OD-characterization of Almost Simple Groups Related to $D_4(4)$

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**Abstract.** Let $G$ be a finite group and $\pi_e(G)$ be the set of orders of all elements in $G$. The set $\pi_e(G)$ determines the prime graph (or Grunberg-Kegel graph) $\Gamma(G)$ whose vertex set is $\pi(G)$. The set of primes dividing the order of $G$, and two vertices $p$ and $q$ are adjacent if and only if $pq \in \pi_e(G)$. The degree $\text{deg}(p)$ of a vertex $p \in \pi(G)$, is the number of edges incident on $p$. Let $\pi(G) = \{p_1, p_2, ..., p_k\}$ with $p_1 < p_2 < ... < p_k$. We define $D(G) := (\text{deg}(p_1), \text{deg}(p_2), ..., \text{deg}(p_k))$, which is called the degree pattern of $G$. The group $G$ is called $k$-fold OD-characterizable if there exist exactly $k$ non-isomorphic groups $M$ satisfying conditions $|G| = |M|$ and $D(G) = D(M)$. Usually a 1-fold OD-characterizable group is simply called OD-characterizable. In this paper, we classify all finite groups with the same order and degree pattern as an almost simple groups related to $D_4(4)$.

**Keywords:** Degree pattern, $k$-fold OD-characterizable, Almost simple group.

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1. Introduction

Let $G$ be a finite group, $\pi(G)$ the set of all prime divisors of $|G|$ and $\pi_e(G)$ be the set of orders of elements in $G$. The prime graph (or Grunberg-Kegel graph) $\Gamma(G)$ of $G$ is a simple graph with vertex set $\pi(G)$ in which two vertices $p$ and $q$ are joined by an edge (and we write $p \sim q$) if and only if $G$ contains an element of order $pq$ (i.e. $pq \in \pi_e(G)$).

The degree $\text{deg}(p)$ of a vertex $p \in \pi(G)$ is the number of edges incident on $p$. If $\pi(G) = \{p_1, p_2, ..., p_k\}$ with $p_1 < p_2 < ... < p_k$, then we define $D(G):=(\text{deg}(p_1), \text{deg}(p_2), ..., \text{deg}(p_k))$, which is called the degree pattern of $G$, and leads a following definition.

Definition 1.1. The finite group $G$ is called $k$-fold OD-characterizable if there exist exactly $k$ non-isomorphic groups $H$ satisfying conditions $|G| = |H|$ and $D(G) = D(H)$. In particular, a 1-fold OD-characterizable group is simply called OD-characterizable.

The interest in characterizing finite groups by their degree patterns started in [7] by M. R. Darafsheh and et. all, in which the authors proved that the following simple groups are uniquely determined by their order and degree patterns: All sporadic simple groups, the alternating groups $A_p$ with $p$ and $p - 2$ primes and some simple groups of Lie type. Also in a series of articles (see [4, 6, 8, 9, 14, 17]), it was shown that many finite simple groups are OD-characterizable.

Let $A$ and $B$ be two groups then a split extension is denoted by $A : B$. If $L$ is a finite simple group and $\text{Aut}(L) \cong L : A$, then if $B$ is a cyclic subgroup of $A$ of order $n$ we will write $L : n$ for the split extension $L : B$. Moreover if there are more than one subgroup of orders $n$ in $A$, then we will denote them by $L : n_1$, $L : n_2$, etc.

Definition 1.2. A group $G$ is said to be an almost simple group related to $S$ if and only if $S \leq G \leq \text{Aut}(S)$, for some non-abelian simple group $S$.

In many papers (see [2, 3, 10, 13, 15, 16]), it has been proved, up to now, that many finite almost simple groups are OD-characterizable or $k$-fold OD-characterizable for certain $k \geq 2$.

We denote the socle of $G$ by $\text{Soc}(G)$, which is the subgroup generated by the set of all minimal normal subgroups of $G$. For $p \in \pi(G)$, we denote by $G_p$ and $\text{Syl}_p(G)$ a Sylow $p$-subgroup of $G$ and the set of all Sylow $p$-subgroups of $G$ respectively, all further unexplained notation are standard and can be found in [11].

In this article our main aim is to show the recognizability of the almost simple groups related to $L := D_4(4)$ by degree pattern in the prime graph and
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order of the group. In fact, we will prove the following theorem.

Main Theorem Let $M$ be an almost simple group related to $L := D_4(4)$. If $G$ is a finite group such that $D(G) = D(M)$ and $|G| = |M|$, then the following assertions hold:

(a) If $M = L$, then $G \cong L$.
(b) If $M = L : 2^1$, then $G \cong L : 2_1$ or $L : 2_3$.
(c) If $M = L : 2^2$, then $G \cong L : 2_2$ or $\mathbb{Z}_2 \times L$.
(d) If $M = L : 2^3$, then $G \cong L : 2_3$ or $L : 2_1$.
(e) If $M = L : 3$, then $G \cong L : 3$ or $\mathbb{Z}_3 \times L$.
(f) If $M = L : 2^4$, then $G \cong L : 2^2 \times (L : 2_1), \mathbb{Z}_2 \times (L : 2_2), \mathbb{Z}_2 \times (L : 2_3), \mathbb{Z}_4 \times L$ or $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times L$.
(g) If $M = L : (D_6)_1$, then $G \cong L : (D_6)_1, L : 6, \mathbb{Z}_3 \times (L : 2_1), \mathbb{Z}_3 \times (L : 2_3)$ or $(\mathbb{Z}_3 \times L).\mathbb{Z}_2$.
(h) If $M = L : (D_6)_2$, then $G \cong L : (D_6)_2, \mathbb{Z}_2 \times (L : 3), \mathbb{Z}_3 \times (L : 2_2), (\mathbb{Z}_3 \times L).\mathbb{Z}_2, \mathbb{Z}_6 \times L$ or $D_6 \times L$.
(i) If $M = L : 6$, then $G \cong L : 6, L : (D_6)_1, \mathbb{Z}_3 \times (L : 2_1), \mathbb{Z}_3 \times (L : 2_3)$ or $(\mathbb{Z}_3 \times L).\mathbb{Z}_2$.

2. Preliminary Results

It is well-known that $\text{Aut}(D_4(4)) \cong D_4(4) : D_{12}$ where $D_{12}$ denotes the dihedral group of order 12. We remark that $D_{12}$ has the following non-trivial proper subgroups up to conjugacy: three subgroups of order 2, one cyclic subgroup each of order 3 and 6, two subgroups isomorphic to $D_6 \cong S_3$ and one subgroup of order 4 isomorphic to the Klein four group denoted by $2^2$. The field and the duality automorphisms of $D_4(4)$ are denoted by $2_1$ and $2_2$ respectively, and we set $2_3 = 2_1.2_2$ (field-duality which is called the diagonal automorphism). Therefore up to conjugacy we have the following almost simple groups related to $D_4(4)$.

Lemma 2.1. If $G$ is an almost simple group related to $L := D_4(4)$, then $G$ is isomorphic to one of the following groups: $L, L : 2^1, L : 2^2, L : (D_6)_1, L : (D_6)_2, L : 6, L : D_{12}$.

Lemma 2.2 ([5]). Let $G$ be a Frobenius group with kernel $K$ and complement $H$. Then:

(a) $K$ is a nilpotent group.
(b) $|K| \equiv 1 \text{ (mod } |H|)$.
Let $p \geq 5$ be a prime. We denote by $\mathcal{S}_p$ the set of all simple groups with prime divisors at most $p$. Clearly, if $q \leq p$, then $\mathcal{S}_q \subseteq \mathcal{S}_p$. We list all the simple groups in class $\mathcal{S}_{17}$ with their order and the order of their outer automorphisms in TABLE 1, taken from [12].

