Iranian Journal of Mathematical Sciences and Informatics
Vol. 10, No. 1 (2015), pp 23-43
DOI: 10.7508/ijmsi.2015.01.003

OD-characterization of Almost Simple Groups Related to $D_4(4)$

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Abstract. Let G be a finite group and $\pi_e(G)$ be the set of orders of all elements in G. The set $\pi_e(G)$ determines the prime graph (or Grunberg-Kegel graph) $\Gamma(G)$ whose vertex set is $\pi(G)$. The set of primes dividing the order of G, and two vertices p and q are adjacent if and only if $pq \in \pi_e(G)$. The degree $deg(p)$ of a vertex $p \in \pi(G)$, is the number of edges incident on p. Let $\pi(G) = \{p_1, p_2, ..., p_k\}$ with $p_1 < p_2 < ... < p_k$. We define $D(G) := (deg(p_1), deg(p_2), ..., deg(p_k))$, which is called the degree pattern of G. The group G is called k-fold OD-characterizable if there exist exactly k non-isomorphic groups M satisfying conditions $|G| = |M|$ and $D(G) = D(M)$. Usually a 1-fold OD-characterizable group is simply called OD-characterizable. In this paper, we classify all finite groups with the same order and degree pattern as an almost simple groups related to $D_4(4)$.

Keywords: Degree pattern, k-fold OD-characterizable, Almost simple group.

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Received 12 February 2013; Accepted 08 August 2013
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[DOI: 10.7508/ijmsi.2015.01.003]

2000 Mathematics subject classification: 20D05, 20D60, 20D06.

1. Introduction

Let G be a finite group, $\pi(G)$ the set of all prime divisors of $|G|$ and $\pi_e(G)$ be the set of orders of elements in G . The prime graph (or Grunberg-Kegel graph) $\Gamma(G)$ of G is a simple graph with vertex set $\pi(G)$ in which two vertices p and q are joined by an edge (and we write $p \sim q$) if and only if G contains an element of order pq (i.e. $pq \in \pi_e(G)$).

The degree $\deg(p)$ of a vertex $p \in \pi(G)$ is the number of edges incident on p. If $\pi(G) = \{p_1, p_2, ..., p_k\}$ with $p_1 < p_2 < ... < p_k$, then we define D(G):=(deg(p_1),deg(p_2),...,deg(p_k)), which is called the degree pattern of G, and leads a following definition.

Definition 1.1. The finite group G is called k-fold OD-characterizable if there exist exactly k non-isomorphic groups H satisfying conditions $|G| = |H|$ and $D(G) = D(H)$. In particular, a 1-fold OD-characterizable group is simply called OD-characterizable.

The interest in characterizing finite groups by their degree patterns started in [7] by M. R. Darafsheh and et. all, in which the authors proved that the following simple groups are uniquely determined by their order and degree patterns: All sporadic simple groups, the alternating groups A_p with p and $p-2$ primes and some simple groups of Lie type. Also in a series of articles (see [4, 6, 8, 9, 14, 17]), it was shown that many finite simple groups are OD-characterizable.

Let A and B be two groups then a split extension is denoted by $A : B$. If L is a finite simple group and A$ut(L) \cong L : A$, then if B is a cyclic subgroup of A of order n we will write $L : n$ for the split extension $L : B$. Moreover if there are more than one subgroup of orders n in A, then we will denote them by $L : n_1$, $L : n_2$, etc.

Definition 1.2. A group G is said to be an almost simple group related to S if and only if $S \leq G \lesssim$ A$ut(S)$, for some non-abelian simple group S.

In many papers (see [2, 3, 10, 13, 15, 16]), it has been proved, up to now, that many finite almost simple groups are OD-characterizable or k-fold OD-characterizable for certain $k \geq 2$.

We denote the socle of G by Soc(G), which is the subgroup generated by the set of all minimal normal subgroups of G. For $p \in \pi(G)$, we denote by G_p and Syl$_p(G)$ a Sylow p-subgroup of G and the set of all Sylow p-subgroups of G respectively, all further unexplained notation are standard and can be found in [11].

In this article our main aim is to show the recognizability of the almost simple groups related to $L := D_4(4)$ by degree pattern in the prime graph and

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[DOI: 10.7508/ijmsi.2015.01.003]

order of the group. In fact, we will prove the following Theorem.

Main Theorem Let M be an almost simple group related to $L := D_4(4)$. If G is a finite group such that $D(G) = D(M)$ and $|G| = |M|$, then the following assertions hold:

(a) If $M = L$, then $G \cong L$.

(b) If $M = L : 2_1$, then $G \cong L : 2_1$ or $L : 2_3$.

(c) If $M = L : 2_2$, then $G \cong L : 2_2$ or $\mathbb{Z}_2 \times L$.

(d) If $M = L : 2_3$, then $G \cong L : 2_3$ or $L : 2_1$.

(e) If $M = L : 3$, then $G \cong L : 3$ or $\mathbb{Z}_3 \times L$.

(f) If $M = L : 2^2$, then $G \cong L : 2^2$, $\mathbb{Z}_2 \times (L : 2_1)$, $\mathbb{Z}_2 \times (L : 2_2)$, $\mathbb{Z}_2 \times (L : 2_3)$, $\mathbb{Z}_4 \times L$ or $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times L$.

(g) If $M = L : (D_6)_1$, then $G \cong L : (D_6)_1$, $L : 6$, $\mathbb{Z}_3 \times (L : 2_1)$, $\mathbb{Z}_3 \times (L : 2_3)$ or $(\mathbb{Z}_3 \times L).\mathbb{Z}_2$.

(h) If $M = L : (D_6)_2$, then $G \cong L : (D_6)_2$, $\mathbb{Z}_2 \times (L : 3)$, $\mathbb{Z}_3 \times (L : 2_2)$, $(\mathbb{Z}_3 \times L).\mathbb{Z}_2$, $\mathbb{Z}_6 \times L$ or $D_6 \times L$.

(i) If $M = L : 6$, then $G \cong L : 6$, $L : (D_6)_1$, $\mathbb{Z}_3 \times (L : 2_1)$, $\mathbb{Z}_3 \times (L : 2_3)$ or $(\mathbb{Z}_3 \times L).\mathbb{Z}_2$.

(j) If $M = L : D_{12}$, then $G \cong L : D_{12}$, $\mathbb{Z}_2 \times (L : (D_6)_1)$, $\mathbb{Z}_2 \times (L : (D_6)_2)$, $\mathbb{Z}_2 \times (L : 6)$, $\mathbb{Z}_3 \times (L : 2^2)$, $(\mathbb{Z}_3 \times (L : 2_1)).\mathbb{Z}_2$, $(\mathbb{Z}_3 \times (L : 2_2)).\mathbb{Z}_2$, $(\mathbb{Z}_3 \times (L : 2_3)).\mathbb{Z}_2$, $\mathbb{Z}_4 \times (L : 3)$, $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times (L : 3)$, $(\mathbb{Z}_4 \times L).\mathbb{Z}_3$, $((\mathbb{Z}_2 \times \mathbb{Z}_2) \times L).\mathbb{Z}_3$, $\mathbb{Z}_6 \times (L : 2_1)$, $\mathbb{Z}_6 \times (L : 2_2)$, $\mathbb{Z}_6 \times (L : 2_3)$, $(\mathbb{Z}_6 \times L).\mathbb{Z}_2$, $D_6 \times (L : 2_1)$, $D_6 \times (L : 2_2)$, $D_6 \times (L : 2_3)$, $\mathbb{Z}_{12} \times L$, $(\mathbb{Z}_2 \times \mathbb{Z}_6) \times L$, $(\mathbb{Z}_2 \times L).D_6$, $\mathbb{A}_4 \times L$, $L.\mathbb{A}_4$, $D_{12} \times L$ or $T \times L$.

2. Preliminary Results

It is well-known that $\mathrm{Aut}(D_4(4)) \cong D_4(4) : D_{12}$ where D_{12} denotes the dihedral group of order 12. We remark that D_{12} has the following non-trivial proper subgroups up to conjugacy: three subgroups of order 2, one cyclic subgroup each of order 3 and 6, two subgroups isomorphic to $D_6 \cong \mathbb{S}_3$ and one subgroup of order 4 isomorphic to the Klein's four group denoted by 2^2. The field and the duality automorphisms of $D_4(4)$ are denoted by 2_1 and 2_2 respectively, and we set $2_3 = 2_1.2_2$ (field$*$duality which is called the diagonal automorphism). Therefore up to conjugacy we have the following almost simple groups related to $D_4(4)$.

Lemma 2.1. *If G is an almost simple group related to $L{:=}D_4(4)$, then G is isomorphic to one of the following groups: $L, L : 2_1, L : 2_2, L : 2_3, L : 3, L : 2^2, L : (D_6)_1, L : (D_6)_2, L : 6, L : D_{12}$.*

Lemma 2.2 ([5])**.** *Let G be a Frobenius group with kernel K and complement H. Then:*

(a) K is a nilpotent group.

