( K \) by \( L \). Now, since \( \text{Hom}(L, \text{Aut}(D_{12})) \) is trivial, we have \( G \cong D_{12} \times L \).

If \( K \cong \mathbb{A}_4 \), then \( G/C_G(K) \cong \mathbb{A}_4 \). As \( G = C_G(K)K \), it follows that \( C_G(K) \cong L \). Therefore \( G \cong L \times \mathbb{A}_4 \) or \( L.\mathbb{A}_4 \).

If \( K \cong T \), then by the similar way in case \( K \cong D_{12} \), we can conclude that \( G \) is a split extension of \( K \) by \( L \). Also, since \( \text{Hom}(L, \text{Aut}(T)) \) is trivial, we have \( G \cong T \times L \).

\( \square \)

According to what we said before the proof, here we depict \( \Gamma(M) \) by \(|M|\) and \( \pi_e(M) \), where \( M \) is an almost simple group related to \( L = D_4(4) \).
OD-characterization of Almost Simple Groups Related to $D_4(4)$

- $\Gamma(L : 2_3)$
- $\Gamma(L : 3)$
- $\Gamma(L : 2^2)$
- $\Gamma(L : (D_6)_1)$
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References