**TABLE 1: Simple groups in $\mathcal{S}_p$, $p \leq 17$.**

| $S$ | $[S]$ | $|\text{Out}(S)|$ | $S$ | $[S]$ | $|\text{Out}(S)|$ |
|-----|--------|---------------|-----|--------|---------------|
| $A_5$ | $2^2 \cdot 3 \cdot 5$ | 2 | $G_2(3)$ | $2^6 \cdot 3^6 \cdot 7 \cdot 13$ | 2 |
| $A_6$ | $2^3 \cdot 3^2 \cdot 5$ | 4 | $3^3 D_4(2)$ | $2^{12} \cdot 3^4 \cdot 7^2 \cdot 13$ | 3 |
| $S_4(3)$ | $2^6 \cdot 3^4 \cdot 5$ | 2 | $L_2(64)$ | $2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$ | 6 |
| $L_2(7)$ | $2^3 \cdot 3 \cdot 7$ | 2 | $U_4(5)$ | $2^7 \cdot 3^4 \cdot 5^6 \cdot 7 \cdot 13$ | 4 |
| $L_2(8)$ | $2^3 \cdot 3^2 \cdot 7$ | 3 | $L_2(9)$ | $2^7 \cdot 3^6 \cdot 5 \cdot 7 \cdot 13$ | 4 |
| $U_3(3)$ | $2^3 \cdot 3^3 \cdot 7$ | 2 | $S_6(3)$ | $2^9 \cdot 3^8 \cdot 5 \cdot 7 \cdot 13$ | 2 |
| $A_7$ | $2^3 \cdot 3^2 \cdot 5 \cdot 7$ | 2 | $O_7(3)$ | $2^9 \cdot 3^3 \cdot 5 \cdot 7 \cdot 13$ | 2 |
| $L_2(49)$ | $2^4 \cdot 3 \cdot 5^2 \cdot 7^2$ | 4 | $G_2(4)$ | $2^{12} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13$ | 2 |
| $U_5(5)$ | $2^6 \cdot 3^2 \cdot 5^3 \cdot 7$ | 6 | $S_4(8)$ | $2^{12} \cdot 3^4 \cdot 7 \cdot 13$ | 6 |
| $L_3(4)$ | $2^8 \cdot 3^2 \cdot 5 \cdot 7$ | 12 | $O_7^+(3)$ | $2^{12} \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 13$ | 24 |
| $A_8$ | $2^6 \cdot 3^2 \cdot 5 \cdot 7$ | 2 | $L_3(3)$ | $2^8 \cdot 3^3 \cdot 5 \cdot 13$ | 12 |
| $A_9$ | $2^6 \cdot 3^4 \cdot 5 \cdot 7$ | 2 | $A_{13}$ | $2^9 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$ | 2 |
| $J_2$ | $2^7 \cdot 3^3 \cdot 5^2 \cdot 7$ | 2 | $A_{14}$ | $2^{10} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$ | 2 |
| $A_{10}$ | $2^7 \cdot 3^4 \cdot 5 \cdot 7$ | 2 | $A_{15}$ | $2^{10} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$ | 2 |
| $U_4(3)$ | $2^7 \cdot 3^6 \cdot 5 \cdot 7$ | 8 | $L_6(3)$ | $2^{11} \cdot 3^{15} \cdot 5 \cdot 7 \cdot 11 \cdot 13^2$ | 4 |
| $S_4(7)$ | $2^8 \cdot 3 \cdot 5^2 \cdot 7^4$ | 2 | $S_4(2)$ | $2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$ | 2 |
| $S_6(2)$ | $2^9 \cdot 3^4 \cdot 5 \cdot 7$ | 1 | $A_{16}$ | $2^{14} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$ | 2 |
| $O_7^+(2)$ | $2^{12} \cdot 3^3 \cdot 5^2 \cdot 7$ | 6 | $F_{4}(2)$ | $2^{17} \cdot 3^8 \cdot 5 \cdot 7 \cdot 11 \cdot 13$ | 2 |
| $L_3(11)$ | $2^2 \cdot 3 \cdot 5 \cdot 11$ | 2 | $L_2(17)$ | $2^4 \cdot 3^2 \cdot 17$ | 2 |
| $M_{11}$ | $2^3 \cdot 3^2 \cdot 5 \cdot 11$ | 1 | $L_2(16)$ | $2^4 \cdot 3 \cdot 5 \cdot 17$ | 2 |
| $M_{12}$ | $2^6 \cdot 3^3 \cdot 5 \cdot 11$ | 2 | $S_4(4)$ | $2^8 \cdot 3^3 \cdot 5^2 \cdot 17$ | 4 |
| $U_5(2)$ | $2^{10} \cdot 3^5 \cdot 5 \cdot 11$ | 2 | $H_6$ | $2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$ | 2 |
| $M_{22}$ | $2^7 \cdot 3^2 \cdot 5 \cdot 11$ | 2 | $O_7^+(2)$ | $2^{12} \cdot 3^3 \cdot 5 \cdot 7 \cdot 13$ | 2 |
| $A_{11}$ | $2^7 \cdot 3^4 \cdot 5^2 \cdot 7$ | 11 | $L_4(4)$ | $2^{12} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 13$ | 4 |
| $M^* L$ | $2^7 \cdot 3^6 \cdot 5 \cdot 7^2$ | 11 | $S_6(2)$ | $2^{16} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 17$ | 1 |
| $H_8$ | $2^9 \cdot 3^2 \cdot 5^3 \cdot 7$ | 11 | $U_4(4)$ | $2^{12} \cdot 3^2 \cdot 5^3 \cdot 13 \cdot 17$ | 4 |
| $A_{12}$ | $2^{10} \cdot 3^3 \cdot 5 \cdot 11$ | 11 | $U_5(17)$ | $2^{16} \cdot 3^2 \cdot 5 \cdot 7 \cdot 13 \cdot 17^3$ | 6 |
| $U_6(2)$ | $2^{15} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11$ | 6 | $O_7^{(2)}$ | $2^{20} \cdot 3^6 \cdot 5^2 \cdot 7 \cdot 11 \cdot 17$ | 2 |
| $L_3(3)$ | $2^2 \cdot 3^2 \cdot 13$ | 2 | $L_2(13^2)$ | $2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 13^2 \cdot 17$ | 4 |
| $L_2(25)$ | $2^3 \cdot 3 \cdot 5^2 \cdot 13$ | 4 | $S_4(13)$ | $2^6 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13^4 \cdot 17$ | 2 |
| $L_2(23)$ | $2^6 \cdot 3 \cdot 5^2 \cdot 13$ | 4 | $L_3(16)$ | $2^{12} \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 13 \cdot 17$ | 24 |
| $S_4(5)$ | $2^8 \cdot 3 \cdot 5^4 \cdot 13$ | 2 | $S_6(4)$ | $2^{18} \cdot 3^4 \cdot 5^3 \cdot 7 \cdot 13 \cdot 17$ | 2 |
| $L_4(3)$ | $2^7 \cdot 3^6 \cdot 5 \cdot 13$ | 4 | $O_7^+(4)$ | $2^{24} \cdot 3^3 \cdot 5^4 \cdot 7 \cdot 13 \cdot 17^2$ | 12 |
| $^2 F_4(2)'$ | $2^{11} \cdot 3^3 \cdot 5 \cdot 13$ | 2 | $F_4(2)$ | $2^{24} \cdot 3^6 \cdot 5^2 \cdot 7^2 \cdot 13 \cdot 17$ | 2 |
| $L_2(13)$ | $2^2 \cdot 3^2 \cdot 7 \cdot 13$ | 2 | $A_{17}$ | $2^{14} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17$ | 2 |
| $L_2(27)$ | $2^2 \cdot 3^3 \cdot 7 \cdot 13$ | 6 | $A_{18}$ | $2^{15} \cdot 3^8 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17$ | 2 |
Definition 2.3. A completely reducible group will be called a CR-group. The center of a CR-group is a direct product of the abelian factor in the decomposition. Hence, a CR-group is centerless, that is, has trivial center, if and only if it is a direct product of non-abelian simple groups. The following Lemma determines the structure of the automorphism group of a centerless CR-group.

Lemma 2.3 ([11]). Let $R$ be a finite centerless CR-group and write $R = R_1 \times R_2 \times \ldots \times R_k$, where $R_i$ is a direct product of $n_i$ isomorphic copies of a simple group $H_i$, and $H_i$ and $H_j$ are not isomorphic if $i \neq j$. Then $\text{Aut}(R) = \text{Aut}(R_1) \times \text{Aut}(R_2) \times \ldots \times \text{Aut}(R_k)$ and $\text{Aut}(R_i) \cong \text{Aut}(H_i) \wr S_{n_i}$, where in this wreath product $\text{Aut}(H_i)$ appears in its right regular representation and the symmetric group $S_{n_i}$ in its natural permutation representation. Moreover, these isomorphisms induce isomorphisms $\text{Out}(R) \cong \text{Out}(R_1) \times \text{Out}(R_2) \times \ldots \times \text{Out}(R_k)$ and $\text{Out}(R_i) \cong \text{Out}(H_i) \wr S_{n_i}$.

3. OD-Characterization of Almost Simple Groups Related to $D_4(4)$

In this section, we study the problem of characterizing almost simple groups by order and degree pattern. Especially we will focus our attention on almost simple groups related to $L = D_4(4)$, namely, we will prove the Main Theorem of Sec. 1. We break the proof into a number of separate propositions.