(b) $|K| \equiv 1 (\mathrm{mod} |H|)$.

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[DOI: 10.7508/ijmsi.2015.01.003]

Let $p \geq 5$ be a prime. We denote by $\mathfrak{S}_p$ the set of all simple groups with prime divisors at most p. Clearly, if $q \leq p$, then $\mathfrak{S}_q \subseteq \mathfrak{S}_p$. We list all the simple groups in class $\mathfrak{S}_{17}$ with their order and the order of their outer automorphisms in TABLE 1, taken from [12].

TABLE 1: Simple groups in $\mathfrak{S}_p$, $p \leq 17$.

S	$\lvert S\rvert$	$\lvert\mathrm{Out}(S)\rvert$	S	$\lvert S\rvert$	$\lvert\mathrm{Out}(S)\rvert$
A_5	$2^2 \cdot 3 \cdot 5$	2	$G_2(3)$	$2^6 \cdot 3^6 \cdot 7 \cdot 13$	2
A_6	$2^3 \cdot 3^2 \cdot 5$	4	${}^3D_4(2)$	$2^{12} \cdot 3^4 \cdot 7^2 \cdot 13$	3
$S_4(3)$	$2^6 \cdot 3^4 \cdot 5$	2	$L_2(64)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$	6
$L_2(7)$	$2^3 \cdot 3 \cdot 7$	2	$U_4(5)$	$2^7 \cdot 3^4 \cdot 5^6 \cdot 7 \cdot 13$	4
$L_2(8)$	$2^3 \cdot 3^2 \cdot 7$	3	$L_3(9)$	$2^7 \cdot 3^6 \cdot 5 \cdot 7 \cdot 13$	4
$U_3(3)$	$2^5 \cdot 3^3 \cdot 7$	2	$S_6(3)$	$2^9 \cdot 3^9 \cdot 5 \cdot 7 \cdot 13$	2
A_7	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	2	$O_7(3)$	$2^9 \cdot 3^9 \cdot 5 \cdot 7 \cdot 13$	2
$L_2(49)$	$2^4 \cdot 3 \cdot 5^2 \cdot 7^2$	4	$G_2(4)$	$2^{12} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13$	2
$U_3(5)$	$2^4 \cdot 3^2 \cdot 5^3 \cdot 7$	6	$S_4(8)$	$2^{12} \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 13$	6
$L_3(4)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	12	$O_8^+(3)$	$2^{12} \cdot 3^{12} \cdot 5^2 \cdot 7 \cdot 13$	24
A_8	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	2	$L_5(3)$	$2^9 \cdot 3^{10} \cdot 5 \cdot 11^2 \cdot 13$	2
A_9	$2^6 \cdot 3^4 \cdot 5 \cdot 7$	2	A_{13}	$2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	2
J_2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	2	A_{14}	$2^{10} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13$	2
A_{10}	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7$	2	A_{15}	$2^{10} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$	2
$U_4(3)$	$2^7 \cdot 3^6 \cdot 5 \cdot 7$	8	$L_6(3)$	$2^{11} \cdot 3^{15} \cdot 5 \cdot 7 \cdot 11^2 \cdot 13^2$	4
$S_4(7)$	$2^8 \cdot 3^2 \cdot 5^2 \cdot 7^4$	2	Suz	$2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	2
$S_6(2)$	$2^9 \cdot 3^4 \cdot 5 \cdot 7$	1	A_{16}	$2^{14} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$	2
$O_8^+(2)$	$2^{12} \cdot 3^5 \cdot 5^2 \cdot 7$	6	Fi_{22}	$2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	2
$L_2(11)$	$2^2 \cdot 3 \cdot 5 \cdot 11$	2	$L_2(17)$	$2^4 \cdot 3^2 \cdot 17$	2
M_{11}	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	1	$L_2(16)$	$2^4 \cdot 3 \cdot 5 \cdot 17$	4
M_{12}	$2^6 \cdot 3^3 \cdot 5 \cdot 11$	2	$S_4(4)$	$2^8 \cdot 3^2 \cdot 5^2 \cdot 17$	4
$U_5(2)$	$2^{10} \cdot 3^5 \cdot 5 \cdot 11$	2	He	$2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$	2
M_{22}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	2	$O_8^-(2)$	$2^{12} \cdot 3^4 \cdot 5 \cdot 7 \cdot 17$	2
A_{11}	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$	2	$L_4(4)$	$2^{12} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 17$	4
M^cL	$2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$	2	$S_8(2)$	$2^{16} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17$	1
HS	$2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$	2	$U_4(4)$	$2^{12} \cdot 3^2 \cdot 5^3 \cdot 13 \cdot 17$	4
A_{12}	$2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$	2	$U_3(17)$	$2^6 \cdot 3^4 \cdot 7 \cdot 13 \cdot 17^3$	6
$U_6(2)$	$2^{15} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11$	6	$O_{10}^-(2)$	$2^{20} \cdot 3^6 \cdot 5^2 \cdot 7 \cdot 11 \cdot 17$	2
$L_3(3)$	$2^4 \cdot 3^3 \cdot 13$	2	$L_2(13^2)$	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 13^2 \cdot 17$	4
$L_2(25)$	$2^3 \cdot 3 \cdot 5^2 \cdot 13$	4	$S_4(13)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 13^4 \cdot 17$	2
$U_3(4)$	$2^6 \cdot 3 \cdot 5^2 \cdot 13$	4	$L_3(16)$	$2^{12} \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 13 \cdot 17$	24
$S_4(5)$	$2^6 \cdot 3^2 \cdot 5^4 \cdot 13$	2	$S_6(4)$	$2^{18} \cdot 3^4 \cdot 5^3 \cdot 7 \cdot 13 \cdot 17$	2
$L_4(3)$	$2^7 \cdot 3^6 \cdot 5 \cdot 13$	4	$O_8^+(4)$	$2^{24} \cdot 3^5 \cdot 5^4 \cdot 7 \cdot 13 \cdot 17^2$	12
${}^2F_4(2)'$	$2^{11} \cdot 3^3 \cdot 5^2 \cdot 13$	2	$F_4(2)$	$2^{24} \cdot 3^6 \cdot 5^2 \cdot 7^2 \cdot 13 \cdot 17$	2
$L_2(13)$	$2^2 \cdot 3 \cdot 7 \cdot 13$	2	A_{17}	$2^{14} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17$	2
$L_2(27)$	$2^2 \cdot 3^3 \cdot 7 \cdot 13$	6	A_{18}	$2^{15} \cdot 3^8 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17$	2

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Definition 2.3. A completely reducible group will be called a CR-group. The center of a CR-group is a direct product of the abelian factor in the decomposition. Hence, a CR-group is centerless, that is, has trivial center, if and only if it is a direct product of non-abelian simple groups. The following Lemma determines the structure of the automorphism group of a centerless CR-group.

Lemma 2.3 ([11])**.** *Let R be a finite centerless CR-group and write $R = R_1 \times R_2 \times ... \times R_k$, where R_i is a direct product of n_i isomorphic copies of a simple group H_i, and H_i and H_j are not isomorphic if $i \neq j$. Then* $\mathrm{Aut}(R) = \mathrm{Aut}(R_1) \times \mathrm{Aut}(R_2) \times ... \times \mathrm{Aut}(R_k)$ *and* $\mathrm{Aut}(R_i) \cong \mathrm{Aut}(H_i) \wr \mathbb{S}_{n_i}$*, where in this wreath product* $\mathrm{Aut}(H_i)$ *appears in its right regular representation and the symmetric group* $\mathbb{S}_{n_i}$ *in its natural permutation representation. Moreover, these isomorphisms induce isomorphisms* $\mathrm{Out}(R) \cong \mathrm{Out}(R_1) \times \mathrm{Out}(R_2) \times ... \times \mathrm{Out}(R_k)$ *and* $\mathrm{Out}(R_i) \cong \mathrm{Out}(H_i) \wr \mathbb{S}_{n_i}$*.*

3. OD-Characterization of Almost Simple Groups Related to $D_4(4)$

In this section, we study the problem of characterizing almost simple groups by order and degree pattern. Especially we will focus our attention on almost simple groups related to $L = D_4(4)$, namely, we will prove the Main Theorem of Sec. 1. We break the proof into a number of separate propositions.
By assumption, we depict all possibilities for the prime graph associated with G by use of the variables for some vertices in each proposition. Also, we need to know the structure of $\Gamma(M)$ to determine the possibilities for G in some proposition, therefore we depict the prime graph of all extension of L in pages 18 to 20. Note that the set of order elements in each of the following propositions is calculated using Magma.

Proposition 3.1. *If $M = L$, then $G \cong L$.*

Proof. By TABLE 1 $|L| = 2^{24}.3^5.5^4.7.13.17^2$. $\pi_e(L) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 15, 17, 20, 21, 30, 34, 51, 63, 65, 85, 255\}$, so $D(L) = (3, 4, 4, 1, 1, 3)$. Since $|G| = |L|$ and $D(G) = D(L)$, we conclude that the prime graph of G has following form:

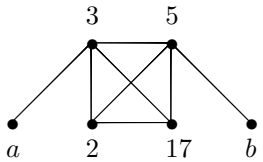

Figure 3.1

where $\{a, b\} = \{7, 13\}$.

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[DOI: 10.7508/ijmsi.2015.01.003]

We will show that G is isomorphic to $L = D_4(4)$. We break up the proof into a several steps.

Step1. Let K be the maximal normal solvable subgroup of G. Then K is a $\{2,3,5\}$-group. In particular, G is non-solvable.

First we show that K is a $17^{'}$-group. Assume the contrary and let $17 \in \pi(K)$. Then 13 dose not divide the order of K. Otherwise, we may suppose that T is a Hall $\{13,17\}$-subgroup of K. It is seen that T is a nilpotent subgroup of order 13.17^i for $i = 1$ or 2. Thus, $13.17 \in \pi_e(K) \subseteq \pi_e(G)$, a contradiction. Thus $\{17\} \subseteq \pi(K) \subseteq \pi(G) - \{13\}$. Let $K_{17} \in \mathrm{Syl}_{17}(K)$. By Frattini argument, $G = KN_G(K_{17})$. Therefore, $N_G(K_{17})$ contains an element x of order 13. Since G has no element of order 13.17, $\langle x \rangle$ should act fixed point freely on K_{17}, that is implying $\langle x \rangle K_{17}$ is a Frobenius group. By Lemma 2.2(b), $|\langle x \rangle||(|K_{17}| - 1)$. It follows that $13|17^i - 1$ for $i = 1$ or 2, which is a contradiction.

Next, we show that K is a $p^{'}$-group for $p \in \{a, b\}$. Let $p||K|$ and $K_p \in \mathrm{Syl}_p(K)$. Now by Frattini argument, $G = KN_G(K_p)$, so 17 must divide the order of $N_G(K_p)$. Therefore, the normalizer $N_G(K_p)$ contains an element of order 17, say x. So $\langle x \rangle K_p$ is a cyclic subgroup of G of order $17.p$, and so $p \sim 17$ in $\Gamma(G)$, which is a contradiction. Therefore K is a $\{2,3,5\}$-group. In addition, since K is a proper subgroup of G, it follows that G is non-solvable.

Step 2. The quotient G/K is an almost simple group. In fact, $S \leq G/K \lesssim \mathrm{Aut(S)}$, where S is a finite non-abelian simple group isomorphic to $L := D_4(4)$.

Let $\overline{G} = G/K$. Then $S := \mathrm{Soc}(\overline{G}) = P_1 \times P_2 \times ... \times P_m$, where $P_i{}^{'}s$ are finite non-abelian simple groups and $S \leq \frac{G}{K} \lesssim \mathrm{Aut}(S)$. If we show that $m = 1$, the proof of Step 2 will be completed.

Suppose that $m \geq 2$. In this case, we claim that 13 does not divide $|S|$. Assume the contrary and let $13 \mid |S|$, on the other hand, $\{2,3\} \subset \pi(P_i)$ for every i (by TABLE 1), hence $2 \sim 13$ and $3 \sim 13$, which is a contradiction. Now, by step 1, we observe that $13 \in \pi(\overline{G}) \subseteq \pi(\mathrm{Aut}(S))$. But $\mathrm{Aut}(S) = \mathrm{Aut}(S_1) \times \mathrm{Aut}(S_2) \times ... \times \mathrm{Aut}(S_r)$, where the groups S_j are direct products of isomorphic $P_i{}^{'}s$ such that $S = S_1 \times S_2 \times ... \times S_r$. Therefore, for some j, 13 divides the order of an automorphism group of a direct product S_j of t isomorphic simple groups P_i. Since $P_i \in \mathfrak{S}_{17}$, it follows that $|\mathrm{Out}(P_i)|$ is not divisible by 13 (see TABLE 1). Now, by Lemma 2.3, we obtain $|\mathrm{Aut}(S_j)| = |\mathrm{Aut}(P_i)|^{t!}.t!$. Therefore, $t \geq 13$ and so 2^{26} must divide the order of G, which is a contradiction. Therefore $m = 1$ and $S = P_1$.

By TABLE 1 and Step 1, it is evident that $|S| = 2^{\alpha}.3^{\beta}.5^{\gamma}.7.13.17^2$, where $2 \leq \alpha \leq 24$, $1 \leq \beta \leq 5$ and $0 \leq \gamma \leq 4$. Now, using collected results contained in TABLE 1, we deduce that $S \cong D_4(4)$ and by Step 2, $L \trianglelefteq G/K \lesssim \mathrm{Aut}(L)$ is completed. As $|G| = |L|$, we deduce $K = 1$, so $G \cong L$ and the proof is completed.

□

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Proposition 3.2. *If $M = L : 2_1$, then $G \cong L : 2_1$ or $L : 2_3$.*

Proof. As $|L : 2_1| = 2^{25}.3^5.5^4.7.13.17^2$ and $\pi_e(L : 2_1) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 24, 30, 34, 51, 63, 65, 85, 255\}$, then $D(L : 2_1) = (4, 4, 4, 2, 1, 3)$. Since $|G| = |L : 2_1|$ and $D(G) = D(L : 2_1)$, we conclude that there exist several possibilities for $\Gamma(G)$:

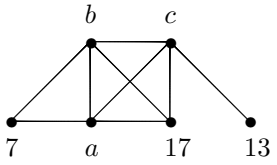

Figure 3.2

where $\{a, b, c\} = \{2, 3, 5\}$.

Step1. Let K be the maximal normal solvable subgroup of G. Then K is a $\{2, 3, 5\}$-group. In particular, G is non-solvable.
By a similar argument to that in Proposition 3.1, we can obtain this assertion.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \leq \frac{G}{K} \lesssim \mathrm{Aut(S)}$, where S is a finite non-abelian simple group.
The proof is similar to Step 2 of Proposition 3.1.

By TABLE 1 and Step 1, it is evident that $|S| = 2^{\alpha}.3^{\beta}.5^{\gamma}.7.13.17^2$, where $2 \leq \alpha \leq 25$, $1 \leq \beta \leq 5$ and $0 \leq \gamma \leq 4$. Now, using collected results contained in TABLE 1, we conclude that $S \cong D_4(4)$ and by Step 2, $L \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut}(L)$. As $|G| = |L : 2_1| = 2|L|$, we deduce $|K| = 1$ or 2.

If $|K| = 1$, then $G \cong L : 2_1, L : 2_2$ or $L : 2_3$. Obviously, $G \cong L : 2_1$ or $L : 2_3$ because $deg(2) = 5$ in $\Gamma(L : 2_2)$ (see page 16).

If $|K| = 2$, then $K \leq Z(G)$ and so $deg(2) = 5$, which is a contradiction. □

Proposition 3.3. *If $M = L : 2_2$, then $G \cong L : 2_2$ or $\mathbb{Z}_2 \times L$.*

Proof. As $|L : 2_2| = 2^{25}.3^5.5^4.7.13.17^2$ and $\pi_e(L : 2_2) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 17, 18, 20, 21, 24, 26, 30, 34, 40, 42, 51, 60, 63, 65, 68, 85, 102, 126, 130, 170, 255\}$, then $D(L : 2_2) = (5, 4, 4, 2, 2, 3)$. By assumption $|G| = |L : 2_2|$ and $D(G) = D(L : 2_2)$, so the prime graph of G has following form:

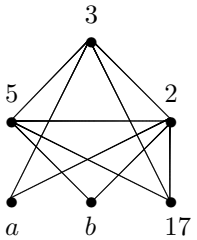

Figure 3.3

where $\{a, b\} = \{7, 13\}$.

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Step1. Let K be the maximal normal solvable subgroup of G. Then K is a $\{2,3,5\}$-group. In particular, G is non-solvable.
By similar arguments as in the proof of Step 1 in Proposition 3.1, we conclude that K is a $\{2,3,5\}$-group and G is non-solvable.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \leq \frac{G}{K} \lesssim$ Aut(S), where S is a finite non-abelian simple group.