By assumption, we depict all possibilities for the prime graph associated with $G$ by use of the variables for some vertices in each proposition. Also, we need to know the structure of $\Gamma(M)$ to determine the possibilities for $G$ in some proposition, therefore we depict the prime graph of all extension of $L$ in pages 18 to 20. Note that the set of order elements in each of the following propositions is calculated using Magma.

Proposition 3.1. If $M = L$, then $G \cong L$.

Proof. By TABLE 1 $|L| = 2^{24} \cdot 3^5 \cdot 5^4 \cdot 7 \cdot 13 \cdot 17^2$. $\pi_e(L) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 15, 17, 20, 21, 30, 34, 51, 63, 65, 85, 255\}$, so $D(L) = (3, 4, 4, 1, 1, 3)$. Since $|G| = |L|$ and $D(G) = D(L)$, we conclude that the prime graph of $G$ has following form:

```
\begin{center}
\begin{tikzpicture}
\tikzset{vertex/.style = {shape=circle,draw,minimum size=1.5em}}
\node[vertex] (a) at (1,1) {$a$};
\node[vertex] (b) at (2,1) {$b$};
\node[vertex] (c) at (3,1) {$c$};

\draw (a) -- (b);
\draw (a) -- (c);
\draw (b) -- (c);
\end{tikzpicture}
\end{center}
```

Figure 3.1

where $\{a, b\} = \{7, 13\}$. 

We will show that $G$ is isomorphic to $L = D_4(4)$. We break up the proof into a several steps.

**Step 1.** Let $K$ be the maximal normal solvable subgroup of $G$. Then $K$ is a $\{2, 3, 5\}$-group. In particular, $G$ is non-solvable.

First we show that $K$ is a $17$'-group. Assume the contrary and let $17 \in \pi(K)$. Then $13$ does not divide the order of $K$. Otherwise, we may suppose that $T$ is a Hall $\{13, 17\}$-subgroup of $K$. It is seen that $T$ is a nilpotent subgroup of order $13.17^i$ for $i = 1$ or $2$. Thus, $13.17 \in \pi_e(K) \subseteq \pi_e(G)$, a contradiction. Thus $\{17\} \subseteq \pi(K) \subseteq \pi(G) - \{13\}$. Let $K_{17} \in \text{Syl}_{17}(K)$. By Frattini argument, $G = KN_G(K_{17})$. Therefore, $N_G(K_{17})$ contains an element $x$ of order $13$. Since $G$ has no element of order $13.17$, ($x$) should act fixed point freely on $K_{17}$, that is implying $\langle x \rangle K_{17}$ is a Frobenius group. By Lemma 2.2(b), $|\langle x \rangle||(|K_{17}| - 1)$. It follows that $13|17^i - 1$ for $i = 1$ or $2$, which is a contradiction.

Next, we show that $K$ is a $p'$-group for $p \in \{a, b\}$. Let $p||K|$ and $K_p \in \text{Syl}_p(K)$. Now by Frattini argument, $G = KN_G(K_p)$, so $17$ must divide the order of $N_G(K_p)$. Therefore, the normalizer $N_G(K_p)$ contains an element of order $17$, say $x$. So $\langle x \rangle K_p$ is a cyclic subgroup of $G$ of order $17.p$, and so $p \sim 17$ in $\Gamma(G)$, which is a contradiction. Therefore $K$ is a $\{2, 3, 5\}$-group. In addition, since $K$ is a proper subgroup of $G$, it follows that $G$ is non-solvable.

**Step 2.** The quotient $G/K$ is an almost simple group. In fact, $S \leq G/K \leq \text{Aut}(S)$, where $S$ is a finite non-abelian simple group isomorphic to $L := D_4(4)$.

Let $G = G/K$. Then $S := \text{Soc}(G) = P_1 \times P_2 \times ... \times P_m$, where $P_i$s are finite non-abelian simple groups and $S \leq \frac{G}{K} \leq \text{Aut}(S)$. If we show that $m = 1$, the proof of Step 2 will be completed.

Suppose that $m \geq 2$. In this case, we claim that $13$ does not divide $|S|$. Assume the contrary and let $13 \mid |S|$, on the other hand, $\{2, 3\} \subset \pi(P_i)$ for every $i$ (by TABLE 1), hence $2 \sim 13$ and $3 \sim 13$, which is a contradiction. Now, by step 1, we observe that $13 \in \pi(G) \subseteq \pi(\text{Aut}(S))$. But $\text{Aut}(S) = \text{Aut}(S_1) \times \text{Aut}(S_2) \times ... \times \text{Aut}(S_t)$, where the groups $S_j$ are direct products of isomorphic $P_i$'s such that $S = S_1 \times S_2 \times ... \times S_t$. Therefore, for some $j$, $13$ divides the order of an automorphism group of a direct product $S_j$ of $t$ isomorphic simple groups $P_i$. Since $P_i \in \mathfrak{S}_{17}$, it follows that $|\text{Out}(P_i)|$ is not divisible by $13$ (see TABLE 1). Now, by Lemma 2.3, we obtain $|\text{Out}(S_j)| = |\text{Out}(P_i)|^{t_i}!$. Therefore, $t \geq 13$ and so $2^{26}$ must divide the order of $G$, which is a contradiction. Therefore $m = 1$ and $S = P_1$.

By TABLE 1 and Step 1, it is evident that $|S| = 2^\alpha.3^\beta.5^\gamma.7.13.17^2$, where $2 \leq \alpha \leq 24$, $1 \leq \beta \leq 5$ and $0 \leq \gamma \leq 4$. Now, using collected results contained in TABLE 1, we deduce that $S \cong D_4(4)$ and by Step 2, $L \leq G/K \leq \text{Aut}(L)$ is completed. As $|G| = |L|$, we deduce $K = 1$, so $G \cong L$ and the proof is completed.

□
Proposition 3.2. If \( M = L : 2_1 \), then \( G \cong L : 2_1 \) or \( L : 2_2 \).

Proof. As \( |L : 2_1| = 2^{25}.3^5.5^4.7.13.17^2 \) and \( \pi_s(L : 2_1) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 24, 30, 34, 40, 42, 51, 60, 63, 65, 68, 85, 102, 126, 130, 170, 255\} \), then \( D(L : 2_1) = (4, 4, 2, 1, 3) \). Since \( |G| = |L : 2_1| \) and \( D(G) = D(L : 2_1) \), we conclude that there exist several possibilities for \( \Gamma(G) \):

\[
\begin{array}{c}
\bullet & \bullet & \bullet \\
\bullet & & \\
3 & 17 & 13 \\
7 & a & b
\end{array}
\]

where \( \{a, b, c\} = \{2, 3, 5\} \).

Step 1. Let \( K \) be the maximal normal solvable subgroup of \( G \). Then \( K \) is a \( \{2, 3, 5\} \)-group. In particular, \( G \) is non-solvable.

By a similar argument to that in Proposition 3.1, we can obtain this assertion.

Step 2. The quotient \( \frac{G}{K} \) is an almost simple group. In fact, \( S \leq \frac{G}{K} \leq \text{Aut}(S) \), where \( S \) is a finite non-abelian simple group.

The proof is similar to Step 2 of Proposition 3.1.

By TABLE 1 and Step 1, it is evident that \( |S| = 2^\alpha.3^\beta.5^\gamma.7.13.17^2 \), where \( 2 \leq \alpha \leq 25, 1 \leq \beta \leq 5 \) and \( 0 \leq \gamma \leq 4 \). Now, using collected results contained in TABLE 1, we conclude that \( S \cong D_4(4) \) and by Step 2, \( L \leq \frac{G}{K} \leq \text{Aut}(L) \). As \( |G| = |L : 2_1| = 2|L| \), we deduce \( |K| = 1 \) or \( 2 \).

If \( |K| = 1 \), then \( G \cong L : 2_1, L : 2_2 \) or \( L : 2_3 \). Obviously, \( G \cong L : 2_1 \) or \( L : 2_3 \) because \( \text{deg}(2) = 5 \) in \( \Gamma(L : 2_2) \) (see page 16).

If \( |K| = 2 \), then \( K \leq Z(G) \) and so \( \text{deg}(2) = 5 \), which is a contradiction. \( \square \)

Proposition 3.3. If \( M = L : 2_2 \), then \( G \cong L : 2_2 \) or \( \mathbb{Z}_2 \times L \).