Let $\overline{G} = \frac{G}{K}$. Then $S := \mathrm{Soc}(\overline{G})$, $S = P_1 \times P_2 \times ... \times P_m$, where P_i's are finite non-abelian simple groups and $S \leq \frac{G}{K} \lesssim \mathrm{Aut}(S)$. We are going to prove that $m = 1$ and $S = P_1$. Suppose that $m \geq 2$. We claim a does not divide $|S|$. Assume the contrary and let $a \mid |S|$, we conclude that a just divide the order of one of the simple groups P_i's. Without loss of generality, we assume that $a||P_1|$. Then the rest of the P_i's must be $\{2,3\}$-group (because only 2 and 3 are adjacent to a in $\Gamma(G)$), this is a contradiction because P_i's are finite non-abelian simple groups. Now, by Step 1, we observe that $a \in \pi(\overline{G}) \subseteq \pi(\mathrm{Aut}(S))$. But $\mathrm{Aut}(S) = \mathrm{Aut}(S_1) \times \mathrm{Aut}(S_2) \times ... \times \mathrm{Aut}(S_r)$, where the groups S_j are direct products of isomorphic P_i's such that $S = S_1 \times S_2 \times ... \times S_r$. Therefore, for some j, a divides the order of an automorphism group of a direct product S_j of t isomorphic simple groups P_i. Since $P_i \in \mathfrak{S}_{17}$, it follows that $|\mathrm{Out}(P_i)|$ is not divisible by a (see TABLE 1), so a does not divide the order of $\mathrm{Aut}(P_i)$. Now, by Lemma 2.3, we obtain $|\mathrm{Aut}(S_j)| = |\mathrm{Aut}(P_i)|^{t!}.t!$. Therefore, $t \geq a$ and so 3^a must divide the order of G, which is a contradiction. Therefore $m = 1$ and $S = P_1$.

By TABLE 1 and Step 1, it is evident that $|S| = 2^{\alpha}.3^{\beta}.5^{\gamma}.7.13.17^2$, where $2 \leq \alpha \leq 25$, $1 \leq \beta \leq 5$ and $0 \leq \gamma \leq 4$. Now, using collected results contained in TABLE 1, we conclude that $S \cong D_4(4)$ and by Step 2, $L \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut}(L)$. As $|G| = |L : 2_2| = 2|L|$, we deduce $|K| = 1$ or 2.

If $|K| = 1$, then $G \cong L : 2_1, L : 2_2$ or $L : 2_3$ because $|G| = 2|L|$. It is obvious that $G \cong L : 2_2$, because $deg(13) = 1$ in $\Gamma(L : 2_1)$ and $\Gamma(L : 2_3)$ (see page 17).

If $|K| = 2$, then $G/K \cong L$ and $K \leq Z(G)$. It follows that G is a central extension of K by L. If G is a non-split extension of K by L, then $|K|$ must divide the Schur multiplier of L, which is 1. But this is a contradiction, so we obtain that G split over $|K|$. Hence $G \cong \mathbb{Z}_2 \times L$. □

Proposition 3.4. *If $M = L : 2_3$, then $G \cong L : 2_3$ or $L : 2_1$.*

Proof. As $|L : 2_3| = 2^{25}.3^5.5^4.7.13.17^2$ and $\pi_e(L : 2_3) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 24, 30, 34, 51, 63, 65, 85, 255\}$, then $D(L : 2_3) = (4, 4, 4, 2, 1, 3)$. Since $|G| = |L : 2_3|$ and $D(G) = D(L : 2_3)$, we conclude that $\Gamma(G)$ has the following form similarly to Proposition 3.2:

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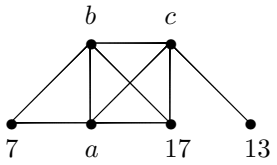

Figure 3.4

where $\{a, b, c\} = \{2, 3, 5\}$.

Step1. Let K be the maximal normal solvable subgroup of G. Then K is a $\{2, 3, 5\}$-group. In particular, G is non-solvable.
We can prove this by the similar way to that in Proposition 3.2.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \leq \frac{G}{K} \lesssim \mathrm{Aut(S)}$, where S is a finite non-abelian simple group.
By using a similar argument, as in the proof of Proposition 3.2, we can verify that $\frac{G}{K}$ is an almost simple group.

By TABLE 1 and Step 1, it is evident that $|S| = 2^{\alpha}.3^{\beta}.5^{\gamma}.7.13.17^2$, where $2 \leq \alpha \leq 25$, $1 \leq \beta \leq 5$ and $0 \leq \gamma \leq 4$. Now, using collected results contained in TABLE 1, we conclude that $S \cong D_4(4)$ and by Step 2, $L \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut}(L)$. As $|G| = |L : 2_3| = 2|L|$, we deduce $|K| = 1$ or 2.

If $|K| = 1$, then $G \cong L : 2_1, L : 2_2$ or $L : 2_3$ because $|G| = 2|L|$. Obviously, $G \cong L : 2_3$ or $L : 2_1$, because $deg(2) = 5$ in $\Gamma(L : 2_2)$ (see page 16).

If $|K| = 2$, then $K \leq Z(G)$ and so $deg(2) = 5$, which is a contradiction. □

Proposition 3.5. *If $M = L : 3$, then $G \cong L : 3$ or $\mathbb{Z}_3 \times L$.*

Proof. As $|L : 3| = 2^{24}.3^6.5^4.7.13.17^2$ and $\pi_e(L : 3) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 15, 17, 18, 20, 21, 24, 30, 34, 39, 45, 51, 63, 65, 85, 255\}$, then $D(L : 3) = (3, 5, 4, 1, 2, 3)$. since $|G| = |L : 3|$ and $D(G) = D(L : 3)$, we conclude that $\Gamma(G)$ has the following form (like $\Gamma(L : 3)$):

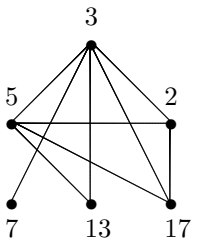

Figure 3.5

Step1. Let K be the maximal normal solvable subgroup of G. Then K is a $\{2, 3\}$-group. In particular, G is non-solvable.

First, we show that K is a p'-group for $p = 7, 13$ and 17. Since the proof is quite similar to the proof of Step 1 in Proposition 3.1, so we avoid here full explanation of all details.

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Next we consider K is a $5'$-group. Assume the contrary, $5 \in \pi_e(K)$. Let $K_5 \in \mathrm{Syl}_5(K)$. By Frattini argument, $G = KN_G(K_5)$. Therefore, $N_G(K_5)$ has an element x of order 7. Since G has no element of order 5.7, $\langle\ x\rangle$ should act fixed point freely on K_5, implying $\langle x\rangle K_5$ is a Frobenius group. By Lemma 2.2(b), $|\langle x\rangle||(|K_5|-1)$, which is impossible. Therefore K is a $\{2,3\}$-group. In addition since K is a proper subgroup of G, then G is non-solvable and the proof of this step is completed.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \leq \frac{G}{K} \lesssim \mathrm{Aut(S)}$, where S is a finite non-abelian simple group.
In a similar way as in the proof of Step 2 in Proposition 3.1, we conclude that $\frac{G}{K}$ is an almost simple group.

By TABLE 1 and Step 1, it is evident that $|S| = 2^{\alpha}.3^{\beta}.5^4.7.13.17^2$, where $2 \leq \alpha \leq 24$ and $1 \leq \beta \leq 6$. Now, using collected results contained in TABLE 1, we conclude that $S \cong D_4(4)$ and by Step 2, $L \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut}(L)$. As $|G| = |L : 3| = 3|L|$, we deduce $|K| = 1$ or 3.

If $|K| = 1$, then $G \cong L : 3$.

If $|K| = 3$, then $G/K \cong L$. In this case we have $G/C_G(K) \lesssim \mathrm{Aut}(K) \cong \mathbb{Z}_2$. Thus $|G/C_G(K)| = 1$ or 2. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is, G is a central extension of K by L. If G is a non-split extension of K by L, then $|K|$ must divide the Schur multiplier of L, which is 1. But this is a contradiction, so we obtain that G split over K. Hence $G \cong \mathbb{Z}_3 \times L$. If $|G/C_G(K)| = 2$, then $K < C_G(K)$ and $1 \neq C_G(K)/K \trianglelefteq G/K \cong L$, which is a contradiction since L is simple. □

Proposition 3.6. *If $M = L : 2^2$, then $G \cong L : 2^2$, $\mathbb{Z}_2 \times (L : 2_1)$, $\mathbb{Z}_2 \times (L : 2_2)$, $\mathbb{Z}_2 \times (L : 2_3)$, $\mathbb{Z}_4 \times L$ or $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times L$.*

Proof. As $|L : 2^2| = 2^{26}.3^5.5^4.7.13.17^2$ and $\pi_e(L : 2^2) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 24, 26, 30, 34, 42, 51, 60, 63, 65, 68, 85, 102, 126, 130, 170, 255\}$, then $D(L : 2^2) = (5, 4, 4, 2, 2, 3)$. Since $|G| = |L : 2^2|$ and $D(G) = D(L : 2^2)$, so the prime graph of G has following form similarly to Proposition 3.3:

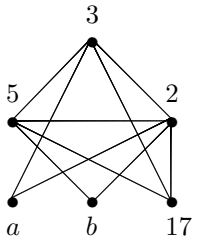

Figure 3.6

where $\{a, b\} = \{7, 13\}$.

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[DOI: 10.7508/ijmsi.2015.01.003]

Step1. Let K be the maximal normal solvable subgroup of G. Then K is a $\{2,3,5\}$-group. In particular, G is non-solvable.
According to Step 1 in Proposition 3.3, we have K is a $\{2,3,5\}$-group and G is non-solvable.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \leq \frac{G}{K} \lesssim \mathrm{Aut(S)}$, where S is a finite non-abelian simple group.
We can prove this by the similar argument in Step 2 in Proposition 3.3.

By TABLE 1 and Step 1, it is evident that $|S| = 2^{\alpha}.3^{\beta}.5^{\gamma}.7.13.17^2$, where $2 \leq \alpha \leq 26$, $1 \leq \beta \leq 5$ and $0 \leq \gamma \leq 4$. Now, using collected results contained in TABLE 1, we conclude that $S \cong D_4(4)$ and by Step 2, $L \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut}(L)$. As $|G| = |L : 2^2| = 4|L|$, we deduce $|K| = 1, 2$ or 4.

If $|K| = 1$, then $G \cong L : 2^2$.

If $|K| = 2$, then $K \leq Z(G)$. In this case G is a central extension of $\mathbb{Z}_2$ by $L : 2_1$, $L : 2_2$ or $L : 2_3$. If G splits over K then $G \cong \mathbb{Z}_2 \times (L : 2_1)$, $\mathbb{Z}_2 \times (L : 2_2)$ or $\mathbb{Z}_2 \times (L : 2_3)$, otherwise we get a contradiction because $|K|$ must divide the Schure multiplier of $L : 2_1, L : 2_2$ and $L : 2_3$, which is impossible.

If $|K| = 4$, then $G/K \cong L$. In this case we have $G/C_G(K) \lesssim \mathrm{Aut}(K) \cong \mathbb{Z}_2$ or S_3. Thus $|G/C_G(K)| = 1, 2, 3$ or 6. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is, G is a central extension of K by L. If G is a non-split extension of K by L, then $|K|$ must divide the Schur multiplier of L, which is 1, but this is a contradiction. Therefore G splits over K. Hence $G \cong K \times L$. So we have $G \cong \mathbb{Z}_4 \times L$ or $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times L$ because $K \cong \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. If $|G/C_G(K)| = 2, 3$ or 6, then $K < C_G(K)$ and $1 \neq C_G(K)/K \trianglelefteq G/K \cong L$. Which is a contradiction, since L is simple. □

Proposition 3.7. *If $M = L : (D_6)_1$, then $G \cong L : (D_6)_1$, $L : 6$, $\mathbb{Z}_3 \times (L : 2_1)$, $\mathbb{Z}_3 \times (L : 2_3)$ or $(\mathbb{Z}_3 \times L).\mathbb{Z}_2$.*

Proof. As $|L : (D_6)_1| = 2^{25}.3^6.5^4.7.13.17^2$ and $\pi_e(L : (D_6)_1) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 24, 30, 34, 39, 42, 45, 51, 60, 63, 65, 85, 255\}$, then $D(L : (D_6)_1) = (4, 5, 4, 2, 2, 3)$. Since $|G| = |L : (D_6)_1|$ and $D(G) = D(L : (D_6)_1)$, we conclude that there exist several possibilities for $\Gamma(G)$:

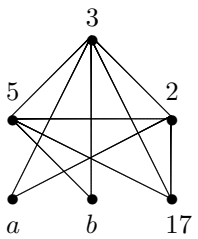

Figure 3.7

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[DOI: 10.7508/ijmsi.2015.01.003]

where $\{a, b\} = \{7, 13\}$.

Step1. Let K be the maximal normal solvable subgroup of G. Then K is a $\{2, 3, 5\}$-group. In particular, G is non-solvable.
By the similar argument to that in Step 1 in Proposition 3.1, we can obtain this assertion.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \leq \frac{G}{K} \lesssim \mathrm{Aut(S)}$, where S is a finite non-abelian simple group.
The proof is similar to Step 2 in Proposition 3.3.

By TABLE 1 and Step 1, it is evident that $|S| = 2^{\alpha}.3^{\beta}.5^{\gamma}.7.13.17^2$, where $2 \leq \alpha \leq 25$, $1 \leq \beta \leq 6$ and $0 \leq \gamma \leq 4$. Now, using collected results contained in TABLE 1, we conclude that $S \cong D_4(4)$ and by Step 2, $L \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut}(L)$. As $|G| = |L : D_6)_1| = 6|L|$, we deduce $|K| = 1, 2, 3$ or 6.

If $|K| = 1$, then $G \cong L : (D_6)_1, L : (D_6)_2$ or $L : 6$ because $|G| = 6|L|$. Obviously, $G \cong L : (D_6)_1$ or $L : 6$ because $deg(2) = 5$ in $\Gamma(L : (D_6)_2)$.

If $|K| = 2$, then $K \leq Z(G)$ and so $deg(2) = 5$, which is a contradiction (see page 18).

If $|K| = 3$, then $G/K \cong L : 2_1, L : 2_2$ or $L : 2_3$. But $G/C_G(K) \lesssim \mathrm{Aut}(K) \cong \mathbb{Z}_2$. Thus $|G/C_G(K)| = 1$ or 2. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is, G is a central extension of K by $L : 2_1, L : 2_2$ or $L : 2_3$. If G splits over K, then $G \cong \mathbb{Z}_3 \times (L : 2_1)$ or $\mathbb{Z}_3 \times (L : 2_3)$ because in $\Gamma(\mathbb{Z}_3 \times (L : 2_2))$ the degree of 2 is 5. Otherwise we get a contradiction because $|K|$ must divide the Schure multiplier of $L : 2_1, L : 2_2$ and $L : 2_3$, which is impossible. If $|G/C_G(K)| = 2$, then $K < C_G(K)$ and $1 \neq C_G(K)/K \trianglelefteq G/K \cong L : 2_1, L : 2_2$ or $L : 2_3$, we obtain $C_G(K)/K \cong L$. Since $K \leq Z(C_G(K))$, $C_G(K)$ is a central extension of K by L. If $C_G(K)$ splits over K, then $C_G(K) \cong \mathbb{Z}_3 \times L$, otherwise we get a contradiction because $|K|$ must divide the Schure multiplier of L, which is impossible. Therefore, $G \cong (\mathbb{Z}_3 \times L).\mathbb{Z}_2$.

If $|K| = 6$, then $G/K \cong L$ and $K \cong \mathbb{Z}_6$ or D_6.
If $K \cong \mathbb{Z}_6$, then $G/C_G(K) \lesssim \mathbb{Z}_2$ and so $|G/C_G(K)| = 1$ or 2. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$. It follows that $deg(2) = 5$, a contradiction. If $|G/C_G(K)| = 2$, then $K < C_G(K)$ and $1 \neq C_G(K)/K \trianglelefteq G/K \cong L$, which is a contradiction because L is simple.
If $K \cong D_6$, then $K \cap C_G(K) = 1$ and $G/C_G(K) \lesssim D_6$. Thus $C_G(K) \neq 1$. Hence, $1 \neq C_G(K) \cong C_G(K)K/K \trianglelefteq G/K \cong L$. It follows that $L \cong G/K \cong C_G(K)$ because L is simple. Therefore, $G \cong D_6 \times L$, which implies that $deg(2) = 5$, a contradiction. □

Proposition 3.8. *If $M = L : (D_6)_2$, then $G \cong L : (D_6)_2$, $\mathbb{Z}_2 \times (L : 3)$, $\mathbb{Z}_3 \times (L : 2_2)$, $(\mathbb{Z}_3 \times L).\mathbb{Z}_2$, $\mathbb{Z}_6 \times L$ or $S_3 \times L$.*

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Proof. As $|L:(D_6)_2| = 2^{25}.3^6.5^4.7.13.17^2$ and $\pi_e(L:(D_6)_2) = \{1,2,3,4,5,6,7,8,9,10,12,13,14,15,17,18,20,21,24,26,30,34,39,40,42,45,51,60,63,65,68,85,102,126,130,170,255\}$, then $D(L:(D_6)_2) = (5,5,4,2,3,3)$. Since $|G| = |L:(D_6)_2|$ and $D(G) = D(L:(D_6)_2)$, we conclude that $\Gamma(G)$ has the following form (like $\Gamma(L:(D_6)_2)$):

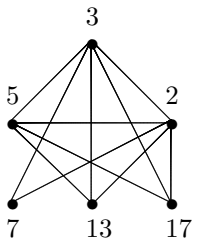

Figure 3.8

Step1. Let K be the maximal normal solvable subgroup of G. Then K is a $\{2,3\}$-group. In particular, G is non-solvable.
The proof is similar to Step 1 in Proposition 3.5.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \leq \frac{G}{K} \lesssim \mathrm{Aut(S)}$, where S is a finite non-abelian simple group.

Let $\overline{G} = \frac{G}{K}$. Then $S := \mathrm{Soc}(\overline{G})$, $S = P_1 \times P_2 \times ... \times P_m$, where P_i's are finite non-abelian simple groups and $S \leq \frac{G}{K} \lesssim \mathrm{Aut}(S)$. We are going to prove that $m = 1$ and $S = P_1$. Suppose that $m \geq 2$. By the same argument in Step 2 of Proposition 3.3 and considering 7 instead of a, we get a contradiction. Therefore $m = 1$ and $S = P_1$.