Proof. As \( |L : 2_2| = 2^{25}.3^5.5^4.7.13.17^2 \) and \( \pi_s(L : 2_2) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 17, 18, 20, 21, 24, 26, 30, 34, 40, 42, 51, 60, 63, 65, 68, 85, 102, 126, 130, 170, 255\} \), then \( D(L : 2_2) = (5, 4, 4, 2, 2, 3) \). By assumption \( |G| = |L : 2_2| \) and \( D(G) = D(L : 2_2) \), so the prime graph of \( G \) has following form:

\[
\begin{array}{c}
\bullet & \bullet & \bullet \\
\bullet & & \\
3 & 2 & 5 \\
a & b & 17
\end{array}
\]

where \( \{a, b\} = \{7, 13\} \).
Step 1. Let $K$ be the maximal normal solvable subgroup of $G$. Then $K$ is a $\{2, 3, 5\}$-group. In particular, $G$ is non-solvable. By similar arguments as in the proof of Step 1 in Proposition 3.1, we conclude that $K$ is a $\{2, 3, 5\}$-group and $G$ is non-solvable.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \leq \frac{G}{K} \leq \text{Aut}(S)$, where $S$ is a finite non-abelian simple group.

Let $\overline{G} = \frac{G}{K}$. Then $S := \text{Soc}(\overline{G})$, $S = P_1 \times P_2 \times \ldots \times P_m$, where $P_i$’s are finite non-abelian simple groups and $S \leq \frac{G}{K} \leq \text{Aut}(S)$. We are going to prove that $m = 1$ and $S = P_1$. Suppose that $m \geq 2$. We claim $a$ does not divide $|S|$. Assume the contrary and let $a \mid |S|$, we conclude that a just divide the order of one of the simple groups $P_i$’s. Without loss of generality, we assume that $a \mid |P_1|$. Then the rest of the $P_i$’s must be $\{2, 3\}$-group (because only 2 and 3 are adjacent to $a$ in $\Gamma(G)$), this is a contradiction because $P_i$’s are finite non-abelian simple groups. Now, by Step 1, we observe that $a \in \pi(\overline{G}) \subseteq \pi(\text{Aut}(S))$. But $\text{Aut}(S) = \text{Aut}(S_1) \times \text{Aut}(S_2) \times \ldots \times \text{Aut}(S_r)$, where the groups $S_j$ are direct products of isomorphic $P_i$’s such that $S = S_1 \times S_2 \times \ldots \times S_r$. Therefore, for some $j$, $a$ divides the order of an automorphism group of a direct product $S_j$ of $t$ isomorphic simple groups $P_i$. Since $P_i \in \mathcal{S}_{17}$, it follows that $|\text{Out}(P_i)|$ is not divisible by $a$ (see TABLE 1), so $a$ does not divide the order of $\text{Aut}(P_i)$. Now, by Lemma 2.3, we obtain $|\text{Aut}(S_j)| = |\text{Aut}(P_i)|^{t!} t!$. Therefore, $t \geq a$ and so $3^a$ must divide the order of $G$, which is a contradiction. Therefore $m = 1$ and $S = P_1$.

By TABLE 1 and Step 1, it is evident that $|S| = 2^a 3^\beta 5^\gamma 7.13.17^2$, where $2 \leq a \leq 25$, $1 \leq \beta \leq 5$ and $0 \leq \gamma \leq 4$. Now, using collected results contained in TABLE 1, we conclude that $S \cong D_4(4)$ and by Step 2, $L \leq \frac{G}{K} \leq \text{Aut}(L)$. As $|G| = |L : 2| = 2|L|$, we deduce $|K| = 1$ or 2.

If $|K| = 1$, then $G \cong L : 2_1$, $L : 2_2$ or $L : 2_3$ because $|G| = 2|L|$. It is obvious that $G \cong L : 2_2$, because $deg(13) = 1$ in $\Gamma(L : 2_1)$ and $\Gamma(L : 2_3)$ (see page 17).

If $|K| = 2$, then $G/K \cong L$ and $K \leq Z(G)$. It follows that $G$ is a central extension of $K$ by $L$. If $G$ is a non-split extension of $K$ by $L$, then $|K|$ must divide the Schur multiplier of $L$, which is 1. But this is a contradiction, so we obtain that $G$ split over $|K|$. Hence $G \cong Z_2 \times L$. $\square$

**Proposition 3.4.** If $M = L : 2_3$, then $G \cong L : 2_3$ or $L : 2_1$.

**Proof.** As $|L : 2_3| = 2^{25} 3^5 5^4 7.13.17^2$ and $\pi_r(L : 2_3) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 24, 30, 34, 51, 63, 65, 85, 255\}$, then $D(L : 2_3) = (4, 4, 4, 2, 1, 3)$. Since $|G| = |L : 2_3|$ and $D(G) = D(L : 2_3)$, we conclude that $\Gamma(G)$ has the following form similarly to Proposition 3.2:
Step 1. Let $K$ be the maximal normal solvable subgroup of $G$. Then $K$ is a $\{2,3,5\}$-group. In particular, $G$ is non-solvable. We can prove this by the similar way to that in Proposition 3.2.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \leq G_K \leq \text{Aut}(S)$, where $S$ is a finite non-abelian simple group. By using a similar argument, as in the proof of Proposition 3.2, we can verify that $G_K$ is an almost simple group.

By TABLE 1 and Step 1, it is evident that $|S| = 2^\alpha 3^\beta 5^\gamma 7^\delta 13^\epsilon 17^\zeta$, where $2 \leq \alpha \leq 25$, $1 \leq \beta \leq 5$ and $0 \leq \gamma \leq 4$. Now, using collected results contained in TABLE 1, we conclude that $S \cong D_4(4)$ and by Step 2, $L \leq \frac{G}{K} \leq \text{Aut}(L)$. As $|G| = |L : 2^3| = 2|L|$, we deduce $|K| = 1$ or $2$.

If $|K| = 1$, then $G \cong L : 2^1$, $L : 2^2$ or $L : 2^3$ because $|G| = 2|L|$. Obviously, $G \cong L : 2^3$ or $L : 2^2$, because $\deg(2) = 5$ in $\Gamma(L : 2^2)$ (see page 16).

If $|K| = 2$, then $K \leq Z(G)$ and so $\deg(2) = 5$, which is a contradiction. □

Proposition 3.5. If $M = L : 3$, then $G \cong L : 3$ or $\mathbb{Z}_3 \times L$.

Proof. As $|L : 3| = 2^{24} 3^6 5^4 7 13 17^2$ and $\pi_c(L : 3) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 15, 17, 18, 20, 21, 24, 30, 34, 39, 45, 51, 63, 65, 85, 255\}$, then $D(L : 3) = (3, 5, 4, 1, 2, 3)$. Since $|G| = |L : 3|$ and $D(G) = D(L : 3)$, we conclude that $\Gamma(G)$ has the following form (like $\Gamma(L : 3)$):

Step 1. Let $K$ be the maximal normal solvable subgroup of $G$. Then $K$ is a $\{2,3\}$-group. In particular, $G$ is non-solvable.

First, we show that $K$ is a $p'$-group for $p = 7, 13$ and 17. Since the proof is quite similar to the proof of Step 1 in Proposition 3.1, so we avoid here full explanation of all details.
Next we consider $K$ is a 5'-group. Assume the contrary, $5 \in \pi_e(K)$. Let $K_5 \in \operatorname{Syl}_5(K)$. By Frattini argument, $G = KN_G(K_5)$. Therefore, $N_G(K_5)$ has an element $x$ of order 7. Since $G$ has no element of order 5, 7, $\langle x \rangle$ should act fixed point freely on $K_5$, implying $\langle x \rangle K_5$ is a Frobenius group. By Lemma 2.2(b), $|\langle x \rangle| |(|K_5| - 1)$, which is impossible. Therefore $K$ is a $\{2, 3\}$-group.

In addition since $K$ is a proper subgroup of $G$, then $G$ is non-solvable and the proof of this step is completed.

**Step 2.** The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \leq \frac{G}{K} \lesssim \operatorname{Aut}(S)$, where $S$ is a finite non-abelian simple group.

In a similar way as in the proof of Step 2 in Proposition 3.1, we conclude that $\frac{G}{K}$ is an almost simple group.

By TABLE 1 and Step 1, it is evident that $|S| = 2^\alpha \cdot 3^2 \cdot 5^4 \cdot 7 \cdot 13 \cdot 17^2$, where $2 \leq \alpha \leq 24$ and $1 \leq \beta \leq 6$. Now, using collected results contained in TABLE 1, we conclude that $S \cong D_4(4)$ and by Step 2, $L \leq \frac{G}{K} \lesssim \operatorname{Aut}(L)$. As $|G| = |L : 3| = 3|L|$, we deduce $|K| = 1$ or 3.

If $|K| = 1$, then $G \cong L : 3$.