By TABLE 1 and Step 1, it is evident that $|S| = 2^{\alpha}.3^{\beta}.5^4.7.13.17^2$, where $2 \leq \alpha \leq 25$ and $1 \leq \beta \leq 6$. Now, using collected results contained in TABLE 1, we conclude that $S \cong D_4(4)$ and by Step 2, $L \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut}(L)$. As $|G| = |L:D_6)_2| = 6|L|$, we deduce $|K| = 1, 2, 3$ or 6.

If $|K| = 1$, then $G \cong L:(D_6)_1, L:(D_6)_2$ or $L:6$ because $|G| = 6|L|$. Obviously $G \cong L:(D_6)_2$ because in $\Gamma(L:(D_6)_1)$ and $\Gamma(L:6)$, we have $deg(13) = 2$(see page 17).

If $|K| = 2$, then $K \leq Z(G)$ and $G/K \cong L:3$. Hence G is a central extension of K by $L:3$. If G splits over K, then $G \cong \mathbb{Z}_2 \times (L:3)$. Otherwise we get a contradiction because $|K|$ must divide the Schure multiplier of $L:3$, which is impossible.

If $|K| = 3$, then $G/K \cong L:2_1, L:2_2$ or $L:2_3$. But $G/C_G(K) \lesssim \mathrm{Aut}(K) \cong \mathbb{Z}_2$. Thus $|G/C_G(K)| = 1$ or 2. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is, G is a central extension of K by $L:2_1, L:2_2$ or $L:2_3$. If G splits over K, then only $G \cong \mathbb{Z}_3 \times (L:2_2)$ because $2 \nsim 13$ in $\Gamma(\mathbb{Z}_3 \times (L:2_1))$ and $\Gamma(\mathbb{Z}_3 \times (L:2_3))$. Otherwise we get a contradiction because $|K|$ must divide the Schure multiplier of $L:2_1, L:2_2$ and $L:2_3$, which is impossible. If

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$|G/C_G(K)| = 2$, then $K < C_G(K)$ and $1 \neq C_G(K)/K \trianglelefteq G/K \cong L : 2_1, L : 2_2$ or $L : 2_3$, we obtain $C_G(K)/K \cong L$. Since $K \leq Z(C_G(K))$, $C_G(K)$ is a central extension of K by L. If $C_G(K)$ splits over K, then $C_G(K) \cong \mathbb{Z}_3 \times L$, otherwise we get a contradiction because $|K|$ must divide the Schure multiplier of L, which is impossible. Therefore, $G \cong (\mathbb{Z}_3 \times L).\mathbb{Z}_2$.

If $|K| = 6$, then $G/K \cong L$ and $K \cong \mathbb{Z}_6$ or D_6. If $K \cong \mathbb{Z}_6$, then $G/C_G(K) \lesssim \mathbb{Z}_2$ and so $|G/C_G(K)| = 1$ or 2. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$ and $G/K \cong L$. Therefore G is a central extension of K by L. If G is a non-split extension of K by L, then $|K|$ must divide the Schure multiplier of L, which is 1. But this is a contradiction. So we obtain that G splits over K. Hence $G \cong \mathbb{Z}_6 \times L$. If $|G/C_G(K)| = 2$, then $K < C_G(K)$ and $1 \neq C_G(K)/K \trianglelefteq G/K \cong L$, which is a contradiction because L is simple. If $K \cong D_6$, then $K \cap C_G(K) = 1$ and $G/C_G(K) \lesssim D_6$. Thus $C_G(K) \neq 1$. Hence, $1 \neq C_G(K) \cong C_G(K)K/K \trianglelefteq G/K \cong L$. It follows that $L \cong G/K \cong C_G(K)$ because L is simple. Therefore $G \cong D_6 \times L$. □

Proposition 3.9. *If $M = L : 6$, then $G \cong L : 6$, $L : (D_6)_1$, $\mathbb{Z}_3 \times (L : 2_1)$, $\mathbb{Z}_3 \times (L : 2_3)$ or $(\mathbb{Z}_3 \times L).\mathbb{Z}_2$.*

Proof. As $|L : 6| = 2^{25}.3^6.5^4.7.13.17^2$ and $\pi_e(L : 6) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 24, 30, 34, 36, 39, 42, 45, 48, 51, 63, 65, 85, 255\}$, then $D(L : 6) = (4, 5, 4, 2, 2, 3)$. Since $|G| = |L : 6|$ and $D(G) = D(L : 6)$, there exist several possibilities for $\Gamma(G)$ similarly to Proposition 3.7:

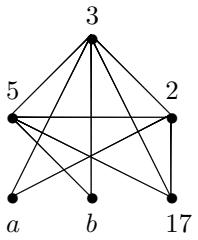

Figure 3.9

where $\{a, b\} = \{7, 13\}$.

Step1. Let K be the maximal normal solvable subgroup of G. Then K is a $\{2, 3, 5\}$-group. In particular, G is non-solvable.
The proof is similar to that in Proposition 3.3.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \leq \frac{G}{K} \lesssim$ Aut(S), where S is a finite non-abelian simple group.
Again we refer to Step 2 of proposition 3.3 to get the proof.

By TABLE 1 and Step 1, it is evident that $|S| = 2^{\alpha}.3^{\beta}.5^{\gamma}.7.13.17^2$, where $2 \leq \alpha \leq 25$, $1 \leq \beta \leq 6$ and $0 \leq \gamma \leq 4$. Now, using collected results contained

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in TABLE 1, we conclude that $S \cong D_4(4)$ and by Step 2, $L \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut}(L)$. As $|G| = |L : 6| = 6|L|$, we deduce $|K| = 1, 2, 3$ or 6.

If $|K| = 1$, then $G \cong L : 6$, $L : (D_6)_1$ or $L : (D_6)_2$ because $|G| = 6|L|$. Obviously, $G \cong L : 6$ or $L : (D_6)_1$ because $deg(2) = 5$ in $\Gamma(L : (D_6)_2)$ (see page 18).

If $|K| = 2$, then $K \leq Z(G)$ and so $deg(2) = 5$, which is a contradiction.

If $|K| = 3$, then $G/K \cong L : 2_1, L : 2_2$ or $L : 2_3$. But $G/C_G(K) \lesssim \mathrm{Aut}(K) \cong \mathbb{Z}_2$. Thus $|G/C_G(K)| = 1$ or 2. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is, G is a central extension of K by $L : 2_1, L : 2_2$ or $L : 2_3$. If G splits over K, then $G \cong \mathbb{Z}_3 \times (L : 2_1)$ or $\mathbb{Z}_3 \times (L : 2_3)$ because in $\Gamma(\mathbb{Z}_3 \times (L : 2_2))$ the degree of 2 is 5. Otherwise we get a contradiction because $|K|$ must divide the Schure multiplier of $L : 2_1, L : 2_2$ and $L : 2_3$, which is impossible. If $|G/C_G(K)| = 2$, then $K < C_G(K)$ and $1 \neq C_G(K)/K \trianglelefteq G/K \cong L : 2_1, L : 2_2$ or $L : 2_3$, we obtain $C_G(K)/K \cong L$. Since $K \leq Z(C_G(K))$, $C_G(K)$ is a central extension of K by L. If $C_G(K)$ splits over K, then $C_G(K) \cong \mathbb{Z}_3 \times L$, otherwise we get a contradiction because $|K|$ must divide the Schure multiplier of L, which is impossible. Therefore, $G \cong (\mathbb{Z}_3 \times L).\mathbb{Z}_2$.

If $|K| = 6$, then $G/K \cong L$ and $K \cong \mathbb{Z}_6$ or D_6. If $K \cong \mathbb{Z}_6$, then $G/C_G(K) \lesssim \mathbb{Z}_2$ and so $|G/C_G(K)| = 1$ or 2. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$. It follows that $deg(2) = 5$, a contradiction. If $|G/C_G(K)| = 2$, then $K < C_G(K)$ and $1 \neq C_G(K)/K \trianglelefteq G/K \cong L$, which is a contradiction because L is simple. If $K \cong D_6$, then $K \cap C_G(K) = 1$ and $G/C_G(K) \lesssim D_6$. Thus $C_G(K) \neq 1$. Hence, $1 \neq C_G(K) \cong C_G(K)K/K \trianglelefteq G/K \cong L$. It follows that $L \cong G/K \cong C_G(K)$ because L is simple. Therefore, $G \cong D_6 \times L$, which implies that $deg(2) = 5$, a contradiction. □

Proposition 3.10. *If $M = L : D_{12}$, then $G \cong L : D_{12}$, $\mathbb{Z}_2 \times (L : (D_6)_1)$, $\mathbb{Z}_2 \times (L : (D_6)_2)$, $\mathbb{Z}_2 \times (L : 6)$, $\mathbb{Z}_3 \times (L : 2^2)$, $(\mathbb{Z}_3 \times (L : 2_1)).\mathbb{Z}_2$, $(\mathbb{Z}_3 \times (L : 2_2)).\mathbb{Z}_2$, $(\mathbb{Z}_3 \times (L : 2_3)).\mathbb{Z}_2$, $\mathbb{Z}_4 \times (L : 3)$, $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times (L : 3)$, $(\mathbb{Z}_4 \times L).\mathbb{Z}_3$, $((\mathbb{Z}_2 \times \mathbb{Z}_2) \times L).\mathbb{Z}_3$, $\mathbb{Z}_6 \times (L : 2_1)$, $\mathbb{Z}_6 \times (L : 2_2)$, $\mathbb{Z}_6 \times (L : 2_3)$, $(\mathbb{Z}_6 \times L).\mathbb{Z}_2$, $S_3 \times (L : 2_1)$, $S_3 \times (L : 2_2)$, $S_3 \times (L : 2_3)$, $\mathbb{Z}_{12} \times L$, $(\mathbb{Z}_2 \times \mathbb{Z}_6) \times L$, $D_{12} \times L$, $(\mathbb{Z}_2 \times L).D_6$, $\mathbb{A}_4 \times L$, $L.\mathbb{A}_4$ or $T \times L$.*

Proof. As $|L : D_{12}| = 2^{26}.3^6.5^4.7.13.17^2$ and $\pi_e(L : (D_{12})) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 24, 26, 30, 34, 39, 40, 42, 45, 48, 51, 60, 63, 65, 68, 85, 102, 126, 130, 170, 255\}$, then $D(L : D_{12}) = (5, 5, 4, 2, 3, 3)$. Since $|G| = |L : D_{12}|$ and $D(G) = D(L : D_{12})$, we conclude that $\Gamma(G)$ has the following form (like $\Gamma(L : D_{12})$):

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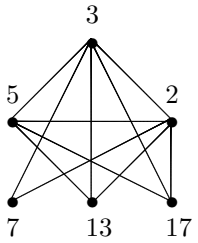

Figure 3.10

Step1. Let K be the maximal normal solvable subgroup of G. Then K is a $\{2,3\}$-group. In particular, G is non-solvable.
The proof is similar to Step 1 in Proposition 3.5.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \leq \frac{G}{K} \lesssim \mathrm{Aut(S)}$, where S is a finite non-abelian simple group.
To get the proof, follow the way in the proof of Step 2 in proposition 3.5.

By TABLE 1 and Step 1, it is evident that $|S| = 2^{\alpha}.3^{\beta}.5^4.7.13.17^2$, where $2 \leq \alpha \leq 26$ and $1 \leq \beta \leq 6$. Now, using collected results contained in TABLE 1, we conclude that $S \cong D_4(4)$ and by Step 2, $L \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut}(L)$. As $|G| = |L : D_{12}| = 12|L|$, we deduce $|K| = 1, 2, 3, 4, 6$ or 12.

If $|K| = 1$, then $G \cong L : D_{12}$.

If $|K| = 2$, then $G/K \cong L : (D_6)_1, L : (D_6)_2$ or $L : 6$ and $K \leq Z(G)$. It follows that G is a central extension of K by $L : (D_6)_1, L : (D_6)_2$ or $L : 6$. If G splits over K, then $G \cong \mathbb{Z}_2 \times (L : (D_6)_1)$, $\mathbb{Z}_2 \times (L : (D_6)_2)$ or $\mathbb{Z}_2 \times (L : 6)$. Otherwise $G \cong \mathbb{Z}_2.(L : (D_6)_1)$ or $\mathbb{Z}_2.(L : (D_6)_2)$.

If $|K| = 3$, then $G/K \cong L : 2^2$. But $G/C_G(K) \lesssim \mathrm{Aut}(K) \cong \mathbb{Z}_2$. Thus $|G/C_G(K)| = 1$ or 2. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is, G is a central extension of K by $L : 2^2$. If G splits over K, then $G \cong \mathbb{Z}_3 \times (L : 2^2)$, Otherwise we get a contradiction because $|K|$ must divide the Schure multiplier of $L : 2^2$, which is impossible. If $|G/C_G(K)| = 2$, then $K < C_G(K)$ and $1 \neq C_G(K)/K \trianglelefteq G/K \cong L : 2^2$, and we obtain $C_G(K)/K \cong L : 2_1, L : 2_2$ or $L : 2_3$. Since $K \leq Z(C_G(K))$, $C_G(K)$ is a central extension of K by $L : 2_1, L : 2_2$ or $L : 2_3$. Thus $C_G(K) \cong \mathbb{Z}_3 \times (L : 2_1)$, $\mathbb{Z}_3 \times (L : 2_2)$ or $\mathbb{Z}_3 \times (L : 2_3)$, otherwise we get a contradiction because 3 must divide the Schure multiplier of $L : 2_1, L : 2_2$ or $L : 2_3$, which is impossible. Therefore, $G \cong (\mathbb{Z}_3 \times (L : 2_1)).\mathbb{Z}_2, (\mathbb{Z}_3 \times (L : 2_2)).\mathbb{Z}_2$ or $(\mathbb{Z}_3 \times (L : 2_3)).\mathbb{Z}_2$.

If $|K| = 4$, then $G/K \cong L : 3$ and $K \cong \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. In this case we have $G/C_G(K) \lesssim \mathrm{Aut}(K) \cong \mathbb{Z}_2$ or S_3. Thus $|G/C_G(K)| = 1, 2, 3$ or 6. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is, G is a central extension of K by $L : 3$. If G split over K by $L : 3$, then $G \cong \mathbb{Z}_4 \times (L : 3)$ or $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times (L : 3)$. Otherwise we get a contradiction because $|K|$ must divide the Schure multiplier of $L : 3$, which is impossible. If $|G/C_G(K)| \neq 1$, since $|G/C_G(K)| = 2, 3$ or 6, it follows that $K < C_G(K)$. As L is simple, we conclude that $1 \neq C_G(K)/K$ must

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be an extension of L. Hence $|G/C_G(K)| = 3$ and therefore $C_G(K)/K \cong L$. Now, since $K \leq Z(C_G(K))$, we conclude that $C_G(K)$ is a central extension of K by L. Thus $C_G(K) \cong \mathbb{Z}_4 \times L$, or $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times L$, otherwise $|K|$ must divide the Schure multiplier of L, which is 1 and it is impossible. Therefore, $G \cong (\mathbb{Z}_4 \times L).\mathbb{Z}_3$ or $((\mathbb{Z}_2 \times \mathbb{Z}_2) \times L).\mathbb{Z}_3$.

If $|K| = 6$, then $G/K \cong L : 2_1$, $L : 2_2$ or $L : 2_3$ and $K \cong \mathbb{Z}_6$ or D_6. If $K \cong \mathbb{Z}_6$, then $G/C_G(K) \lesssim \mathbb{Z}_2$ and so $|G/C_G(K)| = 1$ or 2. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is G is a central extension of $\mathbb{Z}_6$ by $L : 2_1$, $L : 2_2$ or $L : 2_3$. If G splits over K, we obtain $G \cong \mathbb{Z}_6 \times (L : 2_1)$, $\mathbb{Z}_6 \times (L : 2_2)$ or $\mathbb{Z}_6 \times (L : 2_3)$, otherwise we get a contradiction because $|K|$ must divide the Schure multiplier of $L : 2_1, L : 2_2$ or $L : 2_3$, which is impossible. If $|G/C_G(K)| = 2$, then $K < C_G(K)$ and $1 \neq C_G(K)/K \trianglelefteq G/K \cong L : 2_1$, $L : 2_2$ or $L : 2_3$, and we obtain $C_G(K)/K \cong L$. Since $K \leq Z(C_G(K))$, $C_G(K)$ is a central extension of K by L. Thus $C_G(K) \cong \mathbb{Z}_6 \times L$, otherwise we get a contradiction because $|K|$ must divide the Schure multiplier of L. Therefore $G \cong (\mathbb{Z}_6 \times L).\mathbb{Z}_2$. If $K \cong D_6$, then $G/C_G(K) \lesssim D_6$ and so $|G/C_G(K)| = 1, 2, 3$ or 6. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is a contradiction. If $|G/C_G(K)| = 2$, then we have $|KC_G(K)| = 6.|G|/2 = 3|G|$ because $K \cap C_G(K) = 1$, which is a contradiction. If $|G/C_G(K)| = 3$, then we have $|KC_G(K)| = 6.|G|/3 = 2|G|$ because $K \cap C_G(K) = 1$, which is a contradiction. If $|G/C_G(K)| = 6$, then $G/C_G(K) \cong D_6$ and $C_G(K) \neq 1$. Hence, $1 \neq C_G(K) \cong C_G(K)K/K \trianglelefteq G/K \cong L : 2_1, L : 2_2$ or $L : 2_3$. It follows that $C_G(K) \cong L : 2_1, L : 2_2$ or $L : 2_3$ because L is simple. Therefore, $G \cong D_6 \times (L : 2_1)$, $D_6 \times (L : 2_2)$ or $D_6 \times (L : 2_3)$.