If $|K| = 3$, then $G/K \cong L$. In this case we have $G/C_G(K) \lesssim \operatorname{Aut}(K) \cong \mathbb{Z}_3$. Thus $|G/C_G(K)| = 1$ or 2. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is, $G$ is a central extension of $K$ by $L$. If $G$ is a non-split extension of $K$ by $L$, then $|K|$ must divide the Schur multiplier of $L$, which is 1. But this is a contradiction, so we obtain that $G$ split over $K$. Hence $G \cong \mathbb{Z}_3 \times L$. If $|G/C_G(K)| = 2$, then $K < C_G(K)$ and $1 \neq C_G(K)/K \leq G/K \cong L$, which is a contradiction since $L$ is simple. □

**Proposition 3.6.** If $M = L : 2^2$, then $G \cong L : 2^2$, $\mathbb{Z}_2 \times (L : 2^1)$, $\mathbb{Z}_2 \times (L : 2^2)$, $\mathbb{Z}_2 \times (L : 2^3)$, $\mathbb{Z}_3 \times L$ or $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times L$.

**Proof.** As $|L : 2^2| = 2^{26} \cdot 3^5 \cdot 5^4 \cdot 7 \cdot 13 \cdot 17^2$ and $\pi_e(L : 2^2) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 24, 26, 30, 34, 42, 51, 60, 63, 65, 68, 85, 102, 126, 130, 170, 255\}$, then $D(L : 2^2) = (5, 4, 4, 2, 2, 3)$. Since $|G| = |L : 2^2|$ and $D(G) = D(L : 2^2)$, so the prime graph of $G$ has following form similarly to Proposition 3.3:

![Figure 3.6](image)

where $\{a, b\} = \{7, 13\}$.
Step 1. Let $K$ be the maximal normal solvable subgroup of $G$. Then $K$ is a $\{2, 3, 5\}$-group. In particular, $G$ is non-solvable.

According to Step 1 in Proposition 3.3, we have $K$ is a $\{2, 3, 5\}$-group and $G$ is non-solvable.

Step 2. The quotient $G/K$ is an almost simple group. In fact, $S \leq G/K \leq \text{Aut}(S)$, where $S$ is a finite non-abelian simple group.

We can prove this by the similar argument in Step 2 in Proposition 3.3.

By TABLE 1 and Step 1, it is evident that $|S| = 2^\alpha 3^\beta 5^\gamma 7.13.17^2$, where $2 \leq \alpha \leq 26$, $1 \leq \beta \leq 5$ and $0 \leq \gamma \leq 4$. Now, using collected results contained in TABLE 1, we conclude that $S \cong D_4(4)$ and by Step 2, $L \leq G/K \leq \text{Aut}(L)$. As $|G| = |L : 2^2| = 4|L|$, we deduce $|K| = 1, 2$ or $4$.

If $|K| = 1$, then $G \cong L : 2^2$.

If $|K| = 2$, then $K \leq Z(G)$. In this case $G$ is a central extension of $\mathbb{Z}_2$ by $L : 2_1$, $L : 2_2$ or $L : 2_3$. If $G$ splits over $K$ then $G \cong \mathbb{Z}_2 \times (L : 2_1)$, $\mathbb{Z}_2 \times (L : 2_2)$ or $\mathbb{Z}_2 \times (L : 2_3)$, otherwise we get a contradiction because $|K|$ must divide the Schur multiplier of $L : 2_1$, $L : 2_2$ and $L : 2_3$, which is impossible.

If $|K| = 4$, then $G/K \cong L$. In this case we have $G/C_G(K) \cong \text{Aut}(K) \cong \mathbb{Z}_2$ or $S_3$. Thus $|G/C_G(K)| = 1, 2, 3$ or $6$. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is, $G$ is a central extension of $K$ by $L$. If $G$ is a non-split extension of $K$ by $L$, then $|K|$ must divide the Schur multiplier of $L$, which is $1$, but this is a contradiction. Therefore $G$ splits over $K$. Hence $G \cong K \times L$. So we have $G \cong \mathbb{Z}_4 \times L$ or $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times L$ because $K \cong \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. If $|G/C_G(K)| = 2, 3$ or $6$, then $K < C_G(K)$ and $1 \neq C_G(K)/K \leq G/K \cong L$. Which is a contradiction, since $L$ is simple.

Proposition 3.7. If $M = L : (D_6)_1$, then $G \cong L : (D_6)_1$, $L : 6$, $\mathbb{Z}_3 \times (L : 2_1)$, $\mathbb{Z}_3 \times (L : 2_2)$ or $(\mathbb{Z}_3 \times L).\mathbb{Z}_2$.

Proof. As $|L : (D_6)_1| = 2^{25}.3^6.5^4.7.13.17^2$ and $\pi_\ell(L : (D_6)_1) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 24, 30, 34, 39, 42, 45, 51, 60, 63, 65, 85, 255\}$, then $D(L : (D_6)_1) = \{4, 5, 4, 2, 2, 3\}$. Since $|G| = |L : (D_6)_1|$ and $D(G) = D(L : (D_6)_1)$, we conclude that there exist several possibilities for $\Gamma(G)$:

![Figure 3.7](image-url)
where \(\{a, b\} = \{7, 13\}\).

**Step 1.** Let \(K\) be the maximal normal solvable subgroup of \(G\). Then \(K\) is a \(\{2, 3, 5\}\)-group. In particular, \(G\) is non-solvable.

By the similar argument to that in Step 1 in Proposition 3.1, we can obtain this assertion.

**Step 2.** The quotient \(\frac{G}{K}\) is an almost simple group. In fact, \(S \leq \frac{G}{K} \lesssim \text{Aut}(S)\), where \(S\) is a finite non-abelian simple group.

The proof is similar to Step 2 in Proposition 3.3.

By TABLE 1 and Step 1, it is evident that \(|S| = 2^a 3^\beta 5^\gamma 7 13 17^2\), where \(2 \leq a \leq 25, 1 \leq \beta \leq 6\) and \(0 \leq \gamma \leq 4\). Now, using collected results contained in TABLE 1, we conclude that \(S \cong D_4(4)\) and by Step 2, \(L \leq \frac{G}{K} \lesssim \text{Aut}(L)\). As \(|G| = |L : D_6|_1 = 6|L|\), we deduce \(|K| = 1, 2, 3\) or 6.

If \(|K| = 1\), then \(G \cong L : (D_6)_1, L : (D_6)_2\) or \(L : 6\) because \(|G| = 6|L|\).

Obviously, \(G \cong L : (D_6)_1\) or \(L : 6\) because \(\text{deg}(2) = 5\) in \(\Gamma(L : (D_6)_2)\).

If \(|K| = 2\), then \(K \leq Z(G)\) and so \(\text{deg}(2) = 5\), which is a contradiction (see page 18).

If \(|K| = 3\), then \(G/K \cong L : 2_1, L : 2_2\) or \(L : 2_3\). But \(G/C_G(K) \lesssim \text{Aut}(K) \equiv \mathbb{Z}_2\). Thus \(|G/C_G(K)| = 1\) or 2. If \(|G/C_G(K)| = 1\), then \(K \leq Z(G)\), that is, \(G\) is a central extension of \(K\) by \(L : 2_1, L : 2_2\) or \(L : 2_3\). If \(G\) splits over \(K\), then \(G \cong \mathbb{Z}_3 \times (L : 2_1)\) or \(\mathbb{Z}_3 \times (L : 2_2)\) because in \(\Gamma(\mathbb{Z}_3 \times (L : 2_3))\) the degree of 2 is 5. Otherwise we get a contradiction because \(|K|\) must divide the Schur multiplier of \(L : 2_1, L : 2_2\) and \(L : 2_3\), which is impossible. If \(|G/C_G(K)| = 2\), then \(K < C_G(K)\) and \(1 \neq C_G(K)/K \leq G/K \cong L : 2_1, L : 2_2\) or \(L : 2_3\), we obtain \(C_G(K)/K \cong L\). Since \(K \leq Z(C_G(K))\), \(C_G(K)\) is a central extension of \(K\) by \(L\). If \(C_G(K)\) splits over \(K\), then \(C_G(K) \equiv \mathbb{Z}_3 \times L\), otherwise we get a contradiction because \(|K|\) must divide the Schur multiplier of \(L\), which is impossible. Therefore, \(G \cong (\mathbb{Z}_3 \times L)\mathbb{Z}_2\).

If \(|K| = 6\), then \(G/K \cong L\) and \(K \equiv \mathbb{Z}_6\) or \(D_6\).

If \(K \equiv \mathbb{Z}_6\), then \(G/C_G(K) \lesssim \mathbb{Z}_2\) and so \(|G/C_G(K)| = 1\) or 2. If \(|G/C_G(K)| = 1\), then \(K \leq Z(G)\). It follows that \(\text{deg}(2) = 5\), a contradiction. If \(|G/C_G(K)| = 2\), then \(K < C_G(K)\) and \(1 \neq C_G(K)/K \leq G/K \cong L\), which is a contradiction because \(L\) is simple.

If \(K \equiv D_6\), then \(K \cap C_G(K) = 1\) and \(G/C_G(K) \lesssim D_6\). Thus \(C_G(K) \neq 1\).

Hence, \(1 \neq C_G(K) \equiv C_G(K)/K \leq G/K \cong L\). It follows that \(L \cong G/K \cong C_G(K)\) because \(L\) is simple. Therefore, \(G \cong D_6 \times L\), which implies that \(\text{deg}(2) = 5\), a contradiction. \(\Box\)

**Proposition 3.8.** If \(M = L : (D_6)_2\), then \(G \cong L : (D_6)_2, \mathbb{Z}_2 \times (L : 3), \mathbb{Z}_3 \times (L : 2_2), (\mathbb{Z}_3 \times L)\mathbb{Z}_2, \mathbb{Z}_6 \times L\) or \(S_3 \times L\).
Proof: As $|L : (D_6)_2| = 2^{25}.3^6.5^4.7.13.17^2$ and $\pi_e(L : (D_6)_2) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 17, 18, 20, 21, 24, 26, 30, 34, 39, 40, 42, 45, 51, 60, 63, 65, 68, 85, 102, 126, 130, 170, 255\}$, then $D(L : (D_6)_2) = (5, 5, 4, 2, 3, 3)$. Since $|G| = |L : (D_6)_2|$ and $D(G) = D(L : (D_6)_2)$, we conclude that $\Gamma(G)$ has the following form (like $\Gamma(L : (D_6)_2)$):

$\begin{array}{c}
\text{3} \\
\text{5} \\
\text{2} \\
\text{7} \\
\text{13} \\
\text{17}
\end{array}$

Figure 3.8

**Step 1.** Let $K$ be the maximal normal solvable subgroup of $G$. Then $K$ is a $\{2, 3\}$-group. In particular, $G$ is non-solvable.

The proof is similar to Step 1 in Proposition 3.5.

**Step 2.** The quotient $G/K$ is an almost simple group. In fact, $S \leq G/K \leq \text{Aut}(S)$, where $S$ is a finite non-abelian simple group.

Let $G/K = \frac{G}{K}$. Then $S := \text{Soc}(G)$, $S = P_1 \times P_2 \times \ldots \times P_m$, where $P_i$s are finite non-abelian simple groups and $S \leq \frac{G}{K} \leq \text{Aut}(S)$. We are going to prove that $m = 1$ and $S = P_1$. Suppose that $m \geq 2$. By the same argument in Step 2 of Proposition 3.3 and considering 7 instead of a, we get a contradiction. Therefore $m = 1$ and $S = P_1$.

By TABLE 1 and Step 1, it is evident that $|S| = 2^\alpha.3^\beta.5^4.7.13.17^2$, where $2 \leq \alpha \leq 25$ and $1 \leq \beta \leq 6$. Now, using collected results contained in TABLE 1, we conclude that $S \cong D_4(4)$ and by Step 2, $L \leq \frac{G}{K} \leq \text{Aut}(L)$. As $|G| = |L : (D_6)_2| = 6|L|$, we deduce $|K| = 1, 2, 3$ or 6.

If $|K| = 1$, then $G \cong L : (D_6)_1$, $L : (D_6)_2$ or $L : 6$ because $|G| = 6|L|$. Obviously $G \cong L : (D_6)_2$ because in $\Gamma(L : (D_6)_1)$ and $\Gamma(L : 6)$, we have $\text{deg}(13) = 2$ (see page 17).

If $|K| = 2$, then $K \leq Z(G)$ and $G/K \cong L : 3$. Hence $G$ is a central extension of $K$ by $L : 3$. If $G$ splits over $K$, then $G \cong \mathbb{Z}_2 \times (L : 3)$. Otherwise we get a contradiction because $|K|$ must divide the Schure multiplier of $L : 3$, which is impossible.

If $|K| = 3$, then $G/K \cong L : 2_1, L : 2_2$ or $L : 2_3$. But $G/C_G(K) \leq \text{Aut}(K) \cong \mathbb{Z}_2$. Thus $|G/C_G(K)| = 1$ or 2. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is, $G$ is a central extension of $K$ by $L : 2_1, L : 2_2$ or $L : 2_3$. If $G$ splits over $K$, then only $G \cong \mathbb{Z}_3 \times (L : 2_2)$ because $2 \approx 13$ in $\Gamma(\mathbb{Z}_3 \times (L : 2_1))$ and $\Gamma(\mathbb{Z}_3 \times (L : 2_2))$. Otherwise we get a contradiction because $|K|$ must divide the Schure multiplier of $L : 2_1, L : 2_2$ and $L : 2_3$, which is impossible.
$|G/C_G(K)| = 2$, then $K < C_G(K)$ and $1 \neq C_G(K)/K \trianglelefteq G/K \cong L : 2_1, L : 2_2$ or $L : 2_3$, we obtain $C_G(K)/K \cong L$. Since $K \leq Z(C_G(K))$, $C_G(K)$ is a central extension of $K$ by $L$. If $C_G(K)$ splits over $K$, then $C_G(K) \cong Z_3 \times L$, otherwise we get a contradiction because $|K|$ must divide the Schur multiplier of $L$, which is impossible. Therefore, $G \cong (Z_3 \times L) \cdot Z_2$.

If $|K| = 6$, then $G/K \cong L$ and $K \cong Z_6$ or $D_6$. If $K \cong Z_6$, then $G/C_G(K) \leq Z_2$ and so $|G/C_G(K)| = 1$ or $2$. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$ and $G/K \cong L$. Therefore $G$ is a central extension of $K$ by $L$. If $G$ is a non-split extension of $K$ by $L$, then $|K|$ must divide the Schur multiplier of $L$, which is $1$. But this is a contradiction. So we obtain that $G$ splits over $K$. Hence $G \cong Z_6 \times L$. If $|G/C_G(K)| = 2$, then $K < C_G(K)$ and $1 \neq C_G(K)/K \leq G/K \cong L$, which is a contradiction because $L$ is simple. If $K \cong D_6$, then $K \cap C_G(K) = 1$ and $G/C_G(K) \leq D_6$. Thus $C_G(K) \neq 1$. Hence, $1 \neq C_G(K) \cong C_G(K)/K \leq G/K \cong L$. It follows that $L \cong G/K \cong C_G(K)$ because $L$ is simple. Therefore $G \cong D_6 \times L$.

**Proposition 3.9.** If $M = L : 6$, then $G \cong L : 6$, $L : (D_6)_1$, $Z_3 \times (L : 2_1)$, $Z_3 \times (L : 2_3)$ or $(Z_3 \times L).Z_2$.

**Proof.** As $|L : 6| = 2^{25}.3^6.5^4.7.13.17^2$ and $\pi_e(L : 6) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12 \, \, \, 13, 14, 15, 16, 17, 18, 20, 21, 24, 30, 34, 36, 39, 42, 45, 48, 51, 63, 65, 85, 255\}$, then $D(L : 6) = (4, 5, 4, 2, 2, 3)$. Since $|G| = |L : 6|$ and $D(G) = D(L : 6)$, there exist several possibilities for $\Gamma(G)$ similarly to Proposition 3.7:

![Figure 3.9](image-url)

where $\{a, b\} = \{7, 13\}$.

**Step 1.** Let $K$ be the maximal normal solvable subgroup of $G$. Then $K$ is a $\{2, 3, 5\}$-group. In particular, $G$ is non-solvable.

The proof is similar to that in Proposition 3.3.

**Step 2.** The quotient $G/K$ is an almost simple group. In fact, $S \leq G/K \leq \mbox{Aut}(S)$, where $S$ is a finite non-abelian simple group.

Again we refer to Step 2 of proposition 3.3 to get the proof.

By TABLE 1 and Step 1, it is evident that $|S| = 2^\alpha.3^\beta.5^\gamma.7.13.17^2$, where $2 \leq \alpha \leq 25$, $1 \leq \beta \leq 6$ and $0 \leq \gamma \leq 4$. Now, using collected results contained...
in TABLE 1, we conclude that $S \cong D_4(4)$ and by Step 2, $L \leq \frac{G}{K} \leq \text{Aut}(L)$. As $|G| = |L:6| = 6|L|$, we deduce $|K| = 1, 2, 3$ or $6$.

If $|K| = 1$, then $G \cong L : 6, L : (D_6)_1$ or $L : (D_6)_2$ because $|G| = 6|L|$. Obviously, $G \cong L : 6$ or $L : (D_6)_1$ because $\text{deg}(2) = 5$ in $\Gamma(L : (D_6)_2)$ (see page 18).