Before processing the last case, we recall the following facts.

There exist five non-isomorphic groups of order 12. Two of them are abelian and three are non-abelian. The non-abelian groups are: alternating group A_4, dihedral group D_{12} and the dicyclic group T with generators a and b, subject to the relations $a^6 = 1$, $a^3 = b^2$ and $b^{-1}ab = a^{-1}$.

If $|K| = 12$, then $G/K \cong L$ and $K \cong \mathbb{Z}_{12}$, $\mathbb{Z}_2 \times \mathbb{Z}_6$, D_{12}, $\mathbb{A}_4$ or T. But $C_G(K)K/K \trianglelefteq G/K \cong L$. If $C_G(K)K/K = 1$, then $C_G(K) \leq K$ and hence $|L| = |G/K|||G/C_G(K)|||\mathrm{Aut}(K)|$. Thus $|L|||\mathrm{Aut}(K)|$, a contradiction. Therefore, $C_G(K)K/K \neq 1$ and since L is simple group, we conclude that $G = C_G(K)K$ and hence, $G/C_G(K) \cong K/Z(K)$. Now, we should consider the following cases:

If $K \cong \mathbb{Z}_{12}$ or $\mathbb{Z}_2 \times \mathbb{Z}_6$, then $G/C_G(K) = 1$. Therefore $K \leq Z(G)$, that is G is a central extension of $\mathbb{Z}_{12}$ or $\mathbb{Z}_2 \times \mathbb{Z}_6$ by L. If G splits over K, we obtain $G \cong \mathbb{Z}_{12} \times L$ or $(\mathbb{Z}_2 \times \mathbb{Z}_6) \times L$, otherwise we get a contradiction because $|K|$ must divide the Schure multiplier of L, which is 1 and it is impossible.

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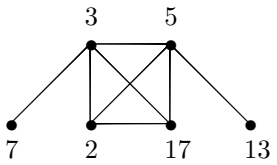
If $K \cong D_{12}$, then $G = K.L$ and $G/C_G(K) \cong D_6$. Since $C_G(K)/Z(K) \cong G/K \cong L$ and $Z(K) \leq Z(C_G(K))$, we conclude that $C_G(K)$ is a central extension of $Z(K) \cong \mathbb{Z}_2$ by L. If $C_G(K)$ is a non-split extension, then 2 must divide the Schure multiplier of L, which is 1 and it is impossible. Thus $C_G(K) \cong \mathbb{Z}_2 \times L$ and hence, G is a split extension of K by L. Now, since $\mathrm{Hom}(L, \mathrm{Aut}(D_{12}))$ is trivial, we have $G \cong D_{12} \times L$.

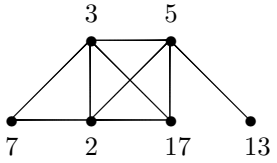
If $K \cong \mathbb{A}_4$, then $G/C_G(K) \cong \mathbb{A}_4$. As $G = C_G(K)K$, It follows that $C_G(K) \cong L$. Therefore $G \cong L \times \mathbb{A}_4$ or $L.\mathbb{A}_4$.

If $K \cong T$, then By the similar way in case $K \cong D_{12}$, we can conclude that G is a split extension of K by L. Also, since $\mathrm{Hom}(L, \mathrm{Aut}(T)$ is trivial, we have $G \cong T \times L$. □

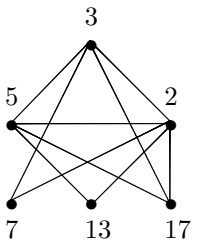
According to what we said before the proof, here we depict $\Gamma(M)$ by $|M|$ and $\pi_e(M)$, where M is an almost simple group related to $L = D_4(4)$.



$\Gamma(L)$



$\Gamma(L : 2_1)$



$\Gamma(L : 2_2)$

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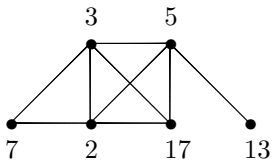
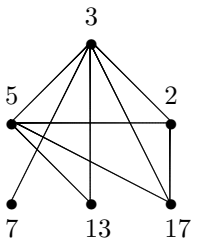
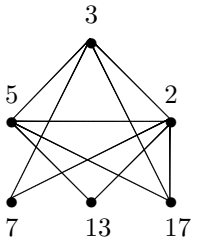


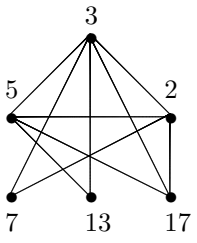
$\Gamma(L : 2_3)$



$\Gamma(L : 3)$



$\Gamma(L : 2^2)$



$\Gamma(L : (D_6)_1)$

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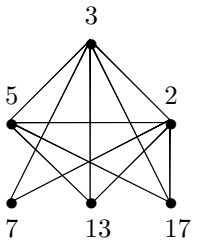
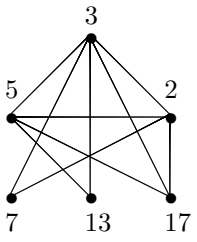
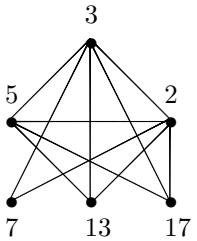


$\Gamma(L : (D_6)_2)$



$\Gamma(L : 6)$



$\Gamma(L : D_{12})$

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4. Acknowledgments

The authors would like to thank professor Derek Holt for sending us the set of element orders of all possible extensions of $D_4(4)$ by subgroups of the outer automorphism. The first author would like to thank Shahrekord University for financial support.

References

1. J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, R. A. Wilson, *Atlas of finite groups*, Clarendon Press (Oxford), London - New York, 1985.
2. M. R. Darafsheh, G. R. Rezaeezadeh, M. bibak, M. Sajjadi, OD-Characterization of almost simple groups related to ${}^2E_6(2)$, *Advances in Aljebra*, **6**, (2013), 45-54.
3. M. R. Darafsheh, G. R. Rezaeezadeh, M. Sajjadi, M. Bibak, OD-Characterization of almost simple groups related to $U_3(17)$, *Quasigroups and related systems*, **21**, (2013), 49-58.
4. M. Foroudi ghasemabadi, N. Ahanjideh, Characterization of the simple groups $D_n(3)$ by prime graph and spectrum, *Iranian Journal of Mathematical Sciences and Informatics*, ,**7**(1), (2012), 91-106.
5. D. Gorenstein, *Finite groups*, New York: Harper and Row, 1980.
6. A. A. Hoseini, A. R. Moghaddamfar, Recognizing alternating groups A_{p+3} for certain primes p by their orders and degree patterns, *Front. Math. China*, **5**(3), (2010), 541-553.
7. A. R. Moghaddamfar, A. R. Zokayi, M. R. Darafsheh, A characterization of finit simole groups by the degree of vertices of their prime graphs, *Algebra Colloquium*, **12**(3), (2005), 431-442.
8. A. R. Moghaddamfar, A. R. Zokayi, Recognizing finite groups through order and degree patterns, *Algebra Colloquium*, **15**(3), (2008), 449-456.
9. G. R. Rezaeezadeh, M. Bibak, M. Sajjadi, Characterization of Projective Special linear Groups in Dimention three by their orders and degree patterns, *Bulletin of the Iranian Mathematical Society*, in press.
10. G. R. Rezaeezadeh, M. R. Darafsheh, M. Sajjadi, M. Bibak, OD-characterization of almost simple groups related to $L_3(25)$, *Bulletin of the Iranian Mathematical Society*, in press.
11. J. S. Robinson Derek, *A course in the theory of groups*, 2nd ed. New York-Heidelberg-Berlin: Springer-Verlag, 2003.
12. A. Zavarnitsin, Finite simple groups with narrow prime spectrum, *Siberian Electronic Mathematical Reports*, **6**, (2009), 1-12.
13. L. C. Zhang, W. J. Shi, OD-characterization of almost simple groups related to $L_2(49)$, *Archivum Mathematicum (Brno) Tomus*, **44**, (2008), 191-199.
14. L. C. Zhang, W. J. Shi, OD-Characterization of simple K_4-groups, *Algebra Colloquium*, **16**(2), (2009), 275-282.
15. L. C. Zhang, W. J. Shi, OD-characterization of almost simple groups related to $U_3(5)$, *Acta Mathematica Scientia*, **26**(1), (2010), 161-168.
16. L. C. Zhang, W. J. Shi, OD-characterization of almost simple groups related to $U_6(2)$, *Acta Mathematica Scientia*, **31**(2), (2011), 441-450.
17. L. C. Zhang, W. J. Shi, OD-Characterization of the projective special linear groups $L_2(q)$, *Acta Mathematica Scientia*, **19**, (2012), 509-524.

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[DOI: 10.7508/ijmsi.2015.01.003]