If $|K| = 2$, then $K \leq Z(G)$ and so $\text{deg}(2) = 5$, which is a contradiction.

If $|K| = 3$, then $G/K \cong L : 2_1$, $L : 2_2$ or $L : 2_3$. But $G/C_G(K) \leq \text{Aut}(K) \cong \mathbb{Z}_2$. Thus $|G/C_G(K)| = 1$ or $2$. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is, $G$ is a central extension of $K$ by $L : 2_1$, $L : 2_2$ or $L : 2_3$. If $G$ splits over $K$, then $G \cong \mathbb{Z}_3 \times (L : 2_1)$ or $\mathbb{Z}_3 \times (L : 2_2)$ because in $\Gamma(\mathbb{Z}_3 \times (L : 2_2))$ the degree of $2$ is $5$. Otherwise we get a contradiction because $|K|$ must divide the Schur multiplier of $L : 2_1, L : 2_2$ and $L : 2_3$, which is impossible. If $|G/C_G(K)| = 2$, then $K < C_G(K)$ and $1 \neq C_G(K)/K \leq G/K \cong L : 2_1, L : 2_2$ or $L : 2_3$, we obtain $C_G(K)/K \cong L$. Since $K \leq Z(C_G(K))$, $C_G(K)$ is a central extension of $K$ by $L$. If $C_G(K)$ splits over $K$, then $C_G(K) \cong \mathbb{Z}_3 \times L$, otherwise we get a contradiction because $|K|$ must divide the Schur multiplier of $L$, which is impossible. Therefore, $G \cong \mathbb{Z}_3 \times L, \mathbb{Z}_2$.

If $|K| = 6$, then $G/K \cong L$ and $K \cong \mathbb{Z}_6$ or $D_6$. If $K \cong \mathbb{Z}_6$, then $G/C_G(K) \leq \mathbb{Z}_2$ and so $|G/C_G(K)| = 1$ or $2$. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$. It follows that $\text{deg}(2) = 5$, a contradiction. If $|G/C_G(K)| = 2$, then $K < C_G(K)$ and $1 \neq C_G(K)/K \leq G/K \cong L$, which is a contradiction because $L$ is simple. If $K \cong D_6$, then $K \cap C_G(K) = 1$ and $G/C_G(K) \cong D_6$. Thus $C_G(K) \neq 1$. Hence, $1 \neq C_G(K) \cong C_G(K)/K \cong G/K \cong L$. It follows that $L \cong G/K \cong C_G(K)$ because $L$ is simple. Therefore, $G \cong D_6 \times L$, which implies that $\text{deg}(2) = 5$, a contradiction.

\textbf{Proposition 3.10.} If $M = L : D_{12}$, then $G \cong L : D_{12}$, $\mathbb{Z}_2 \times (L : (D_6)_1), \mathbb{Z}_2 \times (L : (D_6)_2), \mathbb{Z}_2 \times (L : (D_6)_1), \mathbb{Z}_2 \times (L : (D_6)_2), \mathbb{Z}_3 \times (L : 6), \mathbb{Z}_3 \times (L : 2^2), (\mathbb{Z}_4 \times (L : 2_1)), \mathbb{Z}_4 \times (L : 2_2)), \mathbb{Z}_2, (\mathbb{Z}_3 \times (L : 2_3)), \mathbb{Z}_2, (\mathbb{Z}_4 \times (L : 3)), (\mathbb{Z}_2 \times \mathbb{Z}_2) \times (L : 3), (\mathbb{Z}_4 \times L), \mathbb{Z}_3, (\mathbb{Z}_2 \times \mathbb{Z}_2) \times L, \mathbb{Z}_2, \mathbb{Z}_4 \times (L : 2_1), \mathbb{Z}_6 \times (L : 2_2), \mathbb{Z}_6 \times (L : 2_3), (\mathbb{Z}_6 \times L), \mathbb{Z}_2, S_3 \times (L : 2_1), S_3 \times (L : 2_2), S_3 \times (L : 2_3), \mathbb{Z}_12 \times L, (\mathbb{Z}_2 \times \mathbb{Z}_6) \times L, D_{12} \times L, (\mathbb{Z}_2 \times L), D_6, \mathbb{A}_4 \times L, L.\mathbb{A}_4$ or $T \times L$.

\textbf{Proof.} As $|L : D_{12}| = 2^{26}.3^6.5^4.7.13.17^2$ and $\pi_v(L : (D_{12})) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 24, 26, 30, 34, 39, 40, 42, 45, 51, 60, 63, 65, 68, 85, 102, 126, 130, 170, 255\}$, then $D(L : D_{12}) = (5, 5, 4, 2, 3, 3)$. Since $|G| = |L : D_{12}|$ and $D(G) = D(L : D_{12})$, we conclude that $\Gamma(G)$ has the following form (like $\Gamma(L : D_{12})$):
Step 1. Let $K$ be the maximal normal solvable subgroup of $G$. Then $K$ is a $\{2, 3\}$-group. In particular, $G$ is non-solvable.

The proof is similar to Step 1 in Proposition 3.5.

Step 2. The quotient $G/K$ is an almost simple group. In fact, $S \leq G/K \leq \text{Aut}(S)$, where $S$ is a finite non-abelian simple group.

To get the proof, follow the way in the proof of Step 2 in Proposition 3.5.

By TABLE 1 and Step 1, it is evident that $|S| = 2^6 \cdot 3^2 \cdot 5^4 \cdot 7 \cdot 13 \cdot 17^2$, where $2 \leq \alpha \leq 26$ and $1 \leq \beta \leq 6$. Now, using collected results contained in TABLE 1, we conclude that $S \cong D_4(4)$ and by Step 2, $L \leq G/K \leq \text{Aut}(L)$. As $|G| = |L : D_{12}| = 12|L|$, we deduce $|K| = 1, 2, 3, 4, 6$ or 12.

If $|K| = 1$, then $G \cong L : D_{12}$.

If $|K| = 2$, then $G/K \cong L : (D_6)_1$ or $L : (D_6)_2$ or $L : 6$ and $K \leq Z(G)$. It follows that $G$ is a central extension of $K$ by $L : (D_6)_1$, $L : (D_6)_2$ or $L : 6$. If $G$ splits over $K$, then $G \cong \mathbb{Z}_2 \times (L : (D_6)_1)$, $\mathbb{Z}_2 \times (L : (D_6)_2)$ or $\mathbb{Z}_2 \times (L : 6)$.

Otherwise $G \cong \mathbb{Z}_2 \times (L : (D_6)_1)$ or $\mathbb{Z}_2 \times (L : (D_6)_2)$.

If $|K| = 3$, then $G/K \cong L : 2^2$. But $G/C_G(K) \not\leq \text{Aut}(K) \cong \mathbb{Z}_2$. Thus $|G/C_G(K)| = 1$ or 2. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is, $G$ is a central extension of $K$ by $L : 2^2$. If $G$ splits over $K$, then $G \cong \mathbb{Z}_3 \times (L : 2^2)$.

Otherwise we get a contradiction because $|K|$ must divide the Schur multiplier of $L : 2^2$, which is impossible. If $|G/C_G(K)| = 2$, then $K < C_G(K)$ and $1 \neq C_G(K)/K \cong G/K \cong L : 2^2$, and we obtain $C_G(K)/K \cong L : 2_1$, $L : 2_2$ or $L : 2_3$. Since $K \leq Z(C_G(K))$, $C_G(K)$ is a central extension of $K$ by $L : 2_1$, $L : 2_2$ or $L : 2_3$. Thus $C_G(K) \cong \mathbb{Z}_3 \times (L : 2_1)$, $\mathbb{Z}_3 \times (L : 2_2)$ or $\mathbb{Z}_3 \times (L : 2_3)$, otherwise we get a contradiction because 3 must divide the Schur multiplier of $L : 2_1$, $L : 2_2$ or $L : 2_3$, which is impossible. Therefore, $G \cong (\mathbb{Z}_3 \times (L : 2_1)) \mathbb{Z}_2$, $(\mathbb{Z}_3 \times (L : 2_2)) \mathbb{Z}_2$ or $(\mathbb{Z}_3 \times (L : 2_3)) \mathbb{Z}_2$.

If $|K| = 4$, then $G/K \cong L : 3$ and $K \cong \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. In this case we have $G/C_G(K) \not\leq \text{Aut}(K) \cong \mathbb{Z}_2$ or $S_3$. Thus $|G/C_G(K)| = 1, 2, 3$ or 6. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is, $G$ is a central extension of $K$ by $L : 3$. If $G$ splits over $K$ by $L : 3$, then $G \cong \mathbb{Z}_4 \times (L : 3)$ or $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times (L : 3)$. Otherwise we get a contradiction because $|K|$ must divide the Schur multiplier of $L : 3$, which is impossible. If $|G/C_G(K)| \neq 1$, since $|G/C_G(K)| = 2, 3$ or 6, it follows that $K < C_G(K)$. As $L$ is simple, we conclude that $1 \neq C_G(K)/K$ must
be an extension of \( L \). Hence \( |G/C_G(K)| = 3 \) and therefore \( C_G(K)/K \cong L \). Now, since \( K \leq Z(C_G(K)) \), we conclude that \( C_G(K) \) is a central extension of \( K \) by \( L \). Thus \( C_G(K) \cong \mathbb{Z}_4 \times L \) or \( (\mathbb{Z}_2 \times \mathbb{Z}_2) \times L \), otherwise \( |K| \) must divide the Schur multiplier of \( L \), which is 1 and it is impossible. Therefore, \( G \cong (\mathbb{Z}_4 \times L) \mathbb{Z}_3 \) or \( ((\mathbb{Z}_2 \times \mathbb{Z}_2) \times L) \mathbb{Z}_3 \).

If \( |K| = 6 \), then \( G/K \cong L : 2_1 \) or \( L : 2_2 \) or \( L : 2_3 \) and \( K \cong \mathbb{Z}_6 \) or \( D_6 \). If \( K \cong \mathbb{Z}_6 \), then \( G/C_G(K) \cong Z_2 \) and so \( |G/C_G(K)| = 1 \) or 2. If \( |G/C_G(K)| = 1 \), then \( K \leq Z(G) \), that is \( G \) is a central extension of \( \mathbb{Z}_6 \) by \( L : 2_1 \), \( L : 2_2 \) or \( L : 2_3 \). If \( G \) splits over \( K \), we obtain \( G \cong \mathbb{Z}_6 \times (L : 2_1) \), \( \mathbb{Z}_6 \times (L : 2_2) \) or \( \mathbb{Z}_6 \times (L : 2_3) \), otherwise we get a contradiction because \( |K| \) must divide the Schur multiplier of \( L \), which is 1 and it is impossible. If \( |G/C_G(K)| = 2 \), then \( K < C_G(K) \) and \( 1 \neq C_G(K)/K \cong G/K \cong L : 2_1 \), \( L : 2_2 \) or \( L : 2_3 \), and we obtain \( C_G(K)/K \cong L \). Since \( K \leq Z(C_G(K)) \), \( C_G(K) \) is a central extension of \( K \) by \( L \). Thus \( C_G(K) \cong \mathbb{Z}_6 \times L \), otherwise we get a contradiction because \( |K| \) must divide the Schur multiplier of \( L \). Therefore \( G \cong (\mathbb{Z}_6 \times L) \mathbb{Z}_2 \). If \( K \cong D_6 \), then \( G/C_G(K) \cong D_6 \) and so \( |G/C_G(K)| = 1, 2, 3 \) or \( 6 \). If \( |G/C_G(K)| = 1 \), then \( K \leq Z(G) \), that is a contradiction. If \( |G/C_G(K)| = 2 \), then we have \( |KC_G(K)| = 6, |G|/2 = 3|G| \) because \( K \cap C_G(K) = 1 \), which is a contradiction. If \( |G/C_G(K)| = 3 \), then we have \( |KC_G(K)| = 6, |G|/3 = 2|G| \) because \( K \cap C_G(K) = 1 \), which is a contradiction. If \( |G/C_G(K)| = 6 \), then \( G/C_G(K) \cong D_6 \) and \( C_G(K) \neq 1 \). Hence, \( 1 \neq C_G(K) \cong C_G(K)/K \cong \mathbb{Z}_6 \times L : 2_1 \) or \( L : 2_2 \) or \( L : 2_3 \). It follows that \( C_G(K) \cong L : 2_3 \) because \( L \) is simple. Therefore, \( G \cong D_6 \times (L : 2_1) \), \( D_6 \times (L : 2_2) \) or \( D_6 \times (L : 2_3) \).

Before processing the last case, we recall the following facts.

There exist five non-isomorphic groups of order 12. Two of them are abelian and three are non-abelian. The non-abelian groups are: alternating group \( A_4 \), dihedral group \( D_{12} \) and the dicyclic group \( T \) with generators \( a \) and \( b \), subject to the relations \( a^6 = 1, a^3 = b^2 \) and \( b^{-1}ab = a^{-1} \).

If \( |K| = 12 \), then \( G/K \cong L \) and \( K \cong \mathbb{Z}_{12}, \mathbb{Z}_2 \times \mathbb{Z}_6, D_{12}, A_4 \) or \( T \). But \( C_G(K)K/K \cong G/K \cong L \). If \( C_G(K)K/K = 1 \), then \( C_G(K) \leq K \) and hence \( |L| = |G/K||G/C_G(K)|||\text{Aut}(K)| \). Thus \( |L||\text{Aut}(K)| \), a contradiction. Therefore, \( C_G(K)K/K \neq 1 \) and since \( L \) is simple group, we conclude that \( G = C_G(K)K \) and hence, \( G/C_G(K) \cong K/Z(K) \). Now, we should consider the following cases:

If \( K \cong \mathbb{Z}_{12} \) or \( \mathbb{Z}_2 \times \mathbb{Z}_6 \), then \( G/C_G(K) = 1 \). Therefore \( K \leq Z(G) \), that is \( G \) is a central extension of \( \mathbb{Z}_{12} \) or \( \mathbb{Z}_2 \times \mathbb{Z}_6 \) by \( L \). If \( G \) splits over \( K \), we obtain \( G \cong \mathbb{Z}_{12} \times L \) or \( \mathbb{Z}_2 \times \mathbb{Z}_6 \times L \), otherwise we get a contradiction because \( |K| \) must divide the Schur multiplier of \( L \), which is 1 and it is impossible.
If $K \cong D_{12}$, then $G = K.L$ and $G/C_G(K) \cong D_6$. Since $C_G(K)/Z(K) \cong G/K \cong L$ and $Z(K) \leq Z(C_G(K))$, we conclude that $C_G(K)$ is a central extension of $Z(K) \cong \mathbb{Z}_2$ by $L$. If $C_G(K)$ is a non-split extension, then 2 must divide the Schure multiplier of $L$, which is 1 and it is impossible. Thus $C_G(K) \cong \mathbb{Z}_2 \times L$ and hence, $G$ is a split extension of $K$ by $L$. Now, since $\text{Hom}(L, Aut(D_{12}))$ is trivial, we have $G \cong D_{12} \times L$.

If $K \cong A_4$, then $G/C_G(K) \cong A_4$. As $G = C_G(K)K$, it follows that $C_G(K) \cong L$. Therefore $G \cong L \times A_4$ or $L.A_4$.

If $K \cong T$, then by the similar way in case $K \cong D_{12}$, we can conclude that $G$ is a split extension of $K$ by $L$. Also, since $\text{Hom}(L, Aut(T))$ is trivial, we have $G \cong T \times L$. □

According to what we said before the proof, here we depict $\Gamma(M)$ by $|M|$ and $\pi_e(M)$, where $M$ is an almost simple group related to $L = D_4(4)$.

\[
\begin{align*}
\Gamma(L) & : 
\begin{array}{c}
3 \\
5 \\
7
\end{array} & 
\begin{array}{c}
2 \\
13 \\
17
\end{array}
\end{align*}
\]

\[
\begin{align*}
\Gamma(L : 2_1) & : 
\begin{array}{c}
3 \\
5 \\
7
\end{array} & 
\begin{array}{c}
2 \\
13 \\
17
\end{array}
\end{align*}
\]

\[
\begin{align*}
\Gamma(L : 2_2) & : 
\begin{array}{c}
5 \\
2 \\
7
\end{array} & 
\begin{array}{c}
13 \\
17
\end{array}
\end{align*}
\]
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\[
\begin{align*}
\text{\Gamma}(L : 2) & : 3 & & 5 & & 13 & & 17 & & 2 & & 5 \\
& & & & & & & & & & & \\
\text{\Gamma}(L : 3) & : 3 & & 17 & & 13 & & 2 & & 5 \\
& & & & & & & & & & & \\
\text{\Gamma}(L : 2^2) & : 3 & & 5 & & 17 & & 13 & & 2 & & 5 \\
& & & & & & & & & & & \\
\text{\Gamma}(L : (D_6)_{1}) & : 3 & & 5 & & 17 & & 13 & & 2 & & 5 \\
\end{align*}
\]
